Categories of orthosets and maps possessing an adjoint

Thomas Vetterlein

Institute for Mathematical Methods in Medicine und Data-Based Modelling
Johannes Kepler University (Linz, Austria)

Joint work with Jan Paseka and Bert Lindenhovius

Summer School on General Algebra and Ordered Sets,
Stará Lesná, 3 September 2023

This work is supported by the Austrian Science Fund (FWF): project I 4579-N.

Orthogonality – the omnipresent notion

Motivation

In the Hilbert space model of quantum physics, the one-dimensional subspaces correspond to the pure states.

The collection of one-dimensional subspaces comes along with the relation $\perp$ of orthogonality.

Given nothing but $\perp$, we are able to reconstruct the Hilbert space model. So we may ask: what is this binary relation like?

Foulis's suggestion

The minimalist program: develop the theory of sets endowed with a symmetric and irreflexive binary relation.

J. R. Dacey, "Orthomodular spaces", Ph.D. Thesis, 1968.

Orthosets

Definition

An orthoset (with 0) is a set X together with a binary relation $\perp$ and a constant 0 such that:

- $\perp$ is symmetric,
- $e \perp e$ if and only if $e = 0$,
- $0 \perp e$ for any $e \in X$.

The guiding examples

Let H be a Hilbert space.

- H, together with $\perp$ and the zero vector 0, is an orthoset.
- Let $P(H) = \{\langle x \rangle : x \in H\}$ and let $\langle x \rangle \perp \langle y \rangle$ if $x \perp y$. Then $P(H)$, together with $\perp$ and the zero subspace $\{0\}$, is an orthoset.

Orthosets and ortholattices

For a subset A of an orthoset X, let

$$A^{\perp} = \{x \in X : x \perp a \text{ for all } a \in A\}.$$

Then $\mathscr{P}(X) \to \mathscr{P}(X),\ A \mapsto A^{\perp\perp}$ is a closure operator.
The sets closed w.r.t. ${}^{\perp\perp}$ are called orthoclosed.

An ortholattice is a lattice equipped with an order-reversing involution ${}^{\perp}$ sending each a to a complement of a.

Lemma

The set $\mathsf{C}(X)$ of orthoclosed subsets of X is a complete ortholattice.

Irredundancy of orthosets

Let X be an orthoset. Two elements $e, f \in X$ are called equivalent, and we write $e \sim f$, if

$$x \perp e \text{ iff } x \perp f \quad \text{for any } x \in X.$$

In other words, $e \sim f$ iff $\{e\}^\perp = \{f\}^\perp$.

Definition

We call an orthoset X irredundant if $\sim$ is the equality.

Let $\tilde{X} = X/{\sim}$, equipped with the induced orthogonality relation and with $0/{\sim} = \{0\}$.

Lemma

For any orthoset, $\tilde{X}$ is an irredundant orthoset.
Moreover, $\mathsf{C}(\tilde{X})$ and $\mathsf{C}(X)$ are isomorphic.

Atomisticity of orthosets

Definition

We call an orthoset X atomistic if, for any $e, f \neq 0$,

$$\{e\}^\perp \subseteq \{f\}^\perp \text{ implies } \{e\}^\perp = \{f\}^\perp.$$

If X is atomistic, so is $\mathsf{C}(X)$.

Lemma

We have a one-to-one correspondence:

$$\begin{array}{c} \text{atomistic, irredundant} \\ \text{orthosets} \end{array} \quad \underset{\text{atom space}}{\overset{\mathsf{C}}{\rightleftarrows}} \quad \begin{array}{c} \text{complete atomistic} \\ \text{ortholattices} \end{array}$$

Hermitian spaces

Definition

A $\star$-sfield is a division ring together with an involutive antiautomorphism.

Let H be a linear space over a $\star$-sfield K.
$(\cdot,\cdot) : H \times H \to K$ is called an anisotropic Hermitian form if:

$$
\begin{aligned}
(\alpha x + \beta y, z) &= \alpha\,(x, z) + \beta\,(y, z)\,, \\
(z, \alpha x + \beta y) &= (z, x)\,\alpha^\star + (z, y)\,\beta^\star, \\
(x, y) &= (y, x)^\star\,, \\
(x, x) = 0 &\Rightarrow\; x = 0.
\end{aligned}
$$

Then H equipped with $(\cdot,\cdot)$ is called a Hermitian space.

Hermitian spaces and ortholattices

For any subset A of a Hermitian space H,

$$A^{\perp} = \{u \in H : u \perp a \text{ for all } a \in A\}$$

is a linear subspace called orthoclosed.

Let $\mathsf{C}(H)$ be the ortholattice of orthoclosed subspaces of H.

Theorem (cf. Maeda-Maeda)

For a Hermitian space H, $\mathsf{C}(H)$ is a complete atomistic ortholattice that is moreover irreducible and has the covering property.

Conversely, let L be a complete atomistic ortholattice that is moreover irreducible and has the covering property. Assume that L has length $\geqslant 4$. Then L is isomorphic to $\mathsf{C}(H)$ for some Hermitian space H.

Theorem (J. Paseka, Th. V.)

Let L be a complete ortholattice such that:

- L is atomistic,
- for any distinct atoms p and q, there is an atom $r \perp p$ such that $p \vee q = p \vee r$,
- for any distinct atoms p and q, there is a third atom $r \leqslant p \vee q$,
- L has length $\geqslant 4$.

Then L is isomorphic to $\mathsf{C}(H)$ for some Hermitian space H.

J. Paseka, Th. Vetterlein, Linear orthosets and orthogeometries, *Int. J. Theor. Phys.* 2023.

Linear orthosets

Definition

An orthoset $(X, \perp)$ is called linear if,
for any distinct $e, f \neq 0$, there is a $g \in X$ such that
exactly one of f and g is orthogonal to e and $\{e, f\}^\perp = \{e, g\}^\perp$.

Theorem (J. Paseka, Th. V.)

For any Hermitian space H, the orthoset $(P(H), \perp)$ is linear.
Conversely, for any linear orthoset X of rank $\geqslant 4$, there is a Hermitian space H such that X is isomorphic to $P(H)$.

Consequently, we have a one-to-one correspondence:

$$\begin{array}{ccc} \text{linear orthosets} & \xrightarrow{} & \text{Hermitian spaces} \\ (\text{rank} \geqslant 4) & \xleftarrow{P} & (\text{dimension} \geqslant 4) \end{array}$$

Categories

Problem

Can we extend the correspondence between orthosets and Hermitian spaces to include structure-preserving maps?

Approaches in this context:

- FAURE, FRÖHLICHER: category of orthogeometries with partial maps preserving collinearity and orthogonality;
- J. PASEKA, TH. V.: category of normal orthosets with maps preserving orthogonality and Boolean substructures;
- C. HEUNEN, A. KORNELL, ET AL.: category of Hilbert spaces with bounded linear maps.

C.-A. FAURE, A. FRÖLICHER, “Modern projective geometry”, Kluwer 2000.
J. PASEKA, TH. VETTERLEIN, Categories of orthogonality spaces, *J. Pure Appl. Algebra* 2021.
CH. HEUNEN, A. KORNELL, Axioms for the category of Hilbert spaces, *Proc. Natl. Acad. Sci. USA* 2022.

Our choice of structure-preserving maps

Definition

Let $f\colon X \to Y$ be a map between orthosets.

Then $g\colon Y \to X$ is called an adjoint of f if,
for any $x \in X$ and $y \in Y$

$$f(x) \perp y \quad \text{if and only if} \quad x \perp g(y).$$

A map possessing an adjoint is called adjointable.

Linear maps induce adjointable maps

Each linear map $S\colon H_1 \to H_2$ between Hermitian spaces induces the map

$$P(S)\colon P(H_1) \to P(H_2),\ \langle u \rangle \mapsto \langle S(u) \rangle$$

between the associated orthosets.

Proposition

Let $S\colon H_1 \to H_2$ be a linear map between Hermitian spaces. Assume that

- either H_1, H_2 are finite-dimensional
- or H_1, H_2 are Hilbert spaces and S is bounded.

Then $P(S)$ is adjointable,
and $P(S^\star)$ is the unique adjoint of $P(S)$.

(Here, $S^\star$ is the adjoint of S in the usual sense).

Towards the converse

Adjointable maps preserve linear dependence:

Proposition

Let $f\colon X \to Y$ be adjointable. Then, for any $x_1, x_2 \in X$,

$$x \in \{x_1, x_2\}^{\perp\perp} \text{ implies } f(x) \in \{f(x_1), f(x_2)\}^{\perp\perp}.$$

Adjointable maps do not in general preserve orthogonality. But:

Proposition

Let $f\colon X \to Y$ be an adjointable bijection. Then f is an isomorphism of orthosets if and only if f^{-1} is an adjoint of f.

Adjointable maps are induced by semilinear maps

Let H_1 and H_2 be Hermitian spaces over a $\star$-sfield F.

$S\colon H_1 \to H_2$ is called semilinear if

- $S(u+v) = S(u) + S(v)$ for $u, v \in H_1$,
- $S(\alpha u) = \alpha^\sigma S(u)$ for $u \in H_1,\ \alpha \in F$, where ${}^\sigma\colon F \to F$ is a homomorphism (of sfields).

From Faure and Frölicher's version of the Fundamental Theorem of Projective Geometry, we infer:

Theorem (J. Paseka, Th. V.)

Let $f\colon P(H_1) \to P(H_2)$ be adjointable and let $\operatorname{im} f$ not be contained in a subspace of rank $\leqslant 2$. Then f is induced by a semilinear map $S\colon H_1 \to H_2$.

Adjoints of inclusion maps

Definition

Let A be a subspace of an orthoset X. A map $\sigma\colon X \to A$ such that, for any $x \in X$,

$$\{a \in A\colon a \perp x\} \;=\; \{a \in A\colon a \perp \sigma(x)\}$$

is called a Sasaki map.

Proposition

Let A be a subspace of an orthoset X.
Then $\sigma\colon X \to A$ is a Sasaki map if and only if σ is an adjoint of the inclusion map $\iota\colon A \to X$.

B. Lindenhovius, Th. Vetterlein, A characterisation of orthomodular spaces by Sasaki maps, *Int. J. Theor. Phys.* 2023.

Adjointability and the linearity of orthosets

Theorem (B. Lindenhovius, Th. V.)

Let X be an irreducible atomistic orthoset of rank $\geqslant 4$. Assume that, for any subspace A of X, the inclusion map $\iota\colon A \to X$ has an adjoint.

Then X is linear. In fact, there is an orthomodular space H such that X is isomorphic to $P(H)$.

Adjoint maps and not irredundant orthosets

Adjointable maps preserve equivalence:

Lemma

Let $f\colon X \to Y$ be an adjointable map between orthosets.

Then $x_1 \sim x_2$ implies $f(x_1) \sim f(x_2)$. Hence we can define

$$\tilde{f}\colon \tilde{X} \to \tilde{Y},\ \tilde{x} \mapsto \widetilde{f(x)}.$$

Moreover, if g is an adjoint of f, then $\tilde{g}$ is the unique adjoint of $\tilde{f}$ and vice versa.

Dagger categories

A dagger category is a category equipped with an operation * that assigns to each morphism $f\colon X \to Y$ a morphism $f^\star\colon Y \to X$ such that $f^{\star\star} = f$, $(g \circ f)^\star = f^\star \circ g^\star$, and $\mathrm{id}^\star = \mathrm{id}$.

Basic examples:

- $\mathscr{PfHer}_F$ for a $\star$-sfield F:
  - orthosets $P(H)$, where H is a finite-dimensional Hermitian space over F,
  - maps $P(S)\colon P(H_1) \to P(H_2)$ for a linear $S\colon H_1 \to H_2$,
  - dagger $P(S)^\star = P(S^\star)$.
- $\mathscr{PHil}_F$ for $F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$:
  - orthosets $P(H)$, where H is a Hilbert space over F,
  - maps $P(S)\colon P(H_1) \to P(H_2)$ for a bounded linear S,
  - dagger $P(S)^\star = P(S^\star)$.
- $\mathscr{fBool}$:
  - finite Boolean algebras;
  - bijections between subsets of their atom spaces,
  - dagger is inversion.

Some definitions for dagger categories

- In a dagger category, 0 is a zero object if there is, for any A, a unique morphism $0 \to A$ (and hence also $A \to 0$). We write $0_{A,B}$ for $A \to 0 \to B$.
- A dagger biproduct of A and B is a coproduct

$$A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$$

  such that $\iota_A{}^\star \circ \iota_A = \mathrm{id}_A$, $\iota_B{}^\star \circ \iota_B = \mathrm{id}_B$, and $\iota_B{}^\star \circ \iota_A = 0_{A,B}$.
- f is a dagger isomorphism if $f^\star \circ f = \mathrm{id}$ and $f \circ f^\star = \mathrm{id}$.
- A is called indecomposable if A is not the dagger biproduct of a pair of non-zero objects.

Ch. Heunen, M. Karvonen, Limits in dagger categories, *Theory Appl. Categ.* 2019.

Theorem (J. Paseka, Th. V.)

Let $\mathcal{C}$ be a dagger category whose objects are orthosets of finite rank, whose morphisms are maps between them, and whose dagger assigns to each morphism an adjoint.

(1) $\mathcal{C}$ has finite dagger biproducts.

(2) Let A be a subspace of X. Then A and $A^\perp$ belong to $\mathcal{C}$ and $A \xrightarrow{\iota_A} X \xleftarrow{\iota_{A^\perp}} A^\perp$ is the dagger biproduct of A and $A^\perp$, where ι_A and $\iota_{A^\perp}$ are the inclusion maps.

(3) Any singleton orthoset in $\mathcal{C}$ is indecomposable.

(4) Any two singleton orthosets in $\mathcal{C}$ are dagger isomorphic.

Let then $\tilde{\mathcal{C}}$ be the category whose objects are $\tilde{X}$ for each orthoset X in $\mathcal{C}$, whose morphisms are $\tilde{f}$ for each morphism f in $\mathcal{C}$, and whose dagger assigns $\widetilde{f^\star}$ to $\tilde{f}$. Then $\tilde{\mathcal{C}}$ is equivalent either to $\mathscr{fBool}$, or to $\mathscr{PfHer}_F$ for some $\star$-sfield F.

Conclusion

- Hilbert spaces – and more generally Hermitian spaces – are characterised by their associated orthosets, so-called linear orthosets.
- Maps $f\colon X \to Y$ and $g\colon Y \to X$ between orthosets form an adjoint pair provided that

$$f(x) \perp y \text{ iff } x \perp g(y), \quad x \in X,\ y \in Y.$$

  Adjointable maps between linear orthosets correspond closely to linear maps of Hermitian spaces.
- The requirement that inclusion maps are adjointable even replaces the linearity condition for orthosets.
- As a consequence, we may describe the category of (projective) finite-dimensional Hermitian spaces by means of orthosets and adjointable maps.
- The characterisation uses a biproduct but no tensor product.

Outlook

Further issues:

- Describe $f\mathcal{H}er_F$ without the formation of a quotient.

  To this end, discover the "internal structure" of the equivalence classes of the orthosets in $\mathcal{C}$ subject to the above conditions.

- Describe the category of Hilbert spaces in a similar manner.

- What happens when dropping the conditions involving singleton orthosets?