The Omega-Reducibility of Certain Pseudovarieties of Ordered Monoids

Jana Volaříková

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1. Pseudovariety of ordered monoids: what is it?

2. $\omega$-reducibility: what is it?

3. Which pseudovarieties are of my interest and why?

4. How to prove the $\omega$-reducibility of a certain pseudovariety $V$?

5. Briefly about the $\omega$-reducibility of more complex pseudovarieties.
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2 \( \omega \)-reducibility – what is it?
Outline

1. Pseudovariety of ordered monoids – what is it?
2. $\omega$-reducibility – what is it?
3. Which pseudovarieties are of my interest and why?
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2. ω-reducibility – what is it?
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4. How to prove the ω-reducibility of a certain pseudovariety $V_{1/2}$?
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2. $\omega$-reducibility – what is it?
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4. How to prove the $\omega$-reducibility of a certain pseudovariety $V_{1/2}$?
5. Briefly about the $\omega$-reducibility of more complex pseudovarieties
Ordered monoid \((M, \cdot, 1, \leq)\): \((M, \cdot, 1)\) monoid, \(\leq\) is a partial order on \(M\), which is compatible with the multiplication, i.e.,
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\forall s_1, s_2, t_1, t_2 \in M: (s_1 \leq t_1, s_2 \leq t_2) \Rightarrow s_1 \cdot s_2 \leq t_1 \cdot t_2.
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**Example**

A bounded semilattice:
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A bounded semilattice: \((M, \wedge, 1, \leq)\)
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A bounded semilattice: \((M, \wedge, 1, \leq)\) \(\ldots\) \(s \leq t \iff s \wedge t = s\)

- \(1 = \text{the biggest element} = \text{the neutral element}:\)
  \[\forall s \in M : s \leq 1 \iff s \wedge 1 = s\]
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- $1 =$ the biggest element $= =$ the neutral element:
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- A pseudovariety of ordered monoids is a class of finite monoids which is closed under taking submonoids, finite direct products and images in homomorphisms of ordered monoids.
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**Example**

Pseudovariety of finite (bounded) semilattices
Pseudowords and $\omega$-Words

- A ... a finite set (alphabet)
A ... a finite set (alphabet), $A^*$ the free monoid over $A$, ...
Pseudowords and $\omega$-Words

- A ... a finite set (alphabet), $A^*$ the free monoid over $A$,
  $\widehat{A}^*$ the free profinite monoid over $A$ (the metric completion of $(A^*, d)$ with respect to a specific metric $d$ on $A^*$)

$\omega$: $x \mapsto x_\omega = \lim_{n \to \infty} x^n$

An unary operation on $\widehat{A}^*$, on a finite monoid $M$ equipped with the discrete metric.

Note that $x_\omega$ is the unique idempotent power of $x$.

Denote by $\omega$ a signature $\omega = \{_\cdot_\cdot, 1, _\omega\}$.

Then the monoids $\widehat{A}^*$ and $M$ can be viewed as $\omega$-algebras.

Elements of the $\omega$-subalgebra of the $\omega$-algebra $\widehat{A}^*$ generated by $A$ are called $\omega$-words.

Example $A = \{a, b\}$ ($ab_\omega bba_\omega aaa_\omega b$ is an $\omega$-word over $A$)

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Omega-Reducibility
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**Example**

- $A = \{a, b\}$
- $(ab^\omega bba)^\omega aaa^\omega b$ is an $\omega$-word over $A$
Validity of Inequalities

- Let $V$ be a pseudovariety of ordered monoids, $u, v \in \hat{A}^*$. 

Example: Pseudovariety $S_l$ of finite meet-semilattices:

$$S_l = \{a, ba = ab, a \leq 1\}$$

$$S_l | = \langle ab \omega bba \omega aaaa \omega b \rangle = ba$$
Let $V$ be a pseudovariety of ordered monoids, $u, v \in \hat{A}^*$. Then $V \models u \leq v$ (= inequality $u \leq v$ is valid in $V$) iff
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A class of finite ordered monoids is a pseudovariety iff it is definable by a set of inequalities of pseudowords. (Reiterman, 1982, Pin + Weil, 1996)
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Pseudovariety $\mathsf{SI}$ of finite meet- semilattices:

- $\mathsf{SI} = [aa = a, ab = ba, a \leq 1]$
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- $SI = [aa = a, ab = ba, a \leq 1]$
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for every ordered monoid $M \in V$, 

- for every onto continuous homomorphism $\alpha : \hat{A} \to M$, and
- for every inequality $u \leq v$ of pseudowords that is valid in $V$, there exists an inequality $u' \leq v'$ of $\omega$-words that is also valid in $V$ and has the same $\alpha$-imprint in $M$, i.e., $\alpha(u') = \alpha(u), \alpha(v') = \alpha(v)$. 

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$$\alpha(u') = \alpha(u), \quad \alpha(v') = \alpha(v).$$
Pseudovarieties of My Interest

A pseudovariety of ordered monoids $V$ is said to be locally finite if for every finite alphabet $A$, the relatively free monoid in $V$ is finite, where $u \equiv v \iff \exists m \geq 1 : u = v^m = v$. 

Example: Pseudovariety of finite semilattices:

$A = \{a, b\} (A^* / \equiv_{\text{Sl}} \cdot) \sim = \{\{a\}, \{b\}, \{a, b\}, \emptyset\}, \cup$ 

Pseudovarieties corresponding to half levels of concatenation hierarchies with a locally finite basis:

$V_0$ is an arbitrary locally finite pseudovariety of monoids $V_1 / 2 := J_{\omega + 1} \leq u \omega \vdash V_0 \vdash u = v \star K \forall n \geq 1 : V_{n+1} / 2 := J_{\omega + 1} \leq u \omega \vdash V_n / 2 \vdash v \leq u$. 

$V_n / 2$ is $\omega$-reducible $\Rightarrow$ it suffices to consider $u, v$ to be $\omega$-words $\Rightarrow V_{n+1} / 2$ is denable by inequalities of $\omega$-words.
A pseudovariety of ordered monoids $V$ is said to be *locally finite* if for every finite alphabet $A$, the ”relatively free monoid in $V$” $A^*/\equiv_V$ is finite, where $u \equiv_V v \iff V \models u = v$. 
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  \item $\forall n \geq 1: V_{n+1/2} := \left[ u^{\omega+1} \leq u^\omega v u^\omega \mid V_{n-1/2} \models v \leq u \right]$
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- $V_0$ is an arbitrary *locally finite* pseudovariety of monoids
- $V_{1/2} := \{u^{\omega+1} \leq u^\omega v u^\omega \mid V_0 \models u = v\}$
- $\forall n \geq 1: V_{n+1/2} := \{u^{\omega+1} \leq u^\omega v u^\omega \mid V_{n-1/2} \models v \leq u\}$

$V_{n-1/2}$ is $\omega$-reducible $\Rightarrow$ it suffices to consider $u, v$ to be $\omega$-words.
Pseudovarieties of My Interest

A pseudovariety of ordered monoids $V$ is said to be *locally finite* if for every finite alphabet $A$, the "relatively free monoid in $V" A^*/\equiv_V$ is finite, where $u \equiv_V v \iff V \models u = v$.

Example

Pseudovariety of finite semilattices:

- $A = \{a, b\}$
- $(A^*/\equiv_{SL}, \cdot) \cong (\{\{a\}, \{b\}, \{a, b\}, \emptyset\}, \cup)$

Pseudovarieties corresponding to half levels of "concatenation hierarchies" with a *locally finite* basis:

- $V_0$ is an arbitrary *locally finite* pseudovariety of monoids
- $V_{1/2} := \{u^\omega + 1 \leq u^\omega \nu u^\omega \mid V_0 \models u = v\}$
- $\forall n \geq 1: V_{n+1/2} := \{u^\omega + 1 \leq u^\omega \nu u^\omega \mid V_{n-1/2} \models v \leq u\}$

$V_{n-1/2}$ is $\omega$-reducible $\Rightarrow$ it suffices to consider $u, v$ to be $\omega$-words $\Rightarrow V_{n+1/2}$ is definable by inequalities of $\omega$-words.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

GIVEN: locally finite pseudovariety $V_0$, finite alphabet $A$, finite ordered monoid $M$, continuous homomorphism $\alpha: \hat{A}^* \rightarrow M$.

Construct the $V_0$-completion $\alpha V_0$ of the homomorphism $\alpha$:

$$\alpha V_0: \hat{A}^* \rightarrow M \times A^*/\equiv V_0 \ x \rightarrow (\alpha(x), [x] \equiv V_0)$$

Construct the image $M \alpha V_0$ of $\alpha V_0$.

OUTPUT of Step 1:

$\forall x, y \in \hat{A}^*$:

$$\alpha V_0(x) = \alpha V_0(y) \Rightarrow x \equiv V_0 y.$$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

**GIVEN:**
- locally finite pseudovariety $V_0$,
Proof of the \( \omega \)-Reducibility of \( V_{1/2} \) - Step 1

- Proof of the \( \omega \)-reducibility of \( V_{1/2} \) inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

GIVEN:

- locally finite pseudovariety \( V_0 \),
- finite alphabet \( A \),
- continuous homomorphism \( \alpha : \hat{A}^* \to M \times A^*/\equiv \).
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

GIVEN:
- locally finite pseudovariety $V_0$,
- finite alphabet $A$,
- finite ordered monoid $M$, 

\begin{align*}
\text{OUTPUT of Step 1:} & \\
\text{V}_{0}\text{-compatible onto continuous homomorphism } & \alpha_{V_0} \colon \hat{A}^* \rightarrow M \\
\text{such that } & \forall x, y \in \hat{A}^* : \alpha_{V_0}(x) = \alpha_{V_0}(y) \Rightarrow x \equiv V_0 y.
\end{align*}
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

**GIVEN:**
- locally finite pseudovariety $V_0$,
- finite alphabet $A$,
- finite ordered monoid $M$,
- continuous homomorphism $\alpha : \hat{A}^* \rightarrow M$.

**OUTPUT of Step 1:** $V_0$-compatible onto continuous homomorphism $\alpha : \hat{A}^* \rightarrow M$.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

**GIVEN:**
- locally finite pseudovariety $V_0$,
- finite alphabet $A$,
- finite ordered monoid $M$,
- continuous homomorphism $\alpha : \hat{A}^* \rightarrow M$.

Construct the $V_0$-completion $\alpha_{V_0}$ of the homomorphism $\alpha$:
- $\alpha_{V_0} : \hat{A}^* \rightarrow M \times A^*/\equiv_{V_0}$
- $x \mapsto (\alpha(x), [x]_{\equiv_{V_0}})$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

**GIVEN:**
- locally finite pseudovariety $V_0$,
- finite alphabet $A$,
- finite ordered monoid $M$,
- continuous homomorphism $\alpha : \hat{A}^* \rightarrow M$.

Construct the $V_0$-completion $\alpha_{V_0}$ of the homomorphism $\alpha$:
- $\alpha_{V_0} : \hat{A}^* \rightarrow M \times A^*/\equiv_{V_0}$
  - $x \mapsto (\alpha(x), [x]_{\equiv_{V_0}})$

Construct the image $M_{\alpha_{V_0}}$ of $\alpha_{V_0}$.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1

- Proof of the $\omega$-reducibility of $V_{1/2}$ inspired by the approach of Place and Zeitoun in their papers (2014–2019) on certain properties of levels of concatenation hierarchies.

**GIVEN:**
- locally finite pseudovariety $V_0$,
- finite alphabet $A$,
- finite ordered monoid $M$,
- continuous homomorphism $\alpha : \hat{A}^* \rightarrow M$.

Construct the $V_0$-completion $\alpha_{V_0}$ of the homomorphism $\alpha$:
- $\alpha_{V_0} : \hat{A}^* \rightarrow M \times A^*/\equiv_{V_0}$
- $x \mapsto (\alpha(x), [x]_{\equiv_{V_0}})$

Construct the image $M_{\alpha_{V_0}}$ of $\alpha_{V_0}$.

**OUTPUT of Step 1:**
$V_0$-compatible onto continuous homomorphism $\alpha_{V_0} : \hat{A}^* \rightarrow M_{\alpha_{V_0}}$:
\[ \forall x, y \in \hat{A}^* : \alpha_{V_0}(x) = \alpha_{V_0}(y) \Rightarrow x \equiv_{V_0} y. \]
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example
Proof of the \( \omega \)-Reducibility of \( V_{1/2} \) - Step 1 – Example

- \( V_0 = \text{SI} \) (pseudovariety of finite semilattices),
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = SI$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{Sl}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$, 
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{Sl}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha: \hat{A}^* \to M$ defined by $\alpha(a) = a, \alpha(b) = b$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = Sl$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha : \hat{A}^* \to M$ defined by $\alpha(a) = a$, $\alpha(b) = b$

Cayley graph of monoid $M$:
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{SI}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha: \widehat{A}^* \to M$ defined by $\alpha(a) = a$, $\alpha(b) = b$

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Cayley graph of monoid $M$:  Cayley graph of monoid $M_{\alpha_{\text{SI}}}$:

1. $V_0 = \text{SI}$ (pseudovariety of finite semilattices),
2. $A = \{a, b\}$,
3. $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
4. $\alpha: \widehat{A}^* \to M$ defined by $\alpha(a) = a$, $\alpha(b) = b$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{Sl}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha: \hat{A}^* \rightarrow M$ defined by $\alpha(a) = a, \alpha(b) = b$

Cayley graph of monoid $M$:  
Cayley graph of monoid $M_{\alpha_{\text{Sl}}}$:

- $\text{Sl} \nvdash aa = 1$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{Sl}$ (pseudovariety of finite semilattices),
- $A = \{a, b\},$
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle,$
- $\alpha: \tilde{A}^* \to M$ defined by $\alpha(a) = a, \alpha(b) = b$

Cayley graph of monoid $M$:  

Cayley graph of monoid $M_{\alpha_{\text{Sl}}}$:

$\text{Sl} \nmid aa = 1$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{Sl}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha : \hat{A}^* \to M$ defined by $\alpha(a) = a$, $\alpha(b) = b$

Cayley graph of monoid $M$: Cayley graph of monoid $M_{\alpha_{\text{Sl}}}$:

Sl $\not\models aa = 1$
Sl $\not\models ab = b$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 1 – Example

- $V_0 = \text{SI}$ (pseudovariety of finite semilattices),
- $A = \{a, b\}$,
- $M = \langle a, b \mid aa = 1, ab = bb = b \rangle$,
- $\alpha : \hat{A}^* \to M$ defined by $\alpha(a) = a, \alpha(b) = b$

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Cayley graph of monoid $M$:

Cayley graph of monoid $M_{\alpha_{\text{SI}}}$:

$\text{SI} \not\models aa = 1$

$\text{SI} \not\models ab = b$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V_n$:

$V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq \cdots \subseteq V_{1/2} = \bigcup_{n=1}^{\infty} V_n$

Theorem (J. V.)

Let $u, v \in A^*$ be arbitrary words and $N = 1 + 9 \cdot |M_{\alpha V_0}|^2 |M_{\alpha V_0}|$. Let $V_{N_{1/2}} = u \leq v$. Then there exist $\omega$-words $u', v'$ satisfying $V_{1/2} = u' \leq v'$, $\alpha_{V_0}(u) = \alpha_{V_0}(u')$, $\alpha_{V_0}(v) = \alpha_{V_0}(v')$. This theorem implies that the pseudovariety $V_{1/2}$ is $\omega$-reducible.

Proven by the induction on the height of a factorization tree of word $u$ for $\alpha_{V_0}$. 

Jana Volaříková

Omega-Reducibility
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V^n_{1/2}$.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V_{1/2}^n$:

- $V_{1/2}^0 \subseteq V_{1/2}^1 \subseteq V_{1/2}^2 \subseteq V_{1/2}^3 \subseteq V_{1/2}^4 \subseteq \cdots$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V^n_{1/2}$:

- $V^0_{1/2} \subseteq V^1_{1/2} \subseteq V^2_{1/2} \subseteq V^3_{1/2} \subseteq V^4_{1/2} \subseteq \ldots$
- $V_{1/2} = \bigcup_{n=1}^{\infty} V^n_{1/2}$
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V^n_{1/2}$:

- $V^0_{1/2} \subseteq V^1_{1/2} \subseteq V^2_{1/2} \subseteq V^3_{1/2} \subseteq V^4_{1/2} \subseteq \cdots$
- $V_{1/2} = \bigcup_{n=1}^{\infty} V^n_{1/2}$

**Theorem (J. V.)**

Let $u, v \in A^*$ be arbitrary words and $N = 1 + 9 \cdot |M_{\alpha V_0}| \cdot 2^{|M_{\alpha V_0}|}$. Let $V^N_{1/2} \models u \leq v$. Then there exist $\omega$-words $u', v'$ satisfying

- $V_{1/2} \models u' \leq v'$,
- $\alpha_{V_0}(u) = \alpha_{V_0}(u')$, $\alpha_{V_0}(v) = \alpha_{V_0}(v')$. 

Jan a Volaříková

Omega-Reducibility
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V^n_{1/2}$:

- $V^0_{1/2} \subseteq V^1_{1/2} \subseteq V^2_{1/2} \subseteq V^3_{1/2} \subseteq V^4_{1/2} \subseteq \cdots$
- $V_{1/2} = \bigcup_{n=1}^{\infty} V^n_{1/2}$

**Theorem (J. V.)**

Let $u, v \in A^*$ be arbitrary words and $N = 1 + 9 \cdot |M_{\alpha_{V_0}}| \cdot 2^{|M_{\alpha_{V_0}}|}$. Let $V^N_{1/2} \models u \leq v$. Then there exist $\omega$-words $u', v'$ satisfying

- $V_{1/2} \models u' \leq v'$,
- $\alpha_{V_0}(u) = \alpha_{V_0}(u')$, $\alpha_{V_0}(v) = \alpha_{V_0}(v')$.

This theorem implies that the pseudovariety $V_{1/2}$ is $\omega$-redicible.
Proof of the $\omega$-Reducibility of $V_{1/2}$ - Step 2

Stratification of $V_{1/2}$ into locally finite pseudovarieties $V_{1/2}^n$:

- $V_{1/2}^0 \subseteq V_{1/2}^1 \subseteq V_{1/2}^2 \subseteq V_{1/2}^3 \subseteq V_{1/2}^4 \subseteq \cdots$
- $V_{1/2} = \bigcup_{n=1}^{\infty} V_{1/2}^n$

**Theorem (J. V.)**

Let $u, v \in A^*$ be arbitrary words and $N = 1 + 9 \cdot |M_{\alpha V_0}| \cdot 2^{|M_{\alpha V_0}|}$. Let $V_{1/2}^N \models u \leq v$. Then there exist $\omega$-words $u', v'$ satisfying

- $V_{1/2} \models u' \leq v'$,
- $\alpha_{V_0}(u) = \alpha_{V_0}(u')$, $\alpha_{V_0}(v) = \alpha_{V_0}(v')$.

- This theorem implies that the pseudovariety $V_{1/2}$ is $\omega$-redicible.
- Proven by the induction on the height of a factorization tree of word $u$ for $\alpha_{V_0}$. 
FACTORIZATION TREE of word \( u \in A^* \) for \( \alpha_{SI} \)
FACTORIZATION TREE

of word $u \in A^*$ for $\alpha_{SI}$

- labeled rooted tree
FACTORIZATION TREE

of word \( u \in A^* \) for \( \alpha_{SI} \)

- labeled rooted tree
- root labeled by \( u \)

\[ \text{height of this factorization tree} = 4 \]
FACTORIZATION TREE

of word $u \in A^*$ for $\alpha_{SI}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$

of node $v \in A^*$:

$v = v_1 \ldots v_n$
FACTORIZATION TREE
of word $u \in A^*$ for $\alpha_{SI}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$
  of node $v \in A^*$:
  $$v = v_1 \ldots v_n$$
- $n > 2 \implies \forall i: \alpha_{SI}(v_i) = e,$
  $$e \cdot e = e$$
FACTORIZATION TREE
of word \( u \in A^* \) for \( \alpha_{SI} \)

- labeled rooted tree
- root labeled by \( u \)
- descendants \( v_1, \ldots, v_n \in A^* \)
of node \( v \in A^* \):
  \[ v = v_1 \ldots v_n \]
- \( n > 2 \Rightarrow \forall i: \alpha_{SI}(v_i) = e, \)
  \( e \cdot e = e \)
- leaves labeled by letters \( a, b \)
FACTORIZATION TREE
of word $u \in A^*$ for $\alpha_{\tilde{S}L}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$ of node $v \in A^*$:
  $v = v_1 \ldots v_n$
- $n > 2 \Rightarrow \forall i: \alpha_{\tilde{S}L}(v_i) = e,
  e \cdot e = e$
- leaves labeled by letters $a, b$

monoid $M_{\alpha_{\tilde{S}L}}$

```
  1  a  b
  a  a  b
  b  a  a
```

height of this factorization tree = 4
FACTORIZATION TREE of word $u \in \mathbb{A}^*$ for $\alpha_{\mathbb{A}^*}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in \mathbb{A}^*$ of node $v \in \mathbb{A}^*$:
  $v = v_1 \ldots v_n$
- $n > 2 \Rightarrow \forall i: \alpha_{\mathbb{A}^*}(v_i) = e$,
  $e \cdot e = e$
- leaves labeled by letters $a, b$

monoid $M_{\alpha_{\mathbb{A}^*}}$

$2^{k+1}$
FACTORIZATION TREE of word $u \in A^*$ for $\alpha_{Sl}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$ of node $v \in A^*$:
  $v = v_1 \ldots v_n$
- $n > 2 \Rightarrow \forall i: \alpha_{Sl}(v_i) = e$, $e \cdot e = e$
- leaves labeled by letters $a, b$

monoid $M_{\alpha_{Sl}}$
FACTORIZATION TREE  
of word \( u \in A^* \) for \( \alpha_{\text{Sl}} \)

- labeled rooted tree
- root labeled by \( u \)
- descendants \( v_1, \ldots, v_n \in A^* \) of node \( v \in A^* \):
  \[ v = v_1 \ldots v_n \]
- \( n > 2 \Rightarrow \forall i: \alpha_{\text{Sl}}(v_i) = e, \)
  \[ e \cdot e = e \]
- leaves labeled by letters \( a, b \)

monoid \( M_{\alpha_{\text{Sl}}} \)
FACTORIZATION TREE
of word $u \in A^*$ for $\alpha_{SI}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$ of node $v \in A^*$:
  - $v = v_1 \ldots v_n$
- $n > 2 \Rightarrow \forall i: \alpha_{SI}(v_i) = e$
- leaves labeled by letters $a, b$

monoid $M_{\alpha_{SI}}$

height of this factorization tree = 4
FACTORIZATION TREE
of word $u \in A^*$ for $\alpha_{Sl}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$ of node $v \in A^*$:
  
  \[ v = v_1 \ldots v_n \]

- $n > 2 \Rightarrow \forall i: \alpha_{Sl}(v_i) = e,$
  
  \[ e \cdot e = e \]

- leaves labeled by letters $a, b$

monoid $M_{\alpha_{Sl}}$
FACTORIZATION TREE
of word $u \in A^*$ for $\alpha_{S_1}$

- labeled rooted tree
- root labeled by $u$
- descendants $v_1, \ldots, v_n \in A^*$ of node $v \in A^*$:
  $v = v_1 \ldots v_n$
- $n > 2 \Rightarrow \forall i: \alpha_{S_1}(v_i) = e,$
  $e \cdot e = e$
- leaves labeled by letters $a, b$

monoid $M_{\alpha_{S_1}}$

height of this factorization tree = 4
Theorem (Simon, 1990, Kufleitner, 2008)

Let $A$ be a finite alphabet, $M$ a finite monoid, $\alpha : A^* \rightarrow M$ a homomorphism. Then, for every word $u \in A^*$, there exists a factorization tree of $u$ for $\alpha$ of height at most $3 \cdot |M|$. 
Useful Statements

**Theorem (Simon, 1990, Kufleitner, 2008)**

Let \( A \) be a finite alphabet, \( M \) a finite monoid, \( \alpha: A^* \rightarrow M \) a homomorphism. Then, for every word \( u \in A^* \), there exists a factorization tree of \( u \) for \( \alpha \) of height at most \( 3 \cdot |M| \).

- This theorem allows us to use the induction on the height of a factorization tree.
Theorem (Simon, 1990, Kufleitner, 2008)

Let $A$ be a finite alphabet, $M$ a finite monoid, $\alpha : A^* \to M$ a homomorphism. Then, for every word $u \in A^*$, there exists a factorization tree of $u$ for $\alpha$ of height at most $3 \cdot |M|$.

- This theorem allows us to use the induction on the height of a factorization tree.

The following lemma will be useful.

Lemma (J. V.)

Let $u_1, \ldots, u_k, v \in A^*$ ($k \in \mathbb{N}$), $n \geq k - 1$. Let $V^n_{1/2} \models u_1 \ldots u_k \leq v$. Then there exist $v_1, \ldots, v_k \in A^*$ such that

- $v = v_1 \ldots v_k$,
- $V^{n-(k-1)}_{1/2} \models u_i \leq v_i$ for $i = 1, \ldots, k$. 

Omega-Reducibility
\[ V_{1/2}^N \models a\ldots a b \leq a\ldots a b \]
$V_{1/2}^N \models a \ldots a b \leq a \ldots a b$

$2^{N+1}$

$2^{N+2}$
$V_{1/2}^N \models a\ldots a\, b \leq a\ldots a\, b$

$2^{N+1}$

$2^{N+2}$
$$V_{1/2}^N \models a \ldots a \, b \leq a \ldots a \, b$$
\[ V_{1/2}^N \models a \ldots a \ b \leq a \ldots a \ b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]
\[ V_{1/2}^N \models a \ldots a \ b \leq a \ldots a \ b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]
$V_{1/2}^N \models a\ldots a\ b \leq a\ldots a\ b \rightarrow V_{1/2}^{N-1} \models a\ldots a \leq a\ldots a,\ ab \leq ab$
\( V_{1/2}^N \models a\ldots a b \leq a\ldots a b \rightarrow V_{1/2}^{N-1} \models a\ldots a \leq a\ldots a, \ ab \leq ab \)
\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ \rightarrow V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]
\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ \rightarrow V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

\[
\begin{array}{c}
\text{a} \ldots \text{a} \\
\text{a} \ldots \text{a} \\
\text{a} \\
\text{a}
\end{array}
\]
\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ 2^{N+1} \quad 2^{N+2} \quad 2^N \quad 2^{N+1} \]

\[ V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

\[ 2^{N-4} \quad 2^{N-3} \]
\[ V^{N}_{1/2} \models a \ldots a b \leq a \ldots a b \rightarrow V^{N-1}_{1/2} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ 2^{N+1} \quad 2^{N+2} \quad 2^{N} \quad 2^{N+1} \]

\[ a \ldots a \]

\[ 2^N \]

\[ a \]

\[ aa \]

\[ a \]

\[ a \]

\[ \ldots \]

\[ ab \]

\[ a \]

\[ \]

\[ b \]

\[ a \]

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$V_{1/2}^N \models \underbrace{a \ldots a}_{2^N} b \leq \underbrace{a \ldots a}_{2^N + 2} b \rightarrow V_{1/2}^{N-1} \models \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N + 1}, \ ab \leq ab$

$\rightarrow V_{1/2}^{N-3} \models \underbrace{a a}_{2^N-4} \leq \underbrace{a a}_{2^N-3}, \ \underbrace{a \ldots a}_{2^N} \leq \underbrace{a \ldots a}_{2^N-1}, \ a a \leq a a, \ a \leq a, \ b \leq b$
\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \]
\[
\begin{array}{c}
2^{N+1} \\
2^{N+2}
\end{array}
\]

\[ \rightarrow \quad V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]
\[
\begin{array}{c}
2^N \\
2^{N+1}
\end{array}
\]

\[ \rightarrow \quad V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]
\[
\begin{array}{c}
2^{N-4} \\
2^{N-3}
\end{array}
\]

\[ \rightarrow \quad V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]
\[V_{1/2}^N \models a \ldots a b \leq a \ldots a b\]
\[2^{N+1} \quad 2^{N+2}\]

\[\rightarrow \quad V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab\]
\[2^N \quad 2^{N+1}\]

\[\rightarrow \quad V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b\]
\[2^{N-4} \quad 2^{N-3}\]

\[\rightarrow \quad V_{1/2}^{N-4} \models a \leq a, \ a \leq a\]

• \[V_{1/2} \models a \leq a\]
$V_{1/2}^N \models a \ldots a \ b \leq a \ldots a \ b$

\[ 2^N + 1 \quad 2^N + 2 \]

\[ \rightarrow \ V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ 2^N \quad 2^N + 1 \]

\[ \rightarrow \ V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

\[ 2^{N-4} \quad 2^{N-3} \]

\[ \bullet \ V_{1/2} \models aa \leq aa \]

\[ \rightarrow \ V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]

\[ \bullet \ V_{1/2} \models a \leq a \]
\( V^{N-1}_{1/2} \models a \ldots a \leq a \ldots a, \ ab \leq ab \)

\( V^{N-3}_{1/2} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \)

- \( V_{1/2} \models aa \leq aa \)
- \( V_{1/2} \models a \leq a, \ b \leq b \)

\( V^{N-4}_{1/2} \models a \leq a, \ a \leq a \)

- \( V_{1/2} \models a \leq a \)
\( V_{1/2}^N \models a \ldots a \, b \leq a \ldots a \, b \)

\[ \begin{array}{c}
2^N + 1 \\
2^N + 2
\end{array} \]

\[ \vdash V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \, ab \leq ab \]

\[ \begin{array}{c}
2^N \\
2^N + 1
\end{array} \]

\[ V_{1/2} : = \left[ u^{\omega + 1} \leq u^\omega \, vu^\omega \mid V_0 \models u = v \right] \]

\[ \vdash V_{1/2}^{N-3} \models aa \leq aa, \, a \ldots a \leq a \ldots a, \, aa \leq aa, \quad a \leq a, \, b \leq b \]

\[ \begin{array}{c}
2^{N-4} \\
2^{N-3}
\end{array} \]

- \( V_{1/2} \models aa \leq aa \)
- \( V_{1/2} \models a \leq a, \, b \leq b \)

\[ \vdash V_{1/2}^{N-4} \models a \leq a, \, a \leq a \]

- \( V_{1/2} \models a \leq a \)
\[ V_{1/2} \models a \ldots a b \leq a \ldots a b \]

\[ 2^{N+1} \quad 2^{N+2} \]

→ \[ V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ 2^N \quad 2^{N+1} \]

\[ V_{1/2} := \left[ u^{\omega+1} \leq u^\omega vu^\omega \mid V_0 \models u = v \right] \]

- \[ V_0 = \text{SI} \models aa = a \]

→ \[ V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

- \[ V_{1/2} \models aa \leq aa \]

- \[ V_{1/2} \models a \leq a, \ b \leq b \]

→ \[ V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]

- \[ V_{1/2} \models a \leq a \]
\[ V_{1/2}^{N} \models a \ldots a b \leq a \ldots a b \]
\[ 2^{N+1} \quad 2^{N+2} \]

\[ \rightarrow V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]
\[ 2^{N} \quad 2^{N+1} \]

\[ V_{1/2} := [u^{\omega+1} \leq u^{\omega} vu^{\omega} \mid V_{0} \models u = v] \]

\[ \bullet \ V_{0} = Sl \models aa = a \rightarrow V_{1/2} \models (aa)^{\omega+1} \leq (aa)^{\omega} a(aa)^{\omega} \]

\[ \rightarrow V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

\[ \bullet \ V_{1/2} \models aa \leq aa \]

\[ \bullet \ V_{1/2} \models a \leq a, \ b \leq b \]

\[ \rightarrow V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]

\[ \bullet \ V_{1/2} \models a \leq a \]
\[ V_{1/2}^N \models a \ldots a b \leq a \ldots a b \]

\[ V_{1/2}^{N-1} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]

\[ V_{1/2} := [u^{\omega+1} \leq u^\omega v u^\omega \mid V_0 \models u = v] \]

- \( V_0 = S1 \models aa = a \rightarrow V_{1/2} \models (aa)^{\omega+1} \leq (aa)^\omega a(aa)^\omega \)
- \( V_{1/2} \models ab \leq ab \)

\[ V_{1/2}^{N-3} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

- \( V_{1/2} \models aa \leq aa \)
- \( V_{1/2} \models a \leq a, \ b \leq b \)

\[ V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]

- \( V_{1/2} \models a \leq a \)
\[ V^{N}_{1/2} \models a \ldots a b \leq a \ldots a b \]
\[
\begin{array}{c}
2^N + 1 \\
2^N + 2
\end{array}
\]

- \( V^{N}_{1/2} \models (aa)^{\omega+1} ab \leq (aa)^{\omega} a(aa)^{\omega} ab \)

\[ \rightarrow V^{N-1}_{1/2} \models a \ldots a \leq a \ldots a, \ ab \leq ab \]
\[
\begin{array}{c}
2^N \\
2^N + 1
\end{array}
\]

\[ V^{N-1}_{1/2} \models [u^{\omega+1} \leq u^{\omega} vu^{\omega} \mid V_0 \models u = v] \]

- \( V_0 = S1 \models aa = a \rightarrow V^{N}_{1/2} \models (aa)^{\omega+1} \leq (aa)^{\omega} a(aa)^{\omega} \)
- \( V^{N}_{1/2} \models ab \leq ab \)

\[ \rightarrow V^{N-3}_{1/2} \models aa \leq aa, \ a \ldots a \leq a \ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]
\[
\begin{array}{c}
2^N - 4 \\
2^N - 3
\end{array}
\]

- \( V^{N-3}_{1/2} \models aa \leq aa \)
- \( V^{N-3}_{1/2} \models aa \leq aa \)
- \( V^{N-3}_{1/2} \models a \leq a, \ b \leq b \)

\[ \rightarrow V^{N-4}_{1/2} \models a \leq a, \ a \leq a \]

- \( V^{N-4}_{1/2} \models a \leq a \)
- \( V^{N-4}_{1/2} \models a \leq a \)
- \( V^{N-4}_{1/2} \models a \leq a \)
\[ V_{1/2}^{N} \models a\ldots a b \leq a\ldots a b \]

\[
\begin{array}{c}
2^{N+1} & 2^{N+2} \\
\end{array}
\]

- \[ V_{1/2} \models (aa)^{\omega+1} ab \leq (aa)^{\omega} a(aa)^{\omega} ab \]

\[ \rightarrow V_{1/2}^{N-1} \models a\ldots a \leq a\ldots a, \ ab \leq ab \]

\[
\begin{array}{c}
2^{N} & 2^{N+1} \\
\end{array}
\]

\[ V_{1/2} := [u^{\omega+1} \leq u^{\omega} v u^{\omega} \mid V_0 \models u = v] \]

- \[ V_0 = Sl \models aa = a \rightarrow V_{1/2} \models (aa)^{\omega+1} \leq (aa)^{\omega} a(aa)^{\omega} \]
- \[ V_{1/2} \models ab \leq ab \]

\[ \rightarrow V_{1/2}^{N-3} \models aa \leq aa, \ a\ldots a \leq a\ldots a, \ aa \leq aa, \ a \leq a, \ b \leq b \]

\[
\begin{array}{c}
2^{N-4} & 2^{N-3} \\
\end{array}
\]

- \[ V_{1/2} \models aa \leq aa \]
- \[ V_{1/2} \models a \leq a, \ b \leq b \]

\[ \rightarrow V_{1/2}^{N-4} \models a \leq a, \ a \leq a \]

- \[ V_{1/2} \models a \leq a \]
ω-Reducibility of $V_{3/2}$ and $V_{5/2}$
\( \Omega \)-Reducibility of \( V_{3/2} \) and \( V_{5/2} \)

\( V_{1/2} \)

\( \omega \)-reducibility

of pairs of words

\[
\begin{array}{c}
v \\
1/2 \\
u
\end{array}
\]
\( \omega \)-Reducibility of \( V_{3/2} \) and \( V_{5/2} \)

<table>
<thead>
<tr>
<th>( V_{1/2} )</th>
<th>( V_{3/2} )</th>
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<tbody>
<tr>
<td>( \omega )-reducibility of pairs of words</td>
<td>( \omega )-reducibility of chains of words</td>
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</tbody>
</table>

\[
\begin{array}{c|c}
\frac{v}{1/2} & \frac{w}{3/2} \\
\frac{u}{1/2} & \frac{v}{1/2} \\
\end{array}
\]
$\omega$-Reducibility of $V_{3/2}$ and $V_{5/2}$

$V_{1/2}$

$\omega$-reducibility of pairs of words

$V_{3/2}$

$\omega$-reducibility of chains of words

$V_{5/2}$

$\omega$-reducibility of (finite) ordered sets of words

\[
\begin{array}{ccc}
V_{1/2} & V_{3/2} & V_{5/2} \\
\omega\text{-reducibility of } & \omega\text{-reducibility of } & \omega\text{-reducibility of (finite) ordered sets of words} \\
of \textit{pairs of words} & of \textit{chains of words} & of \textit{sets of words} \\
\hline
v
\begin{array}{c}
1/2 \\
u
\end{array}
& w
\begin{array}{c}
3/2 \\
v
\end{array}
\begin{array}{c}
1/2 \\
u
\end{array}
& \vdots
\begin{array}{c}
1/2 \\
5/2
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1/2 \\
n_5
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\begin{array}{c}
3/2 \\
n_7
\end{array}
\begin{array}{c}
1/2 \\
n_9
\end{array}
\end{array}
\]
Thank you for your attention.