Abstract commutator theory in concrete classes

David Stanovský

Charles University, Prague, Czech Republic

Summer School on Algebras and Ordered Sets, September 2023

Part I:
Solvability and nilpotence: the beginnings and motivation

Part II:
Abelianness and Centrality: examples and module representation

Part III:
The commutator in specific varieties

The commutator

Centralizing relation for congruences α, β, δ of an algebra A:

$C(\alpha, \beta; \delta)$ iff

for every term $t(x_1, \dots, x_m, y_1, \dots, y_n)$ and every $a \overset{\alpha}{\equiv} b$, $u_i \overset{\beta}{\equiv} v_i$

$$t(a, \bar{u}) \overset{\delta}{\equiv} t(a, \bar{v}) \;\Rightarrow\; t(b, \bar{u}) \overset{\delta}{\equiv} t(b, \bar{v})$$

The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

The commutator

Centralizing relation for congruences α, β, δ of an algebra A:

$C(\alpha, \beta; \delta)$ iff

for every term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ and every $a \stackrel{\alpha}{\equiv} b$, $u_i \stackrel{\beta}{\equiv} v_i$

$$t(a, \bar{u}) \stackrel{\delta}{\equiv} t(a, \bar{v}) \;\Rightarrow\; t(b, \bar{u}) \stackrel{\delta}{\equiv} t(b, \bar{v})$$

The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

A congruence α of A is called

- *abelian* if $C(\alpha, \alpha; 0_A)$, i.e., if $[\alpha, \alpha] = 0_A$.
- *central* if $C(\alpha, 1_A; 0_A)$, i.e., if $[\alpha, 1_A] = 0_A$.

An algebra A is called *solvable*, resp. *nilpotent*, if there are congruences α_i such that

$$0_A = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_k = 1_A$$

and α_{i+1}/α_i is an *abelian*, resp. *central* congruence of A/α_i, for all i.

QUANDLES

Quandles

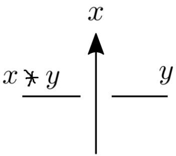
A binary algebra $(Q, *, \backslash)$ is called a *quandle* if

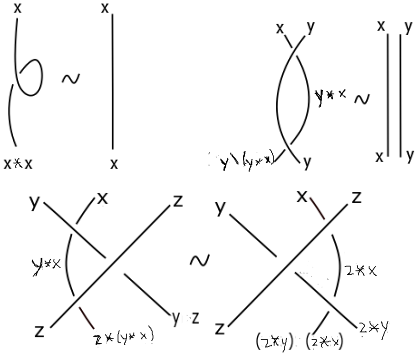
(I) $x * x = x$

(II) for all x, y there is a unique u such that $x * u = y$

(III) $x * (y * z) = (x * y) * (x * z)$

Motivation: knot theory





Quandles

A binary algebra $(Q, *, \backslash)$ is called a *quandle* if

(I) $x * x = x$

(II) for all x, y there is a unique u such that $x * u = y$

(III) $x * (y * z) = (x * y) * (x * z)$

Examples:

- group conjugation: $(G, *)$, $x * y = xyx^{-1}$
- affine forms over abelian groups: $(G, *)$, $x * y = (1 - f)(x) + f(y)$
- ...
- latin quandles = (left) self-distributive quasigroups [since 1923!]

Quandles and associated groups

A binary algebra $(Q, *, \backslash)$ is called a *quandle* if

- $x * x = x$
- all left translations $L_x(y) = x * y$ are automorphisms.

Left multiplication group, displacement group:

$$\mathrm{LMlt}(Q) = \langle L_x : x \in Q \rangle \leq \mathrm{Aut}(Q)$$
$$\mathrm{Dis}(Q) = \langle L_x L_y^{-1} : x, y \in Q \rangle \leq \mathrm{LMlt}(Q)$$

Q is *connected* if $\mathrm{LMlt}(Q)$ acts transitively.

Commutator theory for quandles

[Bonatto, S. 2021]

Let $N(Q) = \{N \leq \mathrm{Dis}(Q) : N \text{ is normal in } \mathrm{LMlt}(Q)\}$

There is a Galois correspondence

$$Con(Q) \longleftrightarrow N(Q)$$

$$\alpha \to \mathrm{Dis}_\alpha = \langle L_x L_y^{-1} : x \, \alpha \, y \rangle$$

$$\alpha_N = \{(x, y) : L_x L_y^{-1} \in N\} \leftarrow N$$

Commutator theory for quandles

[Bonatto, S. 2021]

Let $N(Q) = \{N \leq \mathrm{Dis}(Q) : N \text{ is normal in } \mathrm{LMlt}(Q)\}$

There is a Galois correspondence

$$Con(Q) \longleftrightarrow N(Q)$$
$$\alpha \to \mathrm{Dis}_\alpha = \langle L_x L_y^{-1} : x \,\alpha\, y \rangle$$
$$\alpha_N = \{(x,y) : L_x L_y^{-1} \in N\} \leftarrow N$$

Proposition

TFAE for congruences α, β of a quandle Q:

1. α *centralizes* β *over* 0_Q, *i.e.,* $C(\alpha, \beta; 0_Q)$
2. $[\mathrm{Dis}_\alpha, \mathrm{Dis}_\beta] = 1$ *and* Dis_β *acts* α*-semiregularly on* Q

α-semiregularly means $g(a) = a \;\Rightarrow\; g(b) = b$ for every $b \overset{\alpha}{\equiv} a$

Abelian, nilpotent, and solvable quandles

[Jedlička, Pilitowska, S, Zamojska-Dzienio, 2018] [Bonatto, S, 2021]

quandle		$\mathrm{Dis}(Q)$
affine	$\Leftrightarrow$	abelian, semiregular, "balanced"
$\Downarrow$		$\Downarrow$
abelian	$\Leftrightarrow$	abelian, semiregular
$\Downarrow$		$\Downarrow$
nilpotent	$\Leftrightarrow$	nilpotent
$\Downarrow$		$\Downarrow$
solvable	$\Leftrightarrow$	solvable

Moreover, for finite connected faithful quandles:
nilpotent $\Rightarrow$ direct product of connected quandles of prime power size.

Application: enumeration of quandles

Theorem: [Alexander Stein, 2001]

If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Corollary: Finite latin quandles are solvable.

Application: enumeration of quandles

Theorem: [Alexander Stein, 2001]
If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Corollary: Finite latin quandles are solvable.

Theorem: [Sherman Stein, 1957]
There are no latin quandles of order $\equiv 2 \pmod 4$.

Proof:
1) Assume it is simple. Then
solvable $\Rightarrow$ abelian $\Rightarrow$ affine $\Rightarrow$ order p^k $\Rightarrow$ contradiction [Joyce, 1982]
2) Take the smallest counterexample, take a non-trivial congruence, either the factor or a block are smaller of this order.

Application: enumeration of quandles

Theorem: [Alexander Stein, 2001]
If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Corollary: Finite latin quandles are solvable.

Theorem: [Sherman Stein, 1957]
There are no latin quandles of order $\equiv 2 \pmod 4$.

Proof:
1) Assume it is simple. Then
solvable $\Rightarrow$ abelian $\Rightarrow$ affine $\Rightarrow$ order p^k $\Rightarrow$ contradiction [Joyce, 1982]
2) Take the smallest counterexample, take a non-trivial congruence, either the factor or a block are smaller of this order.

More complicated results: [Bianco, Bonatto, around 2020]
Classification of latin quandles of order pq, connected quandles of order p^3, ...

Application: coloring knots by latin quandles

Theorem: [Alexander Stein, 2001]
If Q is a finite latin quandle, then $\mathrm{LMlt}(Q)$ is solvable.

Corollary: Finite latin quandles are solvable.

Corollary: Knots with trivial Alexander polynomial are not colorable by any finite latin quandle.

Proof idea:
(1) Bae's theorem: a knot admits a non-trivial coloring by an affine quandle if and only if its Alexander polynomial is non-trivial.
(2) Lemma: If c is a non-trivial coloring of K by a quandle Q, and $Q = \langle Im(c) \rangle$, then K is colorable by every simple factor of Q.
$\rightsquigarrow$ If Q is finite latin, it is solvable, hence all simple factors are affine, and the two facts contradict.

LOOPS

Loops

$=$ "non-associative groups"

$= (Q, \cdot, /, \backslash, 1)$ such that 1 is a unit, $/, \backslash$ are division operations wrt $\cdot$

Translations:

$$L_x(y) = x \cdot y, \quad R_x(y) = y \cdot x$$

Multiplication group:

$$\mathrm{Mlt}(Q) = \langle L_x, R_x : x \in Q \rangle \leq S_Q$$

Inner mapping group:

$$\mathrm{Inn}(Q) = (\mathrm{Mlt}(Q))_1 = \langle L_{x,y}, R_{x,y}, T_x : x, y \in Q \rangle \leq \mathrm{Mlt}(Q)$$

where

$$L_{x,y}(z) = (xy)\backslash x(yz), \quad R_{x,y}(z) = (zy)x/(yx), \quad T_x(z) = x\backslash zx$$

"Naive" commutator theory

[Albert, Bruck, 1940s]

Easy fact: *congruences* correspond to *normal subloops*, i.e., subloops invariant with respect to action of $\mathrm{Inn}(Q)$.

The center:

$$Z(Q) = \mathrm{Fix}(\mathrm{Inn}(Q)) =$$

$$= \{a : \ ax = xa,\ a(xy) = (ax)y,\ x(ay) = (xa)y,\ x(ya) = (xy)a \text{ for all } x, y\}.$$

A loop Q is called *(classically) solvable*, resp. *nilpotent*, if there are normal subloops N_i such that

$$1 = N_0 \leq N_1 \leq \ldots \leq N_k = Q$$

and N_{i+1}/N_i is an *abelian group*, resp. *contained in the center* $Z(Q/N_i)$, for all i.

“The” commutator theory

[S, Vojtěchovský, 2014] [Barnes, 2023]

Main theorem:

$$[A, B] = \mathrm{Ng}(I_{u_1,u_2}(a)\backslash I_{v_1,v_2}(a) : \ I \in \{L, R, T\}, u_i\backslash v_i \in B, a \in A).$$

Consequently,

- $[N, Q] = 1$ iff $N \leq Z(Q)$,
- $[N, N] = 1$ iff $I_{u_1,u_2}|_N = I_{v_1,v_2}|_N$ for every $I \in \{L, R, T\}, u_i\backslash v_i \in B$.
- $[N, N] = 1$ iff $\varphi|_N \in \mathrm{Aut}(N)$ for every $\varphi \in \mathrm{Inn}(Q)$ and certain commutators/associators vanish.

Hence, comparing to the “naive” definitions,

- centrality and nilpotence agree,
- abelianness and solvability disagree.

[Drápal 2023] solvability agrees in Moufang loops

Nilpotent loops

- $|Q| = p^k \;\Rightarrow\; Q$ is nilpotent
  - true for groups [quite easy]
  - true for Moufang loops [quite difficult, Glaubermann, Wright 1968]
  - false in general: loops of order p are abelian groups or counterexamples
- Q finite nilpotent $\Rightarrow Q \simeq \prod Q_p$ where Q_p are nilp. loops of order p^k
  - true for groups [not difficult]
  - true for Moufang loops [quite difficult, Glaubermann, Wright 1968]
  - false in general: a loop of order 6 is nilpotent, directly indecomposable

$\Rightarrow$ supernilpotence ???

Higher commutator, supernilpotence

$C_n(\alpha_1, ..., \alpha_{n-1}; \beta; \gamma)$ iff for every term t and every $\bar{a}_i \overset{\alpha_i}{\equiv} \bar{b}_i$, $\bar{u} \overset{\beta}{\equiv} \bar{v}$

$$t(\bar{x}_1, ..., \bar{x}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{x}_1, ..., \bar{x}_n, \bar{v}) \qquad \forall (\bar{x}_1, ..., \bar{x}_n) \in \{\bar{a}_1, \bar{b}_1\} \times ... \times \{\bar{a}_n, \bar{b}_n\} \neq \{(\bar{b}_1, ..., \bar{b}_n)\}$$

$$\Downarrow$$

$$t(\bar{b}_1, ..., \bar{b}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{b}_1, ..., \bar{b}_n, \bar{v}).$$

The *n-ary commutator* $[\alpha_1, ..., \alpha_n]$ is the smallest δ such that $C_n(\alpha_1, \dots, \alpha_{n-1}; \alpha_n; \delta)$.

Fact: $[\alpha_1, ..., \alpha_n] \geq [\alpha_1, [\alpha_2, [..., [\alpha_{n-1}, \alpha_n]]]]$ (in Mal'tsev varieties)

Higher commutator, supernilpotence

$C_n(\alpha_1, ..., \alpha_{n-1}; \beta; \gamma)$ iff for every term t and every $\bar{a}_i \overset{\alpha_i}{\equiv} \bar{b}_i$, $\bar{u} \overset{\beta}{\equiv} \bar{v}$

$$t(\bar{x}_1, ..., \bar{x}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{x}_1, ..., \bar{x}_n, \bar{v}) \qquad \forall (\bar{x}_1, ..., \bar{x}_n) \in \{\bar{a}_1, \bar{b}_1\} \times ... \times \{\bar{a}_n, \bar{b}_n\} \neq \{(\bar{b}_1, ..., \bar{b}_n)\}$$

$$\Downarrow$$

$$t(\bar{b}_1, ..., \bar{b}_n, \bar{u}) \overset{\delta}{\equiv} t(\bar{b}_1, ..., \bar{b}_n, \bar{v}).$$

The *n-ary commutator* $[\alpha_1, ..., \alpha_n]$ is the smallest δ such that $C_n(\alpha_1, \ldots, \alpha_{n-1}; \alpha_n; \delta)$.

Fact: $[\alpha_1, ..., \alpha_n] \geq [\alpha_1, [\alpha_2, [..., [\alpha_{n-1}, \alpha_n]]]]$ (in Mal'tsev varieties)

An algebra is *k-supernilpotent* if $[1_A, ..., 1_A] = 0_A$.

Supernilpotence – a better "definition"

Theorem: [Aichinger, Mudrinski, 2010]
In Mal'tsev varieties,

1. an algebra is k-supernilpotent if and only if all absorbing polynomials of arity $> k$ are constant.
2. a finite algebra is k-supernilpotent if and only if $A \simeq \prod A_p$ where A_p is a nilpotent algebra of order power of p

A polynomial is *absorbing* if $p(a_1, ..., a_n) = 1$ whenever at least one $a_i = 1$.
Examples: $[x, y]$, $[x, y, z]$, $L_{x,y}(z)/z$, ..., $[xy, u]/([x, u][y, u])$, ...

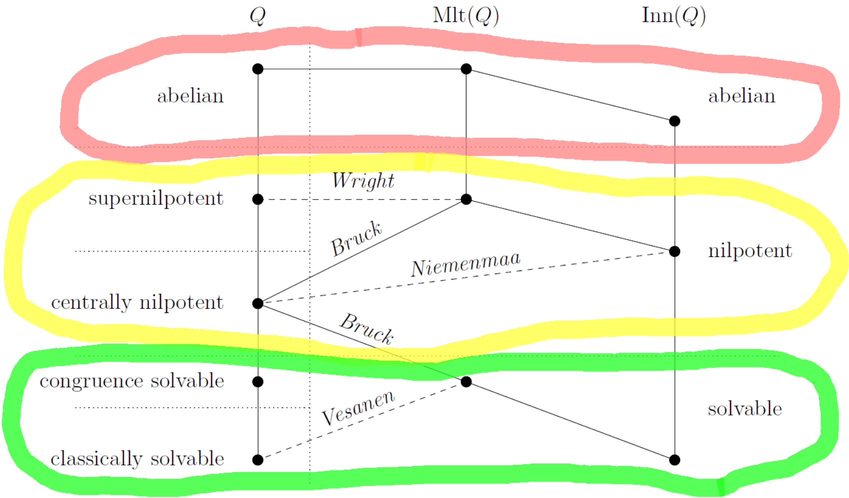
Supernilpotent groups

Theorem: [Aichinger, Ecker, 2006; S, Vojtěchovský 2023]
A group is k-supernilpotent iff k-nilpotent.

In general, not at all.

1. k-supernilpotence $\Rightarrow$ k-nilpotence
2. nilpotence $\not\Rightarrow$ supernilpotence
3. the degree of supernilpotence can be $>>$ degree of nilpotence

Big picture



Q
$\mathrm{Mlt}(Q)$
$\mathrm{Inn}(Q)$
abelian
abelian
supernilpotent
Wright
Bruck
Niemenmaa
nilpotent
centrally nilpotent
Bruck
congruence solvable
Vesanen
solvable
classically solvable

Equational basis for 1,2-supernilpotence

Let $[\![x,y]\!]$ and $[\![x,y,z]\!]$ be any terms such that, in all loops,

$$[\![x,y]\!]=1 \;\Leftrightarrow\; xy=yx$$
$$[\![x,y,z]\!]=1 \;\Leftrightarrow\; x(yz)=(xy)z$$

Example: the standard commutator and associator

$$[\![x,y]\!]=(yx)\backslash(xy),\; [\![x,y,z]\!]=x(yz)\backslash(xy)z$$

Equational basis for 1,2-supernilpotence

Let $[\![x, y]\!]$ and $[\![x, y, z]\!]$ be any terms such that, in all loops,

$$[\![x, y]\!] = 1 \;\Leftrightarrow\; xy = yx$$
$$[\![x, y, z]\!] = 1 \;\Leftrightarrow\; x(yz) = (xy)z$$

Example: the standard commutator and associator

$$[\![x, y]\!] = (yx)\backslash(xy),\; [\![x, y, z]\!] = x(yz)\backslash(xy)z$$

Easy facts:

1-supernilpotence: $[\![x, y]\!] = [\![x, y, z]\!] = 1$ (abelian groups)

2-supernilpotence: $[\![x, [\![y, z]\!]]\!] = [\![x, y, z]\!] = 1$ (2-nilpotent groups)

A group is k-nilpotent if and only if $[x_1, [x_2, [\ldots, [x_k, x_{k+1}]]]] = 1$

Equational basis for 3-supernilpotence

[S, Vojtěchovský, 2023]

TFAE for a loop Q:

- Q is 3-supernilpotent
- Q satisfies the following identities for all $[\![.,.]\!]$, $[\![.,.,.]\!]$
- Q satisfies the following identities for the standard $[\![.,.]\!]$, $[\![.,.,.]\!]$

$$1 = [\![x, [\![y, u, v]\!]]\!] \tag{1}$$

$$1 = [\![x, y, [\![u, v, w]\!]]\!] = [\![x, [\![u, v, w]\!], y]\!] = [\![[\![u, v, w]\!], x, y]\!] \tag{2}$$

$$1 = [\![x, y, [\![u, v]\!]]\!] = [\![x, [\![u, v]\!], y]\!] = [\![[\![u, v]\!], x, y]\!] \tag{3}$$

$$1 = [\![x, [\![y, [\![u, v]\!]]\!]]\!] = [\![x, [\![[\![u, v]\!], y]\!]]\!] \tag{4}$$

$$1 = [\![[\![y, [\![u, v]\!]]\!], x]\!] = [\![[\![[\![u, v]\!], y]\!], x]\!] \tag{5}$$

$$1 = [\![[\![x, y]\!], [\![u, v]\!]]\!] \tag{6}$$

$$[\![xy, u, v]\!] = [\![x, u, v]\!]\, [\![y, u, v]\!] \tag{7}$$

$$[\![u, xy, v]\!] = [\![u, x, v]\!]\, [\![u, y, v]\!] \tag{8}$$

$$[\![u, v, xy]\!] = [\![u, v, x]\!]\, [\![u, v, y]\!] \tag{9}$$