

Completely Join-preserving Maps between Commutator Lattices as Abstractions for Congruence Extensions

George GEORGESCU, Claudia MUREȘAN
University of Bucharest

Leonard KWUIDA
Bern University of Applied Sciences

`cmuresan@fmi.unibuc.ro`

Commutator lattices [7] are complete lattices endowed with an additional binary operation, called *commutator*, which is commutative, smaller than its arguments and completely distributive w.r.t. the join. They serve as abstractions for congruence lattices of algebras whose term-condition commutators are commutative and distributive w.r.t. arbitrary joins, in particular for congruence lattices of members of congruence-modular varieties endowed with the modular commutator.

The (*minimal*) *prime spectrum* of a commutator lattice is its set of (minimal) prime elements w.r.t. the commutator, and its topological structure turns out to be particularly useful. In [2, 3], we have constructed and studied the reticulation of a universal algebra by using the *Stone topology* on the prime spectrum of its congruence lattice. In [4] we have studied the Stone, as well as the *flat topology* on the minimal prime spectrum of congruences, then used these topologies to investigate congruences in extensions of universal algebras, generalizing results from [1] on ideals in ring extensions.

My talk will focus on the abstract case for this theory, which we have developed in [5, 6] by using complete join-semilattice morphisms between commutator lattices as an abstraction for the action on congruences of morphisms between similar algebras. The right adjoint of such a join-semilattice morphism turns out to be continuous and even a homeomorphism between the minimal prime spectra of the two commutator lattices (its domain and codomain) w.r.t. the Stone and flat topologies under certain conditions regarding the prime spectra and compact elements of these lattices.

While this abstraction has allowed us to treat the general case of arbitrary morphisms between similar algebras, it would be interesting to retrieve from it the particular case in [4] of extensions, so that of algebra embeddings. Another correspondence between this abstract case and the one of congruence lattices that remains to be investigated is identifying, among the compact elements of various kinds of algebraic commutator lattices, those that correspond to principal congruences in this concrete case of congruence lattices. Approaching these

open problems would obviously rely on obtaining representations for certain algebraic commutator lattices as congruence lattices endowed with the term-condition commutator.

Keywords: (algebraic) commutator lattice; (minimal) prime element; (Zariski/Stone/spectral, flat/inverse) topology; (algebra, ring) extension.

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