Some new semidirect-product-type constructions of residuated lattices

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Representation theorems for residuated posets and residuated lattices are well understood in the presence of the classical axioms of *divisibility* and *prelinearity*: ordinal sums, poset products and the Γ-functor all rest on them. As soon as these axioms are dropped, the known representation techniques largely fail. We introduce two new associative product-type constructions — inspired by wreath and semidirect products of semigroups — for arbitrary **integral residuated lattices**, where neither commutativity nor divisibility is assumed.

Products via nuclei. A closure operator γ on an integral residuated lattice $\mathbf{A}$ is a *nucleus* if $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$; it is a *strong nucleus* if, in addition, $\gamma(x)\backslash\gamma(y) = \gamma(x\backslash y)$ and $\gamma(x)/\gamma(y) = \gamma(x/y)$ for all $x, y \in A$. We write $\mathrm{SN}\,\mathbf{A}$ for the set of all strong nuclei on $\mathbf{A}$.

Definition (J. Kühr). Let $\mathbf{A}$ and $\mathbf{B}$ be integral residuated lattices. A mapping $\boldsymbol{\alpha}\colon B \to \mathrm{SN}\,\mathbf{A}$ is a *nuclei-morphism* from $\mathbf{B}$ to $\mathbf{A}$, written $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$, if

$$\boldsymbol{\alpha}[1] = I_A \qquad \text{and} \qquad \boldsymbol{\alpha}[x] \circ \boldsymbol{\alpha}[y] = \boldsymbol{\alpha}[x \wedge y] = \boldsymbol{\alpha}[x \cdot y] \qquad (x, y \in B).$$

Theorem. If $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$, then the algebra $\mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}$ with universe $\sum_{x \in B} \boldsymbol{\alpha}[x]A$, unit $(1, 1)$, componentwise lattice operations (taken in the nucleus images) and

$$\begin{aligned}
(a_1, b_1) \cdot (a_2, b_2) &= \big(\boldsymbol{\alpha}[b_1 \cdot b_2](a_1 \cdot a_2),\ b_1 \cdot b_2\big),\\
(a_1, b_1)\backslash(a_2, b_2) &= \big(a_1 \backslash \boldsymbol{\alpha}[b_1](a_2),\ b_1 \backslash b_2\big),\\
(a_2, b_2)/(a_1, b_1) &= \big(\boldsymbol{\alpha}[b_1](a_2)/a_1,\ b_2/b_1\big),
\end{aligned}$$

is again an integral residuated lattice.

Theorem (Associativity). Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be integral residuated lattices.

(i) If $\boldsymbol{\alpha}\colon \mathbf{B} \curvearrowright \mathbf{A}$ and $\boldsymbol{\beta}\colon \mathbf{C} \curvearrowright \mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}$, then there are nuclei-morphisms $\overline{\boldsymbol{\beta}}\colon \mathbf{C} \curvearrowright \mathbf{B}$ and $\overline{\boldsymbol{\alpha}}\colon \mathbf{B} \ltimes_{\overline{\boldsymbol{\beta}}} \mathbf{C} \curvearrowright \mathbf{A}$ such that

$$(\mathbf{A} \ltimes_{\boldsymbol{\alpha}} \mathbf{B}) \ltimes_{\boldsymbol{\beta}} \mathbf{C} \;\cong\; \mathbf{A} \ltimes_{\overline{\boldsymbol{\alpha}}} (\mathbf{B} \ltimes_{\overline{\boldsymbol{\beta}}} \mathbf{C}).$$

(ii) Conversely, every such pair $(\overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ arises from a unique pair $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as in (i), and the two assignments are mutually inverse.

Products via product morphisms. A more general, purely order-theoretic construction avoids nuclei altogether.

Definition. A mapping $\phi\colon B \to A$ between integral residuated lattices $\mathbf{B}$ and $\mathbf{A}$ is a *product morphism* if

$$\phi(1) = 1, \qquad \phi(x) \wedge \phi(y) = \phi(x \wedge y), \qquad \phi(x) \cdot \phi(y) \leq \phi(x \cdot y)$$

for all $x, y \in B$.

Theorem. For a product morphism $\phi\colon \mathbf{B} \to \mathbf{A}$ the algebra

$$\mathbf{A} \ltimes_\phi \mathbf{B} = \Big(\textstyle\sum_{x \in B} (\phi(x)],\ \wedge,\ \vee,\ \cdot,\ \backslash,\ /,\ (1,1) \Big),$$

with componentwise $\wedge, \vee, \cdot$ and

$$(a, x) \bullet (b, y) = \big((a \bullet b) \wedge \phi(x \bullet y),\ x \bullet y \big) \qquad \text{for } \bullet \in \{\backslash, /\},$$

is again an (integral) residuated lattice.

Theorem (Associativity). Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be integral residuated lattices. If $\phi\colon \mathbf{B} \to \mathbf{A}$ and $\psi\colon \mathbf{C} \to \mathbf{A} \ltimes_\phi \mathbf{B}$ are product morphisms, then there are product morphisms $\overline{\psi}\colon \mathbf{C} \to \mathbf{B}$ and $\overline{\phi}\colon \mathbf{B} \ltimes_{\overline{\psi}} \mathbf{C} \to \mathbf{A}$ such that

$$(\mathbf{A} \ltimes_\phi \mathbf{B}) \ltimes_\psi \mathbf{C} \ \cong\ \mathbf{A} \ltimes_{\overline{\phi}} (\mathbf{B} \ltimes_{\overline{\psi}} \mathbf{C}),$$

and, as before, every right-associated pair $(\overline{\phi}, \overline{\psi})$ arises from a unique left-associated pair (ϕ, ψ) in this way.

References

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