Residuated partially ordered algebras, logic and preclones

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In the first lecture we introduce partially ordered algebras with operations that are monotone or antitone in each argument. The standard examples are algebras of binary relations, residuated po-semigroups and residuated lattices, as well as many po-subvarieties and their connection with substructural logic and algebraic proof theory. The second installment concentrates on the structure theory of residuated po-algebras, considering poset decompositions and po-Plonka sums. In the last lecture we focus on the general theory of S-preclones for a finite monoid S, which generalizes clone theory to the po-algebra setting and beyond.

In more detail, we aim to cover the following topics:

Lecture 1: Tonicities, inequational logic, and partially ordered varieties (based on standard sources and joint work with Marta Bilkova and Melissa Sugimoto). We begin with the universal algebraic theory of partially ordered algebras (τ-po-algebras) specified by a set $\mathcal{F}$ of function symbols and a tonicity $\tau : \mathcal{F} \to \{+, -\}^*$. This framework allows primitive operations to be monotone ($+$) or antitone ($-$) at specified argument positions, naturally capturing structures such as left-residuated po-monoids, residuated po-semigroups, residuated lattices, po-groups, tense posets, and algebras of binary relations. In this setting, inequational logic is treated as a 2-dimensional deductive system within Abstract Algebraic Logic (AAL). We consider τ-preorders (the order analogues of congruences) and cover order-theoretic counterparts of the fundamental isomorphism and correspondence theorems, Birkhoff's subdirect representation theorem, and the order H-S-P and S-$\mathsf{L}_{\to}$-P characterization theorems for τ-povarieties and τ-quasivarieties. We also discuss aspects of algebraizability of substructural logics and algebraic proof theory.

Lecture 2: Structure theory via po-Płonka sums of residuated po-semigroups (based on joint work with Stefano Bonzio, José Gil-Férez, Adam Přenosil, Melissa Sugimoto and Giuseppe Zecchini). The second lecture focuses on the structure theory of residuated partially ordered semigroups $\langle A, \leqslant, \cdot, \backslash, / \rangle$, where $x \cdot y \leqslant z \iff x \leqslant z/y \iff y \leqslant x\backslash z$. We investigate the role of positive central idempotents $p \in \mathrm{ZE}^+\mathbf{A}$ ($a \leqslant pa = ap$ and $p^2 = p$) and balanced residuated semigroups (satisfying $x\backslash x \approx x/x =: 1_x$ and $y \leqslant 1_x y$). To reconstruct such algebras from their fibers $\mathbf{A}_p = \{a \in A : 1_a = p\}$, we generalize the classical Płonka sum construction to *semilattice-directed systems of metamorphisms* $\Xi = \{\xi_{pq} : \mathbf{A}_p \looparrowright \mathbf{A}_q : p \preccurlyeq q \in I\}$. Unlike homomorphisms, metamorphisms handle each operation and relation coordinatewise via distinct families of maps $\langle \varphi_{pq}, \psi_{pq} \rangle$. We establish compatibility conditions under which the reconstructed partial order

$$a \leqslant b \iff \varphi_{pr}(a) \leqslant_r \psi_{qr}(b) \quad (\text{where } r = p \vee q)$$

yields a valid po-algebra. This establishes that every *steady* residuated semigroup decomposes into a po-Płonka sum of integrally closed residuated monoids (and into Brouwerian semilattices in the idempotent commutative case).

Lecture 3: S-preclones, S-relations, and the $^S\mathrm{Pol}$–$^S\mathrm{Inv}$ Galois connection (based on joint work with Erkko Lehtonen and Reinhard Pöschel). In the final lecture, we present the general theory of S-preclones for an arbitrary finite monoid S of signa, generalizing classical clone theory to signed operations. An S-operation $f : A^n \to A$ carries an argument signum $\mathrm{sgn}(f) = (s_1, \ldots, s_n) \in S^n$. S-preclones are sets of S-operations containing the identity id_A and closed under cyclic shifts (ζ), argument transpositions (τ), addition of fictitious signed arguments (∇^s), identification of equal-signum arguments (Δ), and composition ($\circ$). Dually, an m-ary S-relation is a family $\boldsymbol{\varrho} = (\varrho_s)_{s \in S}$ with $\varrho_s \subseteq A^m$. We introduce the S-preservation relation

$$f \rhd^S \boldsymbol{\varrho} \iff \forall s \in S : f(\varrho_{s_1 s}, \ldots, \varrho_{s_n s}) \subseteq \varrho_s$$

and study the induced Galois connection $^S\mathrm{Pol}$–$^S\mathrm{Inv}$. For finite A, we prove the Galois closure theorems: $^S\langle F \rangle = {}^S\mathrm{Pol}\, {}^S\mathrm{Inv}\, F$ and $^S[Q] = {}^S\mathrm{Inv}\, {}^S\mathrm{Pol}\, Q$. Furthermore, we analyze the lattice $^S\mathcal{L}_A$ of all S-preclones on A, showing that it is atomic and coatomic with finitely many atoms and coatoms, explore embeddings of the classical clone lattice $\mathcal{L}_A$, and highlight the motivating case of $\pm$-preclones over the group $S = \{+, -\}$ characterizing monotone and antitone operations on posets.

References

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