Square roots and their applications on pseudo MV-algebras

Anatolij Dvurečenskij
Omid Zahiri

Mathematical Institute, Slovak Academy of Sciences,
Bratislava, Slovakia

SSAOS 2023 - Stará Lesná, Slovakia
September 2023

Content

1 - Introduction

Psudo MV-algebra (PMV-algebra)

A non-commutative generalization of MV-algebras was introduced by [Georgescu and Iorgulescu, 2001] as an algebraic counterpart of the non-commutative Łukasiewicz logic.

Definition [Georgescu and Iorgulescu, 2001]

A **PMV-algebra** is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$,

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^- = 1^\sim = 0$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $(x^-)^\sim = x$,

(A7) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$,

(A8) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$,

where $x \odot y = (y^- \oplus x^-)^\sim$.

Additional binary operations (implications) :

$$x \rightarrow y := x^- \oplus y, \quad x \rightsquigarrow y := y \oplus x^\sim.$$

Each PMV-algebra M with the following **partial order relation** is a bounded distributive lattice.

$$x \leq y \Leftrightarrow x \odot y^{-} = 0 \Leftrightarrow y^{\sim} \odot x = 0$$

Boolean skeleton

An element x of a PMV-algebra M is called a *Boolean element* if $x \oplus x = x$.

$$\mathrm{B}(M) = \{x \in M : x \oplus x = x\}$$

$\mathrm{B}(M)$ is a Boolean algebra and a subalgebra of M.

A PMV-algebra $(M; \oplus, ^{\sim}, ^{-}, 0, 1)$ is said to be **symmetric** if $x^{\sim} = x^{-}, \ \forall x \in M$.

Ideals and Representable PMV-algebras

A non-empty subset I of a PMV-algebra M is called an *ideal* if (1) for each $y \in M$, $y \leq x \in I$ implies that $y \in I$; (2) I is closed under $\oplus$.
An ideal I of M is said to be
(i) **prime** if $x \wedge y \in I$ implies $x \in I$ or $y \in I$;
(ii) **normal** if $x \oplus I = I \oplus x$ for any $x \in M$, where $x \oplus I := \{x \oplus i \mid i \in I\}$ and $I \oplus x = \{i \oplus x \mid i \in I\}$.

Representable PMV-algebra

A PMV-algebra M is said to be *representable* if M is a subdirect product of a system of linearly ordered PMV-algebras.

$$M \xrightarrow{\text{subdirect embedding}} \prod_{i \in I} L_i$$

PMV-algebras and unital ℓ-groups

A *lattice-ordered group* (*ℓ-group*) is an algebra $(G; \vee, \wedge, +, -, 0)$ such that $(G; \vee, \wedge)$ is a lattice, $(G; +, -, 0)$ is a group, and $+$ is an order-preserving map.

$0 \leq u \in G$ is said to be a **strong unit** if

$$\forall g \in G,\ \exists n \in \mathbb{N} \colon\ g \leq nu$$

A pair (G, u) with a **fixed** strong unit u, is said to be a *unital ℓ-group*.

$\Gamma(G, c)$ [Georgescu and Iorgulescu, 2001]

If $(G; \vee, \wedge, +, -, 0)$ is an ℓ-group, given $0 \leq c$, an interval

$$[0, c] := \{x \in G \mid 0 \leq x \leq c\}$$

$$x \oplus y = (x + y) \wedge c, \quad x \odot y = (x - c + y) \vee 0,$$

$$x^{-} = c - x, \qquad x^{\sim} = -x + c.$$

form a PMV-algebra which is denoted by $\Gamma(G, c) = ([0, c]; \oplus, ^{-}, ^{\sim}, 0, c)$.

PMV-algebras and ℓ-groups

[Dvurečenskij, 2002], using Bosbach's notion of a semiclan, proved that every PMV-algebra is isomorphic to $\Gamma(G,u)$ for some unital ℓ-group (G,u).

Categorical equivalence [Dvurečenskij, 2002]

The category of unital ℓ-groups ($\mathcal{UG}$) is **categorically equivalent** to the category of PMV-algebras ($\mathcal{PMV}$).

$$\Gamma : \mathcal{UG} \to \mathcal{PMV} \qquad \Psi : \mathcal{PMV} \to \mathcal{UG}$$

$$\Gamma(G,u) = (\Gamma(G,u); \oplus, ^-, ^\sim, 0, u)$$

$$\Gamma(f : G \to H) = f|_{\Gamma(G,u)}.$$

2 - Square roots : Definition and main properties

Definition of square root on PMV-algebras

A square root on MV-algebras was originally defined in [Höhle, 1995]. It is a unary operation on an MV-algebra M satisfying (Sq1) and (Sq2) below.

Definition

Let $(M; \oplus, ^-, ^\sim, 0, 1)$ be a PMV-algebra. A mapping $r : M \to M$ is said to be a **square root** if it satisfies the following conditions :
(Sq1) for all $x \in M$, $r(x) \odot r(x) = x$;
(Sq2) for each $x, y \in M$, $y \odot y \leq x$ implies $y \leq r(x)$;
(Sq3) for each $x \in M$, $r(x^-) = r(x) \to r(0)$ and $r(x^\sim) = r(x) \rightsquigarrow r(0)$.

r is a **weak square root** if it satisfies only (Sq1) and (Sq2). For MV-algebras these notions coincide but for PMV-algebras they could be different.

(1) Each square root is a one-to-one map.
(2) If a PMV-algebra M has a square root, then the square root is unique.

Main Properties of square roots on PMV-algebras

Let r be a square root on a PMV-algebra $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$. For each $x, y \in M$
(1) $x \leq y$ implies that $r(x) \leq r(y)$.
(2) $r(x) \wedge r(y) = r(x \wedge y)$.
(3) $r(x) \rightarrow r(y) \leq r(x \rightarrow y)$ and $r(x) \rightsquigarrow r(y) \leq r(x \rightsquigarrow y)$. Moreover, $r(x) \odot r(y) \leq r(x \odot y)$ for all $x, y \in M$ if and only if $r(x) \rightarrow r(y) = r(x \rightarrow y)$ and $r(x) \rightsquigarrow r(y) = r(x \rightsquigarrow y)$.
(4) $r(x \vee y) = r(x) \vee r(y)$.
(5) $r(x \odot y) \leq (r(x) \odot r(y)) \vee r(0)$ and $r(x \odot x) = (r(x) \odot r(x)) \vee r(0)$.
(6) If $a, b \in M$, $a \leq b$, then $r([a, b]) = [r(a), r(b)]$. So, $r(M) = [r(0), 1]$.
(7) $r(x \oplus y) \geq (r(x) \odot r(0)^{-}) \oplus r(y)$.
(8) $x \leq x \vee r(0) = r(x \odot x) \leq r(x)$.
(9) $r(x) \leq x \oplus r(0) = r(0) \oplus x = r(x \oplus x)$.
(10) $x \oplus r(x) = r(x) \oplus x$.

Some Examples

(i) If M is a Boolean algebra, then the identity map $\mathrm{Id}_M : M \to M$ is a square root.
(ii) Take a linearly ordered (or representable) two-divisible unital group (G, u), where $u/2 \in \mathbb{C}(G)$.

$$s : \Gamma(G,u) \to \Gamma(G,u) \qquad s(x) = \frac{x+u}{2}, \quad x \in M,$$

The example also works if (G, u) is a two-divisible unital ℓ-group which enjoys unique extraction of roots, that is x/n is unique.
(iii) Let M be a direct product of a family $\{M_i\}_{i\in I}$ of PMV-algebra with square roots. Then M has a square root.

$$x = (x_i)_i \in M = \prod_i M_i, \qquad s(x) = (s_i(x_i))_i$$

where s_i is a unique square root on M_i.

Examples

(iv) Let $M = \Gamma(G, u)$, where (G, u) is a linearly ordered (or representable) two-divisible unital ℓ-group. Then

$$r(x) = \frac{x - u}{2} + u, \quad x \in M,$$

is a weak square root on M.

(v) If $1 < |G|$ is an ℓ-group, then $M = \Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ has no weak square roots.

Theorem : Square roots and homomorphisms

Homomorphisms - Normal ideals

Let $f : M \longrightarrow N$ be a homomorphism of PMV-algebras and r be a square root on M. Then $f(M)$ has a square root :

$$\tau : \mathrm{Im}(f) \longrightarrow \mathrm{Im}(f), \quad \tau(f(x)) = f(r(x)), \quad \forall x \in M$$

So, given a normal ideal I of a PMV-algebra M with a square root r

$$r_I : \frac{M}{I} \longrightarrow \frac{M}{I} \qquad r_I(\frac{x}{I}) = \frac{r(x)}{I}, \quad \forall x \in M$$

is a square root on $\dfrac{M}{I}$.

3 - Characterization of PMV-algebras with square roots

Strict PMV-algebra and Boolean point

Proposition

Let r be a square root on a PMV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$. Then there exists a **unique** idempotent element $v \in \mathrm{B}(M)$ such that

$$r(0)^- = v \vee r(0) = r(0)^\sim, \qquad v = r(0)^- \odot r(0)^-.$$

Consider a PMV-algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ with a (week) square root r. From $r(0) \odot r(0) = 0$, we get that $r(0) \leq r(0)^- \wedge r(0)^\sim = r(0)^- = r(0)^\sim$.

Definition

A square root r is called **strict** if $r(0) = r(0)^-$.
A PMV-algebra with a strict square root is called a strict PMV-algebra.

Main characterization theorem

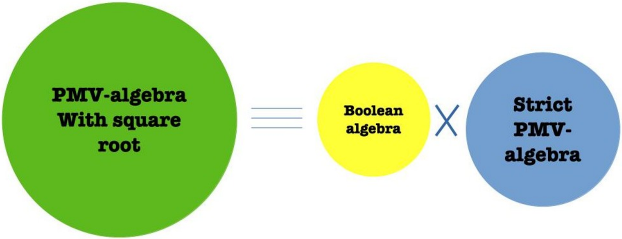
Theorem

Let $r : M \to M$ be a square root on a PMV-algebra $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$. Then M satisfies only one of the following statements :

(i) The PMV-algebra M is a Boolean algebra.

(ii) The PMV-algebra M is a strict PMV-algebra.

(iii) $M \cong M_1 \times M_2$, where M_1 is a Boolean algebra and M_2 is a strict PMV-algebra. Moreover, the Boolean algebra M_1 and the strict PMV-algebra M_2 are uniquely determined by M up to isomorphism.



Main characterization theorem

The detail of the proof takes a long time. Sketch of proof. $v := r(0)^- \odot r(0)^-$

* If $v = 1$, M is a Boolean algebra.
* If $v = 0$, then M is a strict PMV-algebra.
* Otherwise, $M \cong [0, v] \times [0, v^-]$, $[0, v]$ is a Boolean algebra and $[0, v^-]$ is a strict PMV-algebra.

4 - Representations of Square Roots on Representable Symmetric PMV-algebras

Representation

Now, we want to study square root on a representable and symmetric PMV-algebra using group addition.

Theorem 3.1 : Linearly ordered

$M = \Gamma(G, u)$: linearly ordered symmetric PMV-algebra with a square root r.
(i) If $r(0) = 0$, then $r = \mathrm{Id}_M$.
(ii) If $r(0) > 0$, for each element $x \in M$, the element $(x + u)/2$ exists in M, and $r(x) = (x + u)/2$ for $x \in M$, where $+$ denotes the group addition in G.

Theorem 3.2 : Strict

Let $M = \Gamma(G, u)$ be a representable symmetric PMV-algebra with a strict square root r. Then $(x + u)/2$ exists for each $x \in M$, and

$$r(x) = \frac{x + u}{2}, \quad x \in M, \qquad (+ \text{ is the group addition in } G)$$

Representation Theorem

Theorem 3.3 : General case for representing symmetric PMV-algebras

Let $M = \Gamma(G, u)$ be a representable symmetric PMV-algebra with a square root $r : M \to M$, where (G, u) is a unital ℓ-group. Then

$$r(x) = (x \wedge v) \vee \frac{(x \wedge v^-) + v^-}{2}, \qquad \forall x \in M$$

where $v = r(0)^- \odot r(0)^-$.

Theorem

Let $M = \Gamma(G, u)$ be a representable PMV-algebra with a strict square root r. Then M is **symmetric**, two-divisible, and $u/2 \in \mathbb{C}(G)$.

In Theorem 3.1– 3.2, the assumption "M is symmetric" is superfluous

Relation between strongly atomless and strictness

Definition (similar to [Belluce, 1995])

A PMV-algebra M is **strongly atomless** if for each non-zero element $x \in M$, there exists a normal prime ideal P such that $x \notin P$ and x/P is not an atom of M/P.

Each strongly atomless PMV-algebra is representable, since

$$\cap\{P \in \mathrm{Spec}(M) : P \text{ is normal}\} = \{0\}.$$

In addition, each strongly atomless PMV-algebra is atomless.

Theorem : Relation between strict and strongly atomless

If M is strongly atomless PMV-algebra with a square root r, then r is strict.

On the class of PMV-algebras with square roots :

$$\text{Strongly atomless} \rightarrow \text{Strictness}$$

Strongly atomless property

[Höhle, 1995, Thm 6.17] proved that if M is a complete MV-algebra, then

$$\text{strict MV-algebra} \Leftrightarrow \text{strongly atomless}$$

We are ready to show the converse for a more general case, not necessarily for complete MV-algebras.

Theorem

On the class of representable PMV-algebra :

$$\text{Strictness} \rightarrow \text{Strongly atomless}$$

Weak Square Roots that is Not Square Roots

Example 1

Let $G_2 = \mathbb{R}^2$.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \mathrm{e}^{x_2} y_1 + y_2), \quad -(x, y) = (-x, -\mathrm{e}^{-x} y)$$

$(0, 0)$ is the neutral element. We endow G_2 with the lexicographic order.
(i) G_2 is linearly ordered
(ii) $u_2 = (1, 0)$ is a strong unit of G_2
(iii) G_2 is two-divisible with $(x, y)/2 = (x/2, y/(\mathrm{e}^{x/2} + 1))$.
(iv) $M_2 = \Gamma(G_2, u_2)$ is a non-symmetric pseudo MV-algebra

$$(x, y)^- = (1 - x, -e^{-x} y) \quad (x, y)^\sim = (1 - x, -e^{-x+1} y)$$

$r(x, y) = ((x, y) - (1, 0))/2 + (1, 0) = (\dfrac{x+1}{2}, \dfrac{y}{\mathrm{e}^{(x-1)/2} + 1})$ is a weak square and M has no square root.

References

Ambrosio, R. (1999).
Strict mv-algebras.
Journal of mathematical analysis and applications, 237(1) :320–326.

Belluce, L. P. (1995).
α-complete mv-algebras.
pages 7–21.

Bělohlávek, R. (2003).
Some properties of residuated lattices, volume 53, pages 161–171.
Institute of Mathematics, Academy of Sciences of the Czech Republic.

Chang, C. C. (1958).
Algebraic analysis of many valued logics.
Transactions of the American Mathematical society, 88(2) :467–490.

Cignoli, R. L., d'Ottaviano, I. M., and Mundici, D. (2013).
Algebraic foundations of many-valued reasoning, volume 7.
Springer Science & Business Media.

Dvurečenskij, A. (2002).
Pseudo mv-algebras are intervals in -groups.
Journal of the Australian Mathematical Society, 72(3) :427–446.

Georgescu, G. and Iorgulescu, A. (2001).
Pseudo-mv algebras.
Mult.-Valued Log, 6(1-2) :95–135.

Höhle, U. (1995).
Commutative, residuated 1—monoids.
Non-classical logics and their applications to fuzzy subsets, pages 53–106.

Thank You for Your Attention