The Omega-Reducibility of Certain Pseudovarieties of Ordered Monoids

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5. Briefly about the ω-reducibility of more complex pseudovarieties

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there exists an inequality $u' \leq v'$ of *ω-words* that is also valid in V and "has the same α-imprint in M", i.e.,

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Construct the V_0-*completion* α_{V_0} of the homomorphism α:

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OUTPUT of Step 1:
V_0-*compatible* onto continuous homomorphism $\alpha_{V_0} \colon \widehat{A^*} \twoheadrightarrow M_{\alpha_{V_0}}$:
$\forall x, y \in \widehat{A^*} \colon \ \alpha_{V_0}(x) = \alpha_{V_0}(y) \ \Rightarrow \ x \equiv_{V_0} y.$

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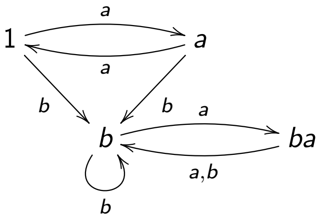
Cayley graph of monoid M:



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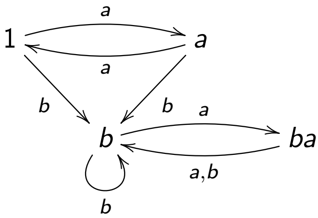
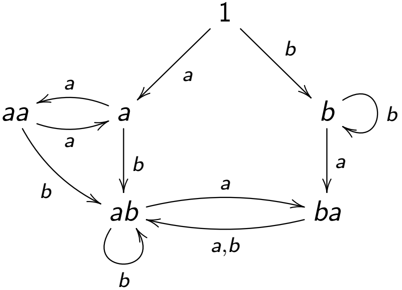


Cayley graph of monoid $M_{\alpha_{Sl}}$:



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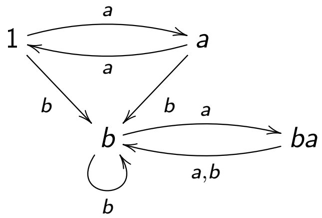

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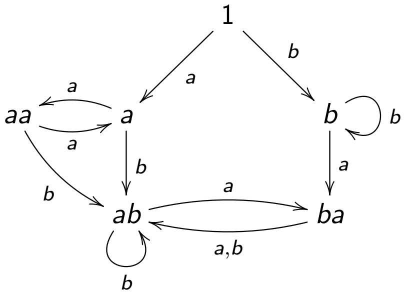

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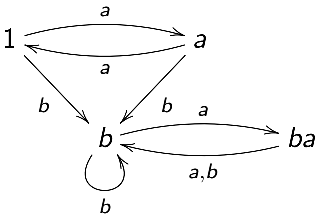

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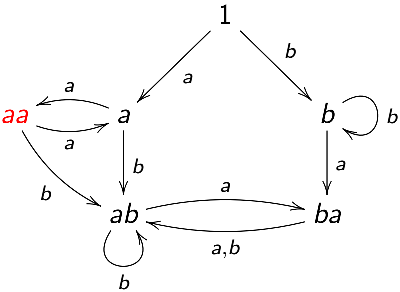

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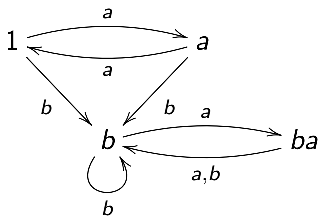

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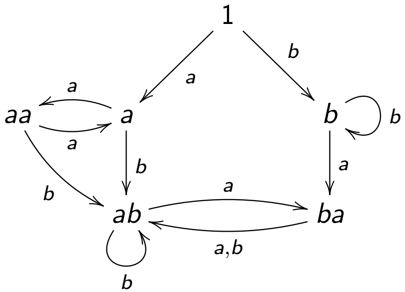

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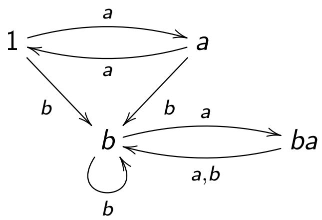

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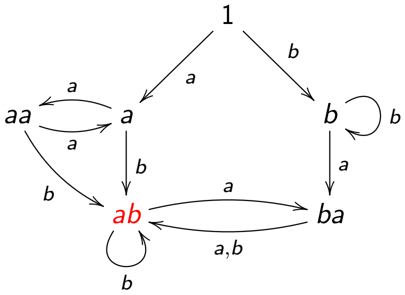

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Theorem (J. V.)

Let $u, v \in A^*$ be arbitrary words and $N = 1 + 9 \cdot |M_{\alpha_{\mathsf{V}_0}}| \cdot 2^{|M_{\alpha_{\mathsf{V}_0}}|}$. Let $\mathsf{V}^N_{1/2} \models u \leq v$. Then there exist ω-words u', v' satisfying

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- Proven by the induction on the height of a *factorization tree* of word u for α_{V_0}.

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of word $u \in A^*$ for α_{SI}

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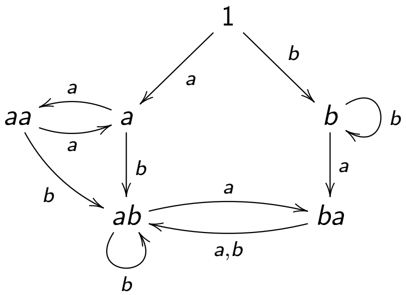
monoid $M_{\alpha_{\mathsf{SI}}}$



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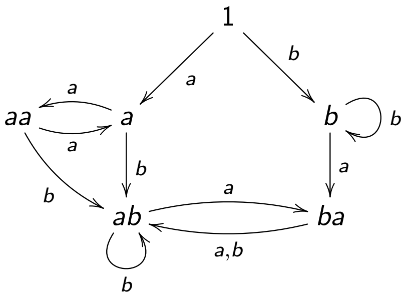

$$\underbrace{a \dots a}_{2^K+1} b$$

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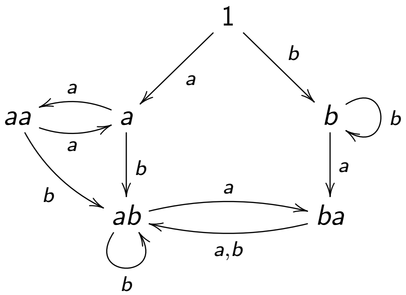

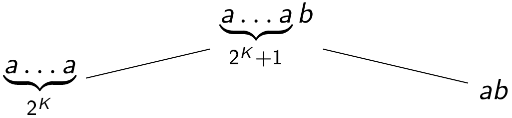

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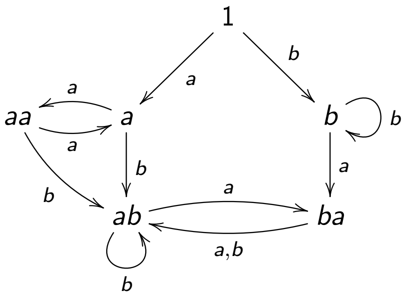

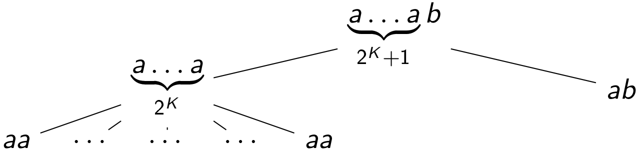

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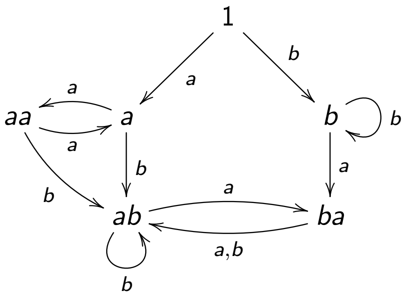

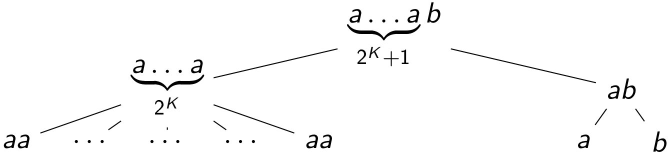

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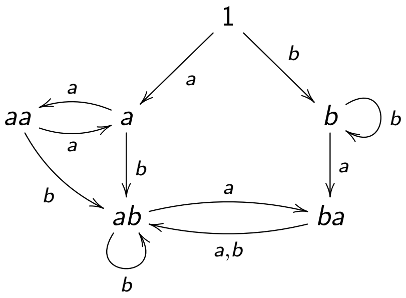

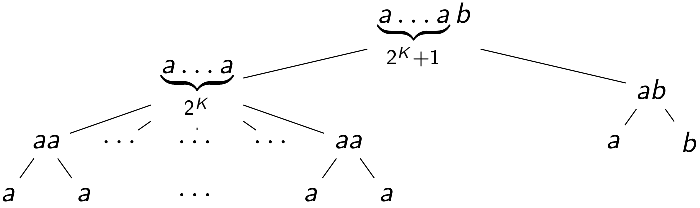

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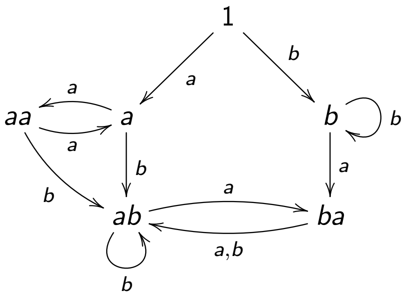

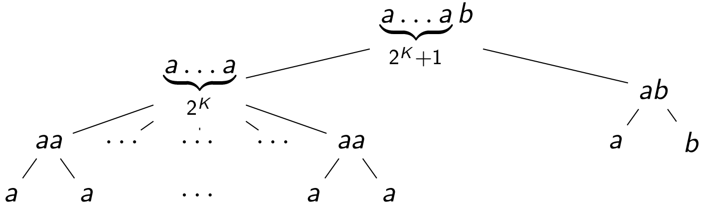

height of this factorization tree = 4

Useful Statements

Theorem (Simon, 1990, Kufleitner, 2008)

Let A be a finite alphabet, M a finite monoid, $\alpha\colon A^* \to M$ a homomorphism. Then, for every word $u \in A^*$, there exists a factorization tree of u for α of height at most $3 \cdot |M|$.

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The following lemma will be useful.

Lemma (J. V.)

Let $u_1, \ldots, u_k, v \in A^*$ $(k \in \mathbb{N})$, $n \geq k-1$. Let $\mathrm{V}^n_{1/2} \models u_1 \ldots u_k \leq v$. Then there exist $v_1, \ldots, v_k \in A^*$ such that

- $v = v_1 \ldots v_k$,
- $\mathrm{V}^{n-(k-1)}_{1/2} \models u_i \leq v_i$ *for* $i = 1, \ldots, k$.

$$\mathsf{V}^N_{1/2} \models \underbrace{a \dots a}_{2^N+1} b \leq \underbrace{a \dots a}_{2^N+2} b$$

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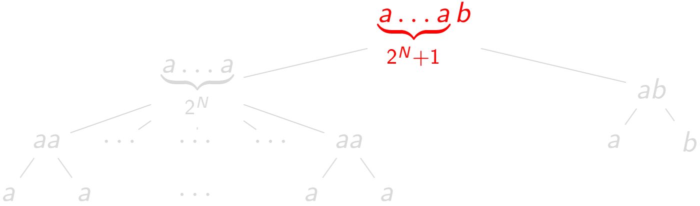

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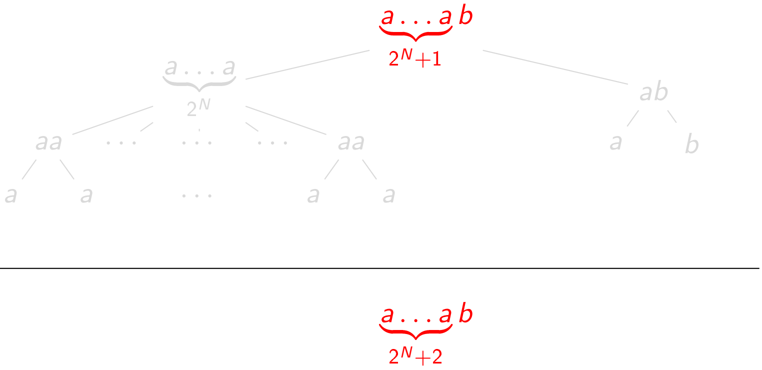
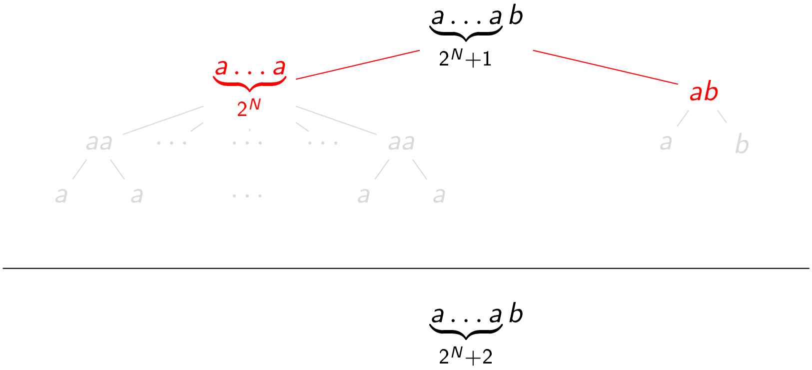
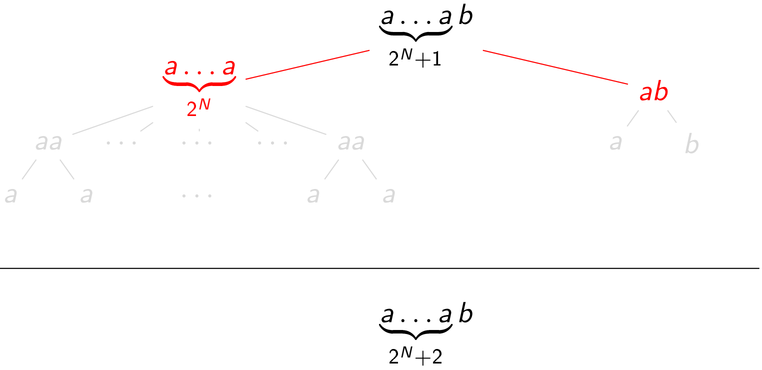


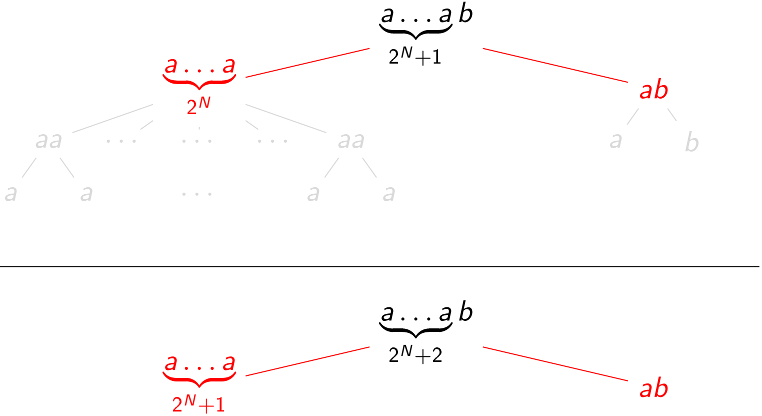
$$\mathsf{V}^{N}_{1/2} \models \underbrace{a \ldots a}_{2^N+1}\, b \leq \underbrace{a \ldots a}_{2^N+2}\, b$$



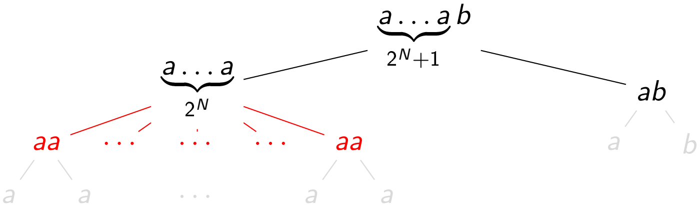
$$V_{1/2}^{N} \models \underbrace{a \ldots a}_{2^N+1} b \leq \underbrace{a \ldots a}_{2^N+2} b \rightarrow V_{1/2}^{N-1} \models \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N+1},\ ab \leq ab$$

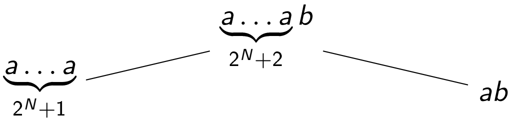


$$\mathrm{V}^{N}_{1/2} \models \underbrace{a\dots a}_{2^N+1} b \leq \underbrace{a\dots a}_{2^N+2} b \to \mathrm{V}^{N-1}_{1/2} \models \underbrace{a\dots a}_{2^N} \leq \underbrace{a\dots a}_{2^N+1},\ ab \leq ab$$

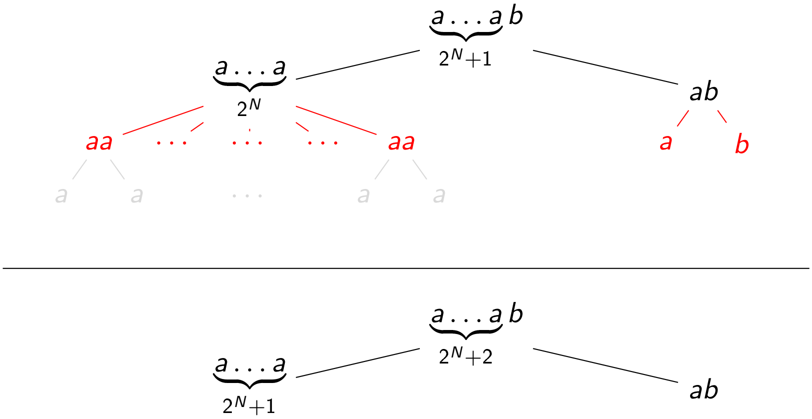


$$\mathsf{V}^{N}_{1/2} \models \underbrace{a\dots a}_{2^N+1}\, b \leq \underbrace{a\dots a}_{2^N+2}\, b \to \mathsf{V}^{N-1}_{1/2} \models \underbrace{a\dots a}_{2^N} \leq \underbrace{a\dots a}_{2^N+1},\ ab \leq ab$$



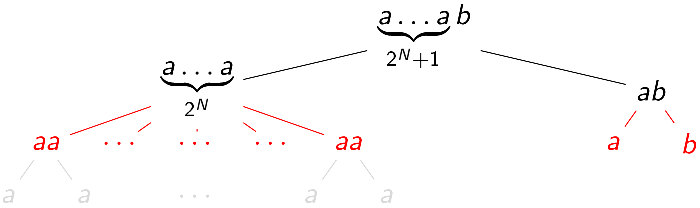


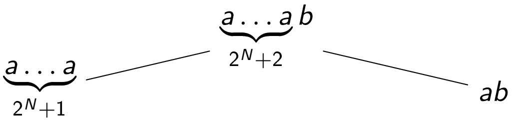
$$\mathsf{V}^N_{1/2} \models \underbrace{a \ldots a}_{2^N+1} b \leq \underbrace{a \ldots a}_{2^N+2} b \;\rightarrow\; \mathsf{V}^{N-1}_{1/2} \models \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N+1},\ ab \leq ab$$



$$V^{N}_{1/2} \models \underbrace{a\dots a}_{2^N+1} b \le \underbrace{a\dots a}_{2^N+2} b \to \; V^{N-1}_{1/2} \models \underbrace{a\dots a}_{2^N} \le \underbrace{a\dots a}_{2^N+1},\; ab \le ab$$

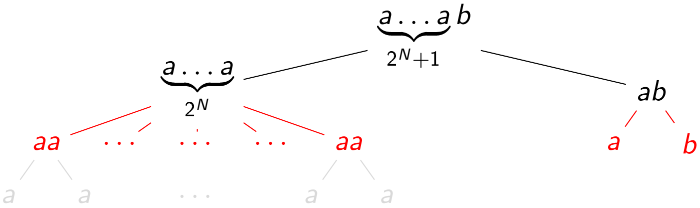
$$\to V^{N-3}_{1/2} \models aa \le aa,\; \underbrace{a\dots a}_{2^N-4} \le \underbrace{a\dots a}_{2^N-3},\; aa \le aa, \quad a \le a,\; b \le b$$

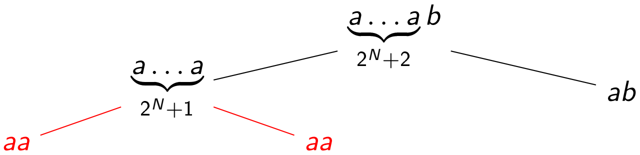




$$V^{N}_{1/2} \models \underbrace{a \dots a}_{2^N+1} b \le \underbrace{a \dots a}_{2^N+2} b \to V^{N-1}_{1/2} \models \underbrace{a \dots a}_{2^N} \le \underbrace{a \dots a}_{2^N+1},\ ab \le ab$$

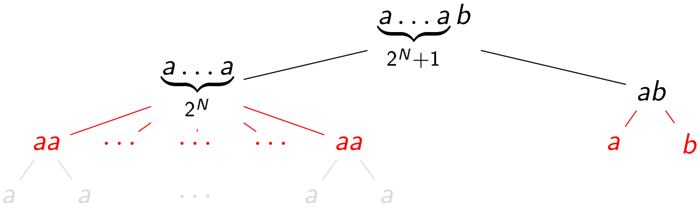
$$\to V^{N-3}_{1/2} \models aa \le aa,\ \underbrace{a \dots a}_{2^N-4} \le \underbrace{a \dots a}_{2^N-3},\ aa \le aa, \quad a \le a,\ b \le b$$

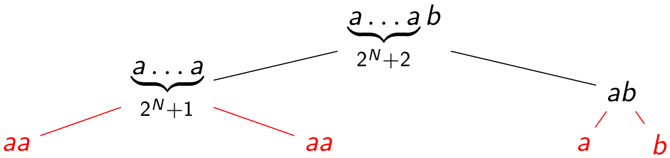




$$V^{N}_{1/2} \models \underbrace{a\dots a}_{2^N+1} b \le \underbrace{a\dots a}_{2^N+2} b \to V^{N-1}_{1/2} \models \underbrace{a\dots a}_{2^N} \le \underbrace{a\dots a}_{2^N+1},\ ab \le ab$$

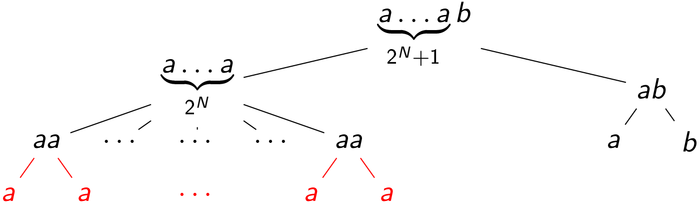
$$\to V^{N-3}_{1/2} \models aa \le aa,\ \underbrace{a\dots a}_{2^N-4} \le \underbrace{a\dots a}_{2^N-3},\ aa \le aa,\quad a \le a,\ b \le b$$

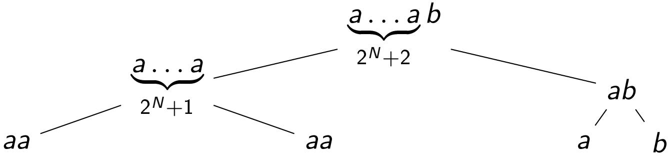




$$V_{1/2}^{N} \models \underbrace{a\ldots a}_{2^N+1} b \leq \underbrace{a\ldots a}_{2^N+2} b \rightarrow V_{1/2}^{N-1} \models \underbrace{a\ldots a}_{2^N} \leq \underbrace{a\ldots a}_{2^N+1},\ ab \leq ab$$

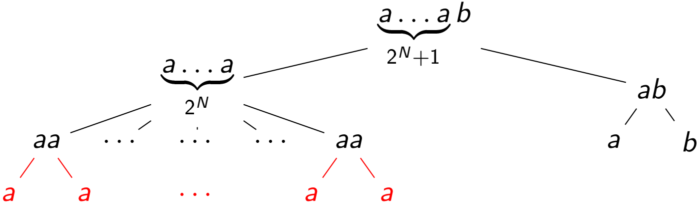
$$\rightarrow V_{1/2}^{N-3} \models aa \leq aa,\ \underbrace{a\ldots a}_{2^N-4} \leq \underbrace{a\ldots a}_{2^N-3},\ aa \leq aa, \quad a \leq a,\ b \leq b$$

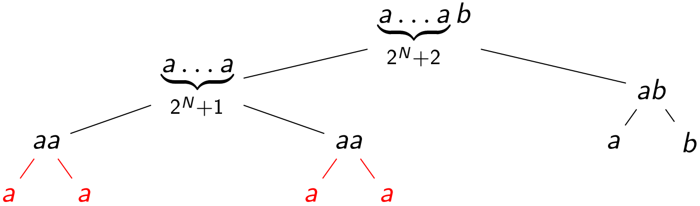




$$V^{N}_{1/2} \models \underbrace{a \dots a}_{2^N+1} b \leq \underbrace{a \dots a}_{2^N+2} b \to V^{N-1}_{1/2} \models \underbrace{a \dots a}_{2^N} \leq \underbrace{a \dots a}_{2^N+1},\ ab \leq ab$$

$$\to V^{N-3}_{1/2} \models aa \leq aa,\ \underbrace{a \dots a}_{2^N-4} \leq \underbrace{a \dots a}_{2^N-3},\ aa \leq aa, \quad a \leq a,\ b \leq b$$





$$V_{1/2}^{N} \models \underbrace{a \dots a}_{2^N+1} b \leq \underbrace{a \dots a}_{2^N+2} b$$

$$\to \; V_{1/2}^{N-1} \models \underbrace{a \dots a}_{2^N} \leq \underbrace{a \dots a}_{2^N+1}, \; ab \leq ab$$

$$\to V_{1/2}^{N-3} \models aa \leq aa, \underbrace{a \dots a}_{2^N-4} \leq \underbrace{a \dots a}_{2^N-3}, aa \leq aa, \quad a \leq a, \; b \leq b$$

$$\to V_{1/2}^{N-4} \models a \leq a, \; a \leq a$$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a\ldots a}_{2^N+1} b \leq \underbrace{a\ldots a}_{2^N+2} b$$

$$\rightarrow\ \mathsf{V}_{1/2}^{N-1} \models \underbrace{a\ldots a}_{2^N} \leq \underbrace{a\ldots a}_{2^N+1},\ ab \leq ab$$

$$\rightarrow \mathsf{V}_{1/2}^{N-3} \models aa \leq aa,\ \underbrace{a\ldots a}_{2^N-4} \leq \underbrace{a\ldots a}_{2^N-3},\ aa \leq aa, \quad a \leq a,\ b \leq b$$

$$\rightarrow \mathsf{V}_{1/2}^{N-4} \models a \leq a,\ a \leq a$$

- $\mathsf{V}_{1/2} \models a \leq a$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a \ldots a}_{2^N+1} b \leq \underbrace{a \ldots a}_{2^N+2} b$$

$$\rightarrow \ \mathsf{V}_{1/2}^{N-1} \models \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N+1}, \ ab \leq ab$$

$$\rightarrow \mathsf{V}_{1/2}^{N-3} \models aa \leq aa, \ \underbrace{a \ldots a}_{2^N-4} \leq \underbrace{a \ldots a}_{2^N-3}, \ aa \leq aa, \quad a \leq a, \ b \leq b$$

- $\mathsf{V}_{1/2} \models aa \leq aa$

$$\rightarrow \mathsf{V}_{1/2}^{N-4} \models a \leq a, \ a \leq a$$

- $\mathsf{V}_{1/2} \models a \leq a$

$$\mathrm{V}_{1/2}^{N} \models \underbrace{a \dots a}_{2^N+1} b \le \underbrace{a \dots a}_{2^N+2} b$$

$$\to \; \mathrm{V}_{1/2}^{N-1} \models \underbrace{a \dots a}_{2^N} \le \underbrace{a \dots a}_{2^N+1}, \; ab \le ab$$

$$\to \mathrm{V}_{1/2}^{N-3} \models aa \le aa, \; \underbrace{a \dots a}_{2^N-4} \le \underbrace{a \dots a}_{2^N-3}, \; aa \le aa, \quad a \le a, \; b \le b$$

- $\mathrm{V}_{1/2} \models aa \le aa$
- $\mathrm{V}_{1/2} \models a \le a, \; b \le b$

$$\to \mathrm{V}_{1/2}^{N-4} \models a \le a, \; a \le a$$

- $\mathrm{V}_{1/2} \models a \le a$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a \dots a}_{2^N+1} b \leq \underbrace{a \dots a}_{2^N+2} b$$

$$\to \ \mathsf{V}_{1/2}^{N-1} \models \underbrace{a \dots a}_{2^N} \leq \underbrace{a \dots a}_{2^N+1}, \ ab \leq ab$$

$$\mathsf{V}_{1/2} := [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid \mathsf{V}_0 \models u = v]\!]$$

$$\to \mathsf{V}_{1/2}^{N-3} \models aa \leq aa, \ \underbrace{a \dots a}_{2^N-4} \leq \underbrace{a \dots a}_{2^N-3}, \ aa \leq aa, \quad a \leq a, \ b \leq b$$

- $\mathsf{V}_{1/2} \models aa \leq aa$
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$$\mathsf{V}_{1/2} := [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid \mathsf{V}_0 \models u = v]\!]$$

- $\mathsf{V}_0 = \mathsf{SI} \models aa = a$

$$\to \mathsf{V}_{1/2}^{N-3} \models aa \leq aa, \; \underbrace{a \dots a}_{2^N-4} \leq \underbrace{a \dots a}_{2^N-3}, \; aa \leq aa, \quad a \leq a, \; b \leq b$$

- $\mathsf{V}_{1/2} \models aa \leq aa$
- $\mathsf{V}_{1/2} \models a \leq a, \; b \leq b$

$$\to \mathsf{V}_{1/2}^{N-4} \models a \leq a, \; a \leq a$$

- $\mathsf{V}_{1/2} \models a \leq a$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a\ldots a}_{2^N+1} b \leq \underbrace{a\ldots a}_{2^N+2} b$$

$$\to\ \mathsf{V}_{1/2}^{N-1} \models \underbrace{a\ldots a}_{2^N} \leq \underbrace{a\ldots a}_{2^N+1},\ ab \leq ab$$

$\mathsf{V}_{1/2} := [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid \mathsf{V}_0 \models u = v]\!]$

- $\mathsf{V}_0 = \mathsf{Sl} \models aa = a \to \mathsf{V}_{1/2} \models (aa)^{\omega+1} \leq (aa)^{\omega} a (aa)^{\omega}$

$$\to \mathsf{V}_{1/2}^{N-3} \models aa \leq aa,\ \underbrace{a\ldots a}_{2^N-4} \leq \underbrace{a\ldots a}_{2^N-3},\ aa \leq aa, \quad a \leq a,\ b \leq b$$

- $\mathsf{V}_{1/2} \models aa \leq aa$
- $\mathsf{V}_{1/2} \models a \leq a,\ b \leq b$

$$\to \mathsf{V}_{1/2}^{N-4} \models a \leq a,\ a \leq a$$

- $\mathsf{V}_{1/2} \models a \leq a$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a \ldots a}_{2^N+1} b \leq \underbrace{a \ldots a}_{2^N+2} b$$

$$\rightarrow \mathsf{V}_{1/2}^{N-1} \models \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N+1}, \ ab \leq ab$$

$\mathsf{V}_{1/2} := [\![u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid \mathsf{V}_0 \models u = v]\!]$

- $\mathsf{V}_0 = \mathsf{SI} \models aa = a \rightarrow \mathsf{V}_{1/2} \models (aa)^{\omega+1} \leq (aa)^{\omega} a (aa)^{\omega}$
- $\mathsf{V}_{1/2} \models ab \leq ab$

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- $\mathsf{V}_{1/2} \models aa \leq aa$
- $\mathsf{V}_{1/2} \models a \leq a, \ b \leq b$

$$\rightarrow \mathsf{V}_{1/2}^{N-4} \models a \leq a, \ a \leq a$$

- $\mathsf{V}_{1/2} \models a \leq a$

$$\mathsf{V}_{1/2}^{N} \models \underbrace{a \dots a}_{2^N+1} b \le \underbrace{a \dots a}_{2^N+2} b$$

- $\mathsf{V}_{1/2} \models (aa)^{\omega+1} ab \le (aa)^{\omega} a (aa)^{\omega} ab$

$$\to \ \mathsf{V}_{1/2}^{N-1} \models \underbrace{a \dots a}_{2^N} \le \underbrace{a \dots a}_{2^N+1}, \ ab \le ab$$

$$\mathsf{V}_{1/2} := [\![u^{\omega+1} \le u^{\omega} v u^{\omega} \mid \mathsf{V}_0 \models u = v]\!]$$

- $\mathsf{V}_0 = \mathsf{SI} \models aa = a \to \mathsf{V}_{1/2} \models (aa)^{\omega+1} \le (aa)^{\omega} a (aa)^{\omega}$
- $\mathsf{V}_{1/2} \models ab \le ab$

$$\to \mathsf{V}_{1/2}^{N-3} \models aa \le aa, \ \underbrace{a \dots a}_{2^N-4} \le \underbrace{a \dots a}_{2^N-3}, \ aa \le aa, \quad a \le a, \ b \le b$$

- $\mathsf{V}_{1/2} \models aa \le aa$
- $\mathsf{V}_{1/2} \models a \le a, \ b \le b$

$$\to \mathsf{V}_{1/2}^{N-4} \models a \le a, \ a \le a$$

- $\mathsf{V}_{1/2} \models a \le a$

$$V_{1/2}^N \models \underbrace{a\dots a}_{2^N+1} b \le \underbrace{a\dots a}_{2^N+2} b$$

- $V_{1/2} \models (aa)^{\omega+1}ab \le (aa)^{\omega}a(aa)^{\omega}ab$

$$\to\ V_{1/2}^{N-1} \models \underbrace{a\dots a}_{2^N} \le \underbrace{a\dots a}_{2^N+1},\ ab \le ab$$

$$V_{1/2} := [\![u^{\omega+1} \le u^{\omega}vu^{\omega} \mid V_0 \models u = v]\!]$$

- $V_0 = \mathrm{Sl} \models aa = a \to V_{1/2} \models (aa)^{\omega+1} \le (aa)^{\omega}a(aa)^{\omega}$
- $V_{1/2} \models ab \le ab$

$$\to V_{1/2}^{N-3} \models aa \le aa,\ \underbrace{a\dots a}_{2^N-4} \le \underbrace{a\dots a}_{2^N-3},\ aa \le aa,\quad a \le a,\ b \le b$$

- $V_{1/2} \models aa \le aa$
- $V_{1/2} \models a \le a,\ b \le b$

$$\to V_{1/2}^{N-4} \models a \le a,\ a \le a$$

- $V_{1/2} \models a \le a$

ω-Reducibility of $V_{3/2}$ and $V_{5/2}$

ω-Reducibility of $V_{3/2}$ and $V_{5/2}$

$V_{1/2}$
ω-reducibility
of *pairs* of words

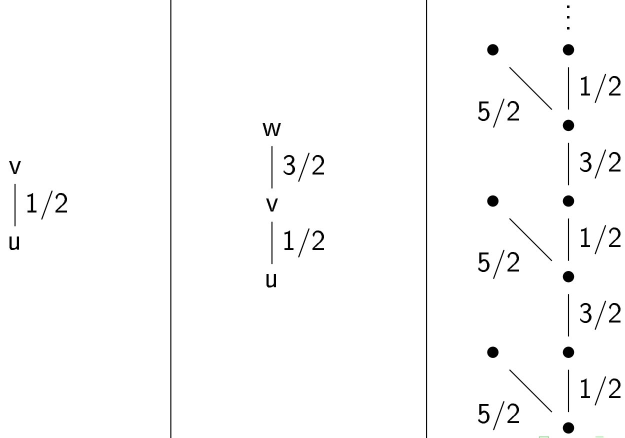
$$\begin{array}{c} v \\ \big| \, 1/2 \\ u \end{array}$$

ω-Reducibility of $V_{3/2}$ and $V_{5/2}$

$V_{1/2}$ ω-reducibility of *pairs* of words	$V_{3/2}$ ω-reducibility of *chains* of words
v | 1/2 u	w | 3/2 v | 1/2 u

ω-Reducibility of $V_{3/2}$ and $V_{5/2}$

$V_{1/2}$ ω-reducibility of *pairs* of words	$V_{3/2}$ ω-reducibility of *chains* of words	$V_{5/2}$ ω-reducibility of (finite) *ordered sets* of words



Thank you for your attention.