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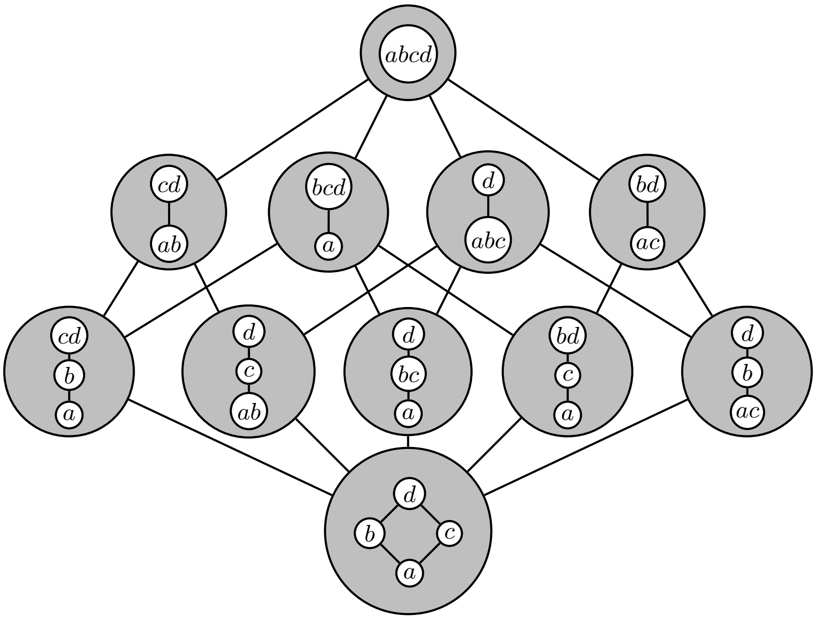

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Invited talks

Completely hereditarily atomic OMLs

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This talk has two parts. The first part is somewhat tutorial in nature. We review aspects of orthomodular lattices (OMLs) and two important methods to construct them, the Kalmbach construction and the constructions of Keller, Gross and Kunzi via orthomodular spaces. In the second part of the talk we introduce the main topic, that of algebraic OMLs and their weakening to completely hereditarily atomic OMLs. Completely hereditarily atomic OMLs can be described in various ways, one of which is that they are complete OMLs whose blocks are all atomic. Motivated by issues in extending quantum set theory to the infinite-dimensional setting, we consider the relationship between algebraicity and the covering property, and their weaker forms, for OMLs. We recall that the covering property says that if a is an atom that does not lie beneath x, then $a \vee x$ covers x.

Theorem 1 *An OML is algebraic and has the covering property iff it is a direct product of finite-height modular ortholattices.*

So to move past the finite-dimensional setting, one cannot retain both algebraicity and the covering property. We use the two construction techniques discussed in the tutorial to provide somewhat involved examples of directly irreducible OMLs of infinite height, one that is algebraic with the 2-covering property, and one that is completely hereditarily atomic with the covering property.

Operads

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Operads are a cousin of algebraic theories. They were first seriously used in algebraic topology in the early 1970s, but have now spread much more widely. Operads have less expressive power than general algebraic theories; for example, there is an operad for monoids, but no operad for groups. The trade-off is that models (algebras) for operads can be considered in categories where models for algebraic theories cannot. I will give an overview of some of the theory of operads and what has been done with it.

An algebraic view of entropy

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Entropy is often seen as belonging to the realm of physics or dynamical systems. However, in this talk, I will explain how entropy arises naturally from very general algebraic considerations - indeed, from the study of operads. The technical heart of this idea goes back to 1950s work on functional equations by Faddeev and others, while the modern presentation is joint work with Baez and Fritz.

Entropy modulo a prime

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As an illustration of the algebraic, axiomatic view of entropy, I will explain a curiosity: the entropy of probability distributions where the "probabilities" are not real numbers but integers modulo a prime p. The entropy, too, is an integer mod p. This entropy, introduced by Kontsevich, has a functional form quite different from ordinary entropy, but there is compelling evidence that it is the right definition. I will explain as much as possible, although limited by the central, unsolved mystery: what does entropy modulo a prime actually mean?

Abstract commutator theory in concrete classes

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The aim of the series is to introduce the abstract commutator theory of universal algebra and to show how it applies in various concrete varieties, such as groups, loops, semigroups or quandles.

(1) *Solvability and nilpotence: the beginnings and motivation.* I will start with the original historical motivation: the Galois theory of field extensions. Perhaps surprisingly, the same concept of solvability appears naturally in other situations, for instance, in the complexity analysis of the identity checking and equation solving problem. Is solvability and nilpotence a common phenomenon throughout the world of algebraic structures?

(2) *Abelianness and centrality.* What makes an algebraic object abelian? One possible answer is that such an object admits some sort of module structure. I will show an abstract syntactic condition that defines abelianness in universal algebra, and I will discuss how it relates to a module representation. In the second part of the lecture, I will discuss the concept of the center. How does it generalize from groups to loops (that is, when losing associativity) or inverse semigroups (that is, when losing inverses)?

(3) *The commutator.* Finally, I will define the commutator of two congruences, and the derived concepts of solvability and nilpotence. I will show how the theory can be adapted in concrete classes, for instance loops or quandles, and how to apply these concepts to obtain conceptually simple inductive proofs. I also plan to briefly discuss central extensions.

Contributed talks

Completeness and topologizability of semigroups

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We will discuss different sorts of completeness of semigroups and find a connection between completeness and topologizability of countable groups. The results are published in [1, 2, 3]

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Towards weak bases of minimal relational clones on all finite sets

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Weak bases of relational clones have been used in the past as a theoretical tool to establish more fine-grained complexity analyses of computational problems, see, e.g., [6, 1, 5, 8, 3]. For the Boolean case weak bases have been determined by Lagerkvist in [7], see also the discussion in [2]. The quest for weak bases on sets of larger size was begun in [4] with a study of weak bases for maximal clones, resulting in a complete description for all maximal clones on a three-element set. We shall report on extending this work to all maximal clones on any finite set.

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Structures with slow unlabelled growth

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For a structure $\mathfrak{A}$ we denote by $f_n(\mathfrak{A})$ the number of orbits of the natural action of the automorphism group of $\mathfrak{A}$ on the n-element subsets of A. The study of the behaviour of the sequences $f_n(\mathfrak{A})$, in the case when it always has finite values, was initiated by Cameron and Macpherson in the 80s, and it has been a subject of active research since. In my talk I will discuss some recent developments on this topic concerning the case when we have an exponential or lower upper bound for the sequence $f_n(\mathfrak{A})$.

I will present a complete classification of structures $\mathfrak{A}$ for which $f_n(\mathfrak{A}) < c^n$ holds for some $c < 2$ in terms or their automorphism groups. As a consequence of this classification we can show that all these structures satisfy some interesting model-theoretical properties: they are all interdefinable with a finitely bounded homogeneous structure, and they all satisfy Thomas' conjecture, i.e., they have finitely many reducts up to interdefinability.

Several questions and notions of loop theory relevant for universal algebra

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Consider a class consisting of loops that have no nontrivial section in a given variety. Each such class is a pseudovariety. If the variety is the variety of groups, then it is not clear if the avoiding pseudovariety is a variety or not. A similar question arises for isotopically invariant classes of loops. This is connected to (so called) Falconer varieties---a notion that will be explained. Reieterman's characterization of finite pseudovarieties induces another class of problems: under which conditions on the pseudovariety of loops does there exist a proper implicit operation?

If time allows, I will also mention the notion of propagating equations and explain how the notion has been derived from loop-theoretical results.

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On subdirectly irreductible members of double Boolean algebras

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Double Boolean algebras are algebras $\underline{D} = (D; \sqcup, \sqcap, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ introduced by Rudolf Wille to capture the equational theory of the algebra of protoconcepts. Every double Boolean algebra $\underline{D}$ contains two Boolean algebras denoted $\underline{D}_\sqcap$ and $\underline{D}_\sqcup$. A double Boolean algebra $\underline{D}$ is said to be pure if $D = D_\sqcap \cup D_\sqcup$ and trivial if $\bot \sqcup \bot = \top \sqcap \top$. In this work, we look at the subdirectly irreducible algebras of the variety of double Boolean algebras.

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Digraph conditions equivalent to certain Mal'tsev conditions

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By a 1973 result of Hagemann and Mitschke a variety is congruence n-permutable iff any edge in any reflexive directed graph compatible with the operations of an algebra of the variety is part of an n-circle. Accordingly, the n-permutability of a variety depends only on the set of digraphs admitted by it. Similarly, the n-permutability of a *locally finite* variety depends only on the *finite* digraphs admitted by it. In this talk, we show similar results for Taylor varieties and for varieties omitting TCT types 1 and 5. The graph conditions appearing here are rather nice, as they can be described by certain connectivity conditions of the admitted digraphs. We also consider the digraphs admitted by Polin's variety, which suggest that a digraph description for congruence modularity is hard to find, and possibly does not exist.

On discrete properties of monotone mappings

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Let $h : A \to A$ and ε be a partial order on A. We deal with properties of oriented graphs which corresponds to the algebra (A, h) in the case that h is monotone with respect to ε. We derive that every mono-unary algebra except connected one with a cycle of odd length has the property that there exists a non-trivial partial order such that (A, h) is monotone with respect to it. All mono-unary algebras such that there exists a linear order such that h is monotone with respect to this order will be described; if the number of components of (A, h) is infinite, then the number of such orders is equal to the cardinality of the power set of A.

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The combinatorics of weak congruences of lattices

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Weak congruences are compatible relations on an algebra that are symmetric and transitive. We provide formulas for the number of weak congruences of some algebras, such as the greatest and second greatest numbers of weak congruences of finite lattices, and we describe the way we obtained them. We analyze the weak congruences of ordinal sums and glued sums of lattices. We provide formulas for special kinds of lattices, such as lanterns and chandeliers.

Minimal closed monoids for the Galois connection End-Con

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The minimal nontrivial endomorphism monoids End Con(A, F) of congruence lattices of algebras (A, F) defined on a finite set A are described. They correspond (via the Galois connection End-Con) to the maximal nontrivial congruence lattices Con(A, F) which have been investigated and characterized previously by the authors. Analogous results are provided for endomorphism monoids of quasiorder lattices Quord(A, F).

Meet-irreducibility of congruence lattices of prime-cycled algebras

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Let A be a given finite set. The system of all congruences of an algebra (A, F), ordered by inclusion, forms a lattice $\mathrm{Con}(A, F)$. Similarly, the system of all lattices $\mathrm{Con}(A, F)$ with a given base set A forms a lattice $\mathcal{E}_A$. It is known that all meet-irreducible elements of $\mathcal{E}_A$ are congruence lattices of monounary algebras. In some cases, necessary and sufficient conditions of meet-irreducibility of $\mathrm{Con}(A, f)$ were already proven. Namely, if (A, f) is a connected algebra, if each element of (A, f) maps into a cycle, or if each cycle of (A, f) contains at most 2 elements. Characterization of all meet-irreducible elements in the $\mathcal{E}_A$ remains an open problem. In this talk, we present our results related to meet-irreducibility of congruence lattices of monounary algebras such that each cycle contains prime number of elements.

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Indecomposable involutive 2-permutable solutions of Yang-Baxter equation

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The Yang-Baxter equation is a fundamental equation occurring in integrable models in statistical mechanics and quantum field theory. Let V be a vector space. A solution of the Yang–Baxter equation is a linear mapping $r : V \otimes V \to V \otimes V$ such that

$$(id \otimes r)(r \otimes id)(id \otimes r) = (r \otimes id)(id \otimes r)(r \otimes id).$$

Description of all possible solutions seems to be extremely difficult and therefore there were some simplifications introduced. Let X be a basis of the space V and let $\sigma : X^2 \to X$ and $\tau : X^2 \to X$ be two mappings. We say that (X, σ, τ) is a set-theoretical solution of the Yang–Baxter equation if the mapping $x \otimes y \to \sigma(x, y) \otimes \tau(x, y)$ extends to a solution of the Yang–Baxter equation. It means that $r : X^2 \to X^2$, where $r = (\sigma, \tau)$ satisfies the braid relation:

$$(id \times r)(r \times id)(id \times r) = (r \times id)(id \times r)(r \times id).$$

A solution is called non-degenerate if the mappings $\sigma_x = \sigma(x, \_)$ and $\tau_y = \tau(\_, y)$ are bijections, for all $x, y \in X$. A solution (X, σ, τ) is involutive if $r^2 = id_{X^2}$.

It can be proved that being an involutive solution reduces to the following equation:

$$\sigma_x \sigma_y = \sigma_{\sigma_x(y)} \sigma_{\tau_y(x)}.$$

Moreover, in this case we have $t_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$ and we need to consider one binary operation only.

An involutive solution is called 2-permutable if it satisfies

$$\sigma_{\sigma_x(z)} = \sigma_{\sigma_y(z)},$$

for all $x, y, z \in X$. An involutive solution is called indecomposable if the permutation group generated by $\{\sigma_x \mid x \in X\}$ acts transitively on X. In our talk we shall speak about a generic construction of indecomposable 2-permutable involutive solutions.

The (in)comparability orthoset of a finite poset

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In his PhD. thesis [1], Dacey explored the notion of "abstract orthogonality", by means of sets equipped with a symmetric, irreflexive relation $\perp$. He named these structures *orthogonality spaces*, nowadays called *orthosets*. Every orthoset has an orthocomplementation operator $X \mapsto X^{\perp}$ defined on the set of all its subsets. Dacey proved that $X \mapsto X^{\perp\perp}$ is a closure operator and that the set of all closed subsets of an orthoset forms a complete ortholattice, which we call *the logic of an orthoset*. Moreover, Dacey gave a characterization of orthosets such that their logic is an orthomodular lattice. The orthosets of this type are nowadays called *Dacey spaces*.

In [3], we constructed an orthoset $(Q^{+}(P), \perp)$ from every poset P. The elements of $Q^{+}(P)$ are pairs (a, b) of elements of P with $a < b$, which we called *quotients*.

Theorem 1. *[3, Theorem 4.10] For every finite bounded poset P, P is a lattice if and only if $Q^{+}(P)$ is Dacey.*

We are continuing this line of research in a natural direction. To every poset P we associate two orthosets on the underlying set of P, namely *the strict comparability orthoset $C(P)$* and *the incomparability orthoset $I(P)$*.

In $C(P)$, two elements $a, b \in P$ are orthogonal iff $a > b$ or $b > a$. In $I(P)$, $a, b \in P$ are orthogonal iff a, b are incomparable.

Theorem 2. *Let P be a finite poset. Then the incomparability orthoset of P is Dacey if and only if P is* N-free *in the sense of [2]*

Similarly, the characterization of finite posets for which their strict comparability orthoset is Dacey can be given in terms of their order structure.

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Multiplication of matrices over lattices

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Matrices over the two-element lattice correspond to binary relations. There are many results about the semigroup of binary relations, in this talk we recall a few of them. We give a description of idempotent elements by interpreting the graph corresponding to a matrix as a transportation network.

Multiplication of matrices over a lattice L is associative if and only if L is a distributive lattice. Matrices over distributive lattices can be viewed as multiple-valued analogues of binary relations. We describe idempotent and nilpotent matrices in some special cases. We show that matrix multiplication over nondistributive lattices is antiassociative.

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Special filters in bounded lattices

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M.S. Rao recently investigated some sorts of special filters in distributive pseudocomplemented lattices. We extend this study to lattices which need neither be distributive nor pseudocomplemented. For this sake we define a certain modification of the notion of a pseudocomplement as the set of all maximal elements belonging to the annihilator of the corresponding element. We prove several basic properties of this notion and then define coherent, closed and median filters as well as D-filters. In order to be able to obtain valuable results we often must add some additional assumptions on the underlying lattice, e.g. that this lattice is Stonean or D-Stonean. Our results relate properties of lattices and of corresponding filters. We show how the structure of a lattice influences the form of its filters and vice versa.

Tolerances on posets

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The concept of a tolerance relation, shortly called tolerance, was studied on various algebras since the seventies of the twentieth century (cf. e.g. [1] and [7]). Since tolerances need not be transitive, their blocks may overlap and hence in general the set of all blocks of a tolerance cannot be converted into a quotient algebra in the same way as in the case of congruences. However, G. Czédli [8] showed that lattices can be factorized by means of tolerances in a natural way, and J. Grygiel and S. Radeleczki [9] proved some variant of an Isomorphism Theorem for tolerances on lattices. The aim of the present talk is to extend the concept of a tolerance on a lattice to posets in such a way that results similar to those obtained for tolerances on lattices can be derived.

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Quantum Suplattices

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Discrete quantization is a method of finding noncommutative generalizations of discrete mathematical structures by means of internalizing these structures in a suitable order-enriched dagger compact category **qRel**, whose objects are von Neumann algebras isomorphic to a (possibly infinite) ℓ^∞-sum of matrix algebras called *quantum sets*. The morphisms of **qRel** are noncommutative generalizations of binary relations between sets, called *quantum relations*, and were distilled by Weaver [5] from his work with Kuperberg on the quantization of metric spaces [6]. Other structures that were quantized using discrete quantization include posets [4] and cpos [3].

In this contribution, we apply discrete quantization in order to obtain a noncommutative version of complete lattices (also called suplattices), which we call *quantum suplattices*. We discuss the categorical constructions in the category **Rel** that can be used to define ordinary suplattices, and discuss how to lift these constructions to **qRel** in order to obtain our definition of quantum suplattices. Furthermore, we discuss how classical theorems on suplattices such as the Knaster-Tarski Theorem generalize to the quantum case. We refer to [1] for the concrete constructions of quantum suplattices.

Finally, we discuss how to quantize the concept of a topological space based on the notion of quantum suplattices. Traditionally, C*-algebras form the standard approach to quantum topology, but only generalize locally compact Hausdorff spaces. Our approach is different, and allows for the quantization of specific topological spaces that are not necessarily locally compact or Hausdorff.

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On Congruences of Weakly Dicomplemented Lattices*

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Weakly dicomplemented lattices arise as abstractions of concept algebras, introduced by Rudolf Wille when modelling negation on concept lattices [2]; they are algebras $(L, \wedge, \vee, ^{\Delta}, ^{\nabla}, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ formed of bounded lattices $(L, \wedge, \vee, 0, 1)$ endowed with two unary operations: $^{\Delta}$, called *weak complementation*, and $^{\nabla}$, called *dual weak complementation*, together forming the *weak dicomplementation* $(^{\Delta}, ^{\nabla})$, both of which are order--reversing and that also satisfy, for all $x, y \in L$: $x^{\Delta\Delta} \leq x \leq x^{\nabla\nabla}$ and $(x \wedge y) \vee (x \wedge y^{\Delta}) = x = (x \vee y) \wedge (x \vee y^{\nabla})$, where $\leq$ is the lattice order of L. Their bounded lattice reducts endowed with the weak complementation, $(L, \wedge, \vee, ^{\Delta}, 0, 1)$, are called *weakly complemented lattices*, and their bounded lattice reducts endowed with the dual weak complementation, $(L, \wedge, \vee, ^{\nabla}, 0, 1)$, are called *dual weakly complemented lattices*.

For instance, any Boolean algebra is both a weakly complemented lattice and a dual weakly complemented lattice. Furthermore, any bounded lattice can be endowed with the *trivial weak dicomplementation*, formed of the *trivial weak complementation*, that sends 1 to 0 and all other elements to 1, and the *trivial dual weak complementation*, that sends 0 to 1 and all other elements to 0.

A *context* is a triple (J, M, I), where J and M are sets and $I \subseteq J \times M$. A *subcontext* of (J, M, I) is a context $(H, N, I \cap (H \times N))$, with $H \subseteq J$ and $N \subseteq M$. For every $(A, B) \in \mathcal{P}(J) \times \mathcal{P}(M)$, we denote by: $A' = \{m \in M : (\forall\, a \in A)\,(a\, I\, m)\}$ and $B' = \{j \in J : (\forall\, b \in B)\,(j\, I\, b)\}$. The *concept algebra associated to the context* (J, M, I) is the weakly dicomplemented (complete) lattice $(\mathcal{B}(J, M, I), \wedge, \vee, ^{\Delta}, ^{\nabla}, 0, 1)$, where: $\mathcal{B}(J, M, I) = \{(A, B) \in \mathcal{P}(J) \times \mathcal{P}(M) : A' = B, B' = A\}$ is the set of the *formal concepts* associated to the context (J, M, I), $\wedge$ and $\vee$ are the lattice operations corresponding to the order $\subseteq \times \supseteq$ of $\mathcal{P}(J) \times \mathcal{P}(M)$ restricted to $\mathcal{B}(J, M, I)$, $0 = (\emptyset'', M)$, $1 = (J, J')$ and, for any $(A, B) \in \mathcal{B}(J, M, I)$, $(A, B)^{\Delta} = ((J \setminus A)'', (J \setminus A)')$ and $(A, B)^{\nabla} = ((M \setminus B)', (M \setminus B)'')$. A subcontext (H, N, E) of (J, M, I) is said to be *compatible* iff the map $\Pi_{J,M,H,N} : \mathcal{B}(J, M, I) \to \mathcal{B}(H, N, E)$, $\Pi_{J,M,H,N}(A, B) = (A \cap H, B \cap N)$ for all $(A, B) \in \mathcal{B}(J, M, I)$, is well defined,

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case in which this map is a surjective weakly dicomplemented lattice morphism.

Whenever J is a join--dense subset and M is a meet--dense subset of a complete lattice L, L can be endowed with the weak dicomplementation $({}^{\Delta J}, {}^{\nabla M})$ defined by $x^{\Delta J} = \bigvee(J \setminus (x]_L)$ and $x^{\nabla M} = \bigwedge(M \setminus [x)_L)$ for all $x \in L$, and the map $\varphi_{L,J,M} : L \to \mathcal{B}(J, M, \leq)$, defined by $\varphi_{L,J,M}(x) = (J \cap (x]_L, M \cap [x)_L)$ for all $x \in L$, is a weakly dicomplemented lattice isomorphism. In this case, it follows that, for any compatible subcontext $(H, N, \leq)$ of the context $(J, M, \leq)$, $\Pi_{J,M,H,N} \circ \varphi_{L,J,M} : L \to \mathcal{B}(H, N, \leq)$ is a surjective weakly dicomplemented lattice morphism, hence its kernel, that we denote by $\zeta_{L,H,N}$, is a lattice congruence of L that preserves the weak dicomplementation $({}^{\Delta J}, {}^{\nabla M})$, and the weakly dicomplemented lattices $L/\zeta_{L,H,N}$ and $\mathcal{B}(H, N, \leq)$ are isomorphic; we call $\zeta_{L,H,N}$ the *weakly dicomplemented lattice congruence induced by the compatible subcontext* $(H, N, \leq)$ *of* $(J, M, \leq)$. We say that a weak dicomplementation $({}^{\Delta}, {}^{\nabla})$ on L is *representable* iff ${}^{\Delta} = {}^{\Delta J}$ and ${}^{\nabla} = {}^{\nabla M}$ for some join--dense subset J and some meet--dense subset M of L.

In the paper [1], we study the existence of nontrivial and of representable (dual) weak complementations, along with the lattice congruences that preserve them, in different constructions of bounded lattices, then use this study to determine the finite (dual) weakly complemented lattices with the largest numbers of congruences, along with the structures of their congruence lattices. It turns out that, if $n \geq 7$ is a natural number, then the four largest numbers of congruences of the n--element (dual) weakly complemented lattices are: $2^{n-2} + 1$, $2^{n-3} + 1$, $5 \cdot 2^{n-6} + 1$ and $2^{n-4} + 1$, which yields the fact that, for any $n \geq 5$, the largest and second largest numbers of congruences of the n--element weakly dicomplemented lattices are $2^{n-3} + 1$ and $2^{n-4} + 1$. For smaller numbers of elements, several intermediate numbers of congruences appear between the elements of these sequences. While already published, this research has never been presented at a conference before.

In the same purely lattice--theoretical manner, we study compatible subcontexts and the congruences they induce in various types and constructions of lattices. Out of this part of our ongoing research, I will do my best to select the most interesting results that I can fit into my talk.

Keywords: congruence, (glued/ordinal, horizontal) sum of bounded lattices, (nontrivial, representable) (dual) weak (di)complementation, compatible subcontext.

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A dagger kernel category of orthomodular lattices

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Dagger kernel categories have been introduced in [HeJa] as a simple setting in which one can study categorical quantum logic. The present paper continues the study of dagger kernel categories in relation to orthomodular lattices in the spirit of [Jac].

In particular, we show that the category of orthomodular lattices **OMLatLin** where morphisms are mappings having adjoints is a dagger kernel category. We describe finite dagger biproducts and free objects over finite sets in **OMLatLin**.

A meet semi-lattice $(X, \wedge\ 1)$ is called an *ortholattice* if it comes equipped with a function $(-)^{\perp}\colon X \to X$ satisfying:

- $x^{\perp\perp} = x$;
- $x \leq y$ implies $y^{\perp} \leq x^{\perp}$;
- $x \wedge x^{\perp} = 1^{\perp}$.

One can then define a bottom element as $0 = 1 \wedge 1^{\perp} = 1^{\perp}$ and join by $x \vee y = (x^{\perp} \wedge y^{\perp})^{\perp}$, satisfying $x \vee x^{\perp} = 1$. We write $x \perp y$ if and only if $x \leq y^{\perp}$.

Such an ortholattice is called *orthomodular* if $x \leq y$ implies $y = x \vee (x^{\perp} \wedge y)$.

Definition 1. A *dagger* on a category $\mathcal{C}$ is a functor $^\star\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ that is involutive and the identity on objects. A category equipped with a dagger is called a *dagger category*.

Let $\mathcal{C}$ be a dagger category. A morphism $f\colon A \to B$ is called a *dagger monomorphism* if $f^\star \circ f = \mathrm{id}_A$, and f is called a *dagger isomorphism* if $f^\star \circ f = \mathrm{id}_A$ and $f \circ f^\star = \mathrm{id}_B$.

We now introduce a new way of organising orthomodular lattices into a dagger category.

Definition 2. The category **OMLatLin** has orthomodular lattices as objects. A morphism $f\colon X \to Y$ in **OMLatLin** is a function $f\colon X \to Y$ between the underlying sets such that there is a function $h\colon Y \to X$ and, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp h(y).$$

We say that h is an *adjoint* of a *linear map* f. It is clear that adjointness is a symmetric property: if a map f possesses an adjoint h, then f is also an adjoint of h.

Moreover, a map $f\colon X \to X$ is called *self-adjoint* if f is an adjoint of itself.

The identity morphism on X is the self-adjoint identity map $\mathrm{id}\colon X \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by usual composition of maps.

Lemma 3. *Let $f\colon X \to Y$ be a morphism of orthomodular lattices. Then $\downarrow f^*(1)^\perp = \{x \in X\colon f(x) = 0\}$ is an orthomodular lattice.*

OMLatLin has a *zero object* $\underline{0}$; this means that there is, for any orthomodular lattice X, a unique morphism $\underline{0} \to X$ and hence also a unique morphism $X \to \underline{0}$. The zero object $\underline{0}$ will be one-element orthomodular lattice $\{0\}$.

For objects X and Y, we denote by $0_{X,Y} = X \to \underline{0} \to Y$ the morphism uniquely factoring through $\underline{0}$.

Definition 4. For a morphism $f\colon A \to B$ in a dagger category with zero morphisms, we say that a morphism $k\colon K \to A$ is a *weak dagger kernel* of f if $fk = 0_{K,B}$, and if $m\colon M \to A$ satisfies $fm = 0_{M,B}$ then $kk^*m = m$.

A *dagger kernel category* is a dagger category with a zero object, hence zero morphisms, where each morphism f has a weak dagger kernel k (called *dagger kernel*) that additionally satisfies $k^*k = 1_K$.

Theorem 5. *The category* **OMLatLin** *is a dagger kernel category. The dagger kernel of a morphism $f\colon X \to Y$ is $k\colon \downarrow k \to X$, where $k = f^*(1)^\perp \in X$.*

Corollary 6. *Every morphism $f\colon X \to Y$ in* **OMLatLin** *has a factorisation me where $m = f(1)\colon \downarrow f(1) \to Y$ and $e = f|^{\downarrow f(1)}\colon X \to \downarrow f(1)$.*

By a *dagger biproduct* of objects A, B in a dagger category $\mathcal{C}$ with a zero object, we mean a coproduct $A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$ such that ι_A, ι_B are dagger monomorphisms and $\iota_B{}^\star \circ \iota_A = 0_{A,B}$. The dagger biproduct of an arbitrary set of objects is defined in the expected way.

Proposition 7. *The category* **OMLatLin** *has arbitrary finite dagger biproducts $\bigoplus$. Explicitly, $\bigoplus_{i\in I} X_i$ is the cartesian product of orthomodular lattices X_i, $i \in I$, I finite.*

The coprojections $\kappa_j\colon X_j \to \bigoplus_{i\in I} X_i$ are defined by $(\kappa_j)(x) = x_{j=}$ with $x_{j=}(i) = \begin{cases} x & \text{if } i = j; \\ 0 & \text{otherwise.}\end{cases}$ and $(\kappa_j)^*((x_i)_{i\in I}) = x_j$. The dual product structure is given by $p_j = (\kappa_j)^*$.

Proposition 8. *A free object on a finite set A in* **OMLatLin** *is isomorphic to the finite Boolean algebra $\mathcal{P}A$.*

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S-preclones and the Galois connection ${}^{S}\mathrm{Pol} - {}^{S}\mathrm{Inv}$

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We consider so-called *S-operations* $f : A^n \to A$ for which each variable gets a *signum* $s \in S$ representing ``properties'' like, e.g., order preserving or order reversing with respect to a partial order on A. The set S of such properties has the structure of a monoid reflecting the behaviour of composition of S-operations (e.g., order reversing composed with order reversing is order preserving). The collection of all operations with prescibed properties for their signed variables is not a clone (since it is not closed under arbitrary identification of variables), but it is a preclone with special properties what leads to the notion of *S-preclone*. We introduce *S-relations* $\varrho = (\varrho_s)_{s\in S}$, *$S$-relational clones* and a preservation property ($f \overset{S}{\triangleright} \varrho$), and consider the induced Galois connection ${}^{S}\mathrm{Pol} - {}^{S}\mathrm{Inv}$. The S-preclones turn out to be just the Galois closures. Moreover we can characterize the Galois closures on the relational side as *S-relational clones*.

Classes of groups in lattice framework

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We are studying groups by analyzing their lattices of weak congruences. For an algebra $\mathcal{A}$ these are congruences on subalgebras considered as relations on $\mathcal{A}$. When the algebra is a group G, the algebraic lattice $\mathsf{Wcon}(G)$ encompasses the lattices of subgroups and normal subgroups for every subgroup of G, up to isomorphism.

Subgroup lattices have been used to characterize various classes of groups to some extent, but weak congruence lattices provide much more information. We have obtained characterizations in this context, which are discussed in the papers listed here.

In this presentation, we expand our framework by incorporating systems of subgroups into weak congruence lattices. This advancement enhances our understanding of Kurosh-Chernikov classes of groups and offers a new characterization within this setting.

Furthermore, we address the Algebra of Group Theoretical Classes, which refers to group theoretical properties and the corresponding closure operations on these classes. We demonstrate that these closure operations can be expressed in terms of lattices.

As a consequence, we are able to formulate results of the following type.

A group G belongs to the class $\mathfrak{P}$ if and only if the lattice $\mathsf{Wcon}(G)$ satisfies the lattice theoretic properties $\mathsf{L}_{\mathfrak{P}}$.

If the above holds, we say that $\mathfrak{P}$ is an **L-class** of groups.

Theorem. *Let $\mathfrak{P}$ be an L-class of groups. A group G is a residually $\mathfrak{P}$-group (it belongs to the class $\mathbf{R}\mathfrak{P}$) if and only if the lattice $\mathsf{Wcon}(G)$ fulfils:*

$(*)$ *For each $\Delta_X \in \mathsf{C}(\downarrow\Delta)$, $\Delta_X \neq \{(e,e)\}$, there is $\Delta_N \blacktriangleleft \Delta$, such that $\Delta_N \wedge \Delta_X < \Delta_X$ and the interval $[N^2, G^2]$, as the lattice with normal elements determined by $N^2 \vee \Delta$, satisfies the lattice theoretic properties $\mathsf{L}_{\mathfrak{P}}$.*

We prove that most of the known classes of groups are L-classes. Conversely, we start with a lattice property and analyze (algebraic properties of) the corresponding L-class of groups.

Finally, we prove that Birkhoff's theorem for L-classes of groups can be formulated with purely lattice-theoretic arguments.

Theorem. *An* L*-class* $\mathfrak{P}$ *of groups is a variety if and only if the following hold:*

(i) *if* G *is a* $\mathfrak{P}$*-group, then in the lattice* $\mathsf{Wcon}(G)$ *for every* $\Delta_H \in \downarrow\Delta$*, such that* $\Delta_H \blacktriangleleft \Delta$ *(normal in lattice sense), the interval* $[H^2, G^2]$*, which is a lattice with normal element determined by* $H^2 \vee \Delta$*, satisfies* $\mathsf{L}_{\mathfrak{P}}$*, i.e., the lattice properties determining the class* $\mathfrak{P}$ *and*

(ii) *f* G *is a group, such that the lattice* $\mathsf{Wcon}(G)$ *satisfies* $(*)$*, then* G *belongs to* $\mathfrak{P}$*.*

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More on Tense Operators

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The talk will focus on exploring tense operators in quantale-enriched categories (quantale modules, V-frames, V-F-semilattices) analogously to the categories studied in 'Another look on tense operators' (sup-semilattices, frames, F-semilattices). The goal is to understand the connections between tense operators and functorial constructions in quantum logic.

We'll investigate the quantale-enriched versions of the classical three adjoint situations, highlighting the interplay of tense operators within quantale-enriched structures. Concrete examples will be presented to demonstrate the applications of quantale-enriched categories in algebraic methods in quantum logic.

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Square roots and their applications on pseudo MV-algebras

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A square root as a unary operation on MV-algebras was introduced in [3]. This presentation provides a study of pseudo MV-algebras with square roots on pseudo MV-algebras. We introduce different notions of a square root on a pseudo MV-algebra which coincide on MV-algebras, and present their main properties. We show that the class of pseudo-MV-algebras with square roots is a proper subvariety of the variety of pseudo MV-algebras. We define a strict square root to classify the class of pseudo MV-algebras with square roots. We found a relationship between strongly atomless pseudo MV-algebras and strict pseudo MV-algebras and we investigate square roots on representable symmetric pseudo MV-algebras, and we present a complete characterization of a square root and a weak square root on a representable symmetric pseudo MV-algebra using addition in a unital ℓ-group. In addition, some interesting examples are provided.

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