Star-Free Modules

Milan Lekár, Jan Paseka, Richard Smolka
Faculty of Science, Masaryk University
paseka@math.muni.cz

Dynamic algebra provides an algebraic framework for expressing weakest preconditions of programs and has been widely used in formal program verification. When one wishes to verify *quantum* programs, it is natural to extend dynamic algebra to settings where the underlying propositional logic is not classical but quantum—that is, where the lattice of propositions is an *orthomodular lattice* rather than a Boolean algebra.

Orthomodular lattices arise naturally in quantum mechanics: the closed subspaces of any Hilbert space form a complete orthomodular lattice, the so-called Hilbert lattice. Crucially, orthomodular lattices are in general non-distributive, which distinguishes quantum logic from classical propositional logic and forces a careful re-examination of algebraic constructions that are routine in the Boolean setting.

A *star-free dynamic orthomodular lattice* (introduced in [2]) is an orthomodular lattice equipped with a necessity operator $\Box(a, p)$—read as "p holds after executing program a"—for programs built from sequential composition, nondeterministic choice, and tests, but *without* the Kleene star. The Kleene star is excluded because its treatment in the orthomodular/quantum setting raises substantial additional difficulties.

The main result of [2] is a *Stone-type representation theorem*: every star-free dynamic orthomodular lattice embeds into its canonical extension, a complete star-free dynamic orthomodular lattice constructed from the proper filters of the original one. This mirrors the classical Stone representation of Boolean algebras by set algebras, and it establishes one half of a discrete duality between star-free dynamic orthomodular lattices and the corresponding Kripke-style frames (star-free dynamic orthomodular frames).

The present paper develops a *module-theoretic* perspective on this structure. Rather than the necessity operator $\Box(a, p)$ of [2] (which corresponds to the *weakest precondition*), we work with a dual action $\ell(a, p)$ that propagates information *forward* along programs. The program terms form a term algebra $QA[\Pi, L]$ which, after quotienting by the natural behavioural equivalence, yields a unital involutive m-semilattice $S_{\mathbf{Dl}}$ acting on the orthomodular lattice L (see [1]). This algebraic structure is the module-theoretic analogue of the complex algebras studied in [2].

A central subtlety—illustrated by a detailed example —is that the involution on programs does not descend to the naive quotient by behavioural equivalence of forward actions alone. One must simultaneously track the behaviour of the dual program a^*, leading to the strengthened equivalence $\approx_{\mathbf{Dl}}$. This phenomenon has a direct counterpart in the frame semantics of [2], where the self-adjointness axiom for test relations plays an analogous role.

References

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[2] T. Takagi. *A Stone-type representation theorem for star-free dynamic orthomodular lattice,* In: Proceedings of the 56th International Symposium on Multiple-Valued Logic (ISMVL 2026), pp. 129–134, IEEE (2026).