Abstract commutator theory in concrete classes

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Part I:

Solvability and nilpotence:
the beginnings and motivation

Part II:

Abelianness and Centrality: examples and module representation

Part III:

The commutator in specific varieties

Abelian algebras

Recall:

abelian $=$ "module-like" $(\neq$ commutative, associative)

... abelian groups $=$ the only groups that can be considered as modules

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abelian = "module-like" ($\neq$ commutative, associative)

... abelian groups = the only groups that can be considered as modules

An algebra A is *abelian* if $[1_A, 1_A] = 0_A$. That is,

for every term $t(x, y_1, \ldots, y_n)$ and every $a, b, \bar{u}, \bar{v}$ in A

$$t(a, \bar{u}) = t(a, \bar{v}) \;\Rightarrow\; t(b, \bar{u}) = t(b, \bar{v})$$

[J.D.H. Smith 1970s]

Abelian algebras: examples

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$\rightsquigarrow$ Modules are abelian.

Proof: $t(x, y_1, \dots, y_n) = rx + \sum r_i y_i$, cancel ra, add rb.

Subreducts of modules are also abelian.

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$\rightsquigarrow$ Unary algebras are abelian.

Proof: Terms depend on at most one variable.

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Proof: $t(x, y, z) = yxz, \quad a11 = 11a \;\Rightarrow\; ab1 = 1ba$

(but commutative semigroups need not be abelian)

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$\rightsquigarrow$ An abelian *binary algebra with 1* is associative.

Proof: $t(x, y, z, w) = (xy)(zw)$,
$(1b)(c1) = (11)(bc) \;\Rightarrow\; (ab)(c1) = (a1)(bc)$

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$\rightsquigarrow$ An abelian *idempotent self-distributive* algebra (e.g. *quandle*) is medial.

Proof: $t(x, y, u, v) = (xy)(uv)$,
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(a quandle is abelian iff it is medial and $\mathrm{Dis}(Q)$ acts semiregularly [JPSZ 2018])

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$\rightsquigarrow$ An abelian algebra with a *semilattice* reduct is trivial.

Proof: $t(x, y) = x \wedge y$, $\quad (a \wedge b) \wedge a = (a \wedge b) \wedge b \;\Rightarrow\; a \wedge a = a \wedge b$,
hence $a \leq b$ for all a, b

Abelian algebras vs. modules

Theorem (Gumm-Smith 1970s)

An algebra with a Mal'tsev polynomial is abelian if and only if it is polynomially equivalent to a module.

(examples: groups, quasigroups; non-examples: quandles, monoids)

- [Herrmann 1979] ditto under congruence modularity (hard!)
- [Kearnes-Szendrei 1998] even weaker assumptions
- [tame congruence theory / Barto-Kozik-S 2015] finite and Taylor
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Theorem (Kearnes-Szendrei 1998)

An algebra with a Taylor term is abelian if and only if it is polynomial subreduct of a module.

- also true for finite simple [TCT], idempotent simple [Kearnes 1994], ...
- also true for monoids, quandles [JPSZ 2018]
- fails in general [McKenzie, Quackenbush 1980s], examples rare

Abelian congruences

A congruence α of A is *abelian* if for every term $t(x, y_1, \dots, y_n)$ and every $a \stackrel{\alpha}{\equiv} b$, $\bar{u} \stackrel{\alpha}{\equiv} \bar{v}$

$$t(a, \bar{u}) = t(a, \bar{v}) \ \Rightarrow \ t(b, \bar{u}) = t(b, \bar{v})$$

Fact: A group congruence α is abelian iff $[1]_\alpha$ is an abelian group

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Fact: For loops, it fails. Ex.: a loop of order 8, a Moufang loop of order 16

Theorem (S-Vojtěchovský 2014)

A normal subloop N is abelian iff $\varphi|_N \in \mathrm{Aut}(N)$ for every $\varphi \in \mathrm{Inn}(Q)$, and $\forall\ a, b \in N,\ x, u, v \in Q$ s.t. $u/v \in N$

$$[a, b] = [a, b, x] = [a, x, b] = [x, a, b] = 1, \quad [a, x, u] = [a, x, v]$$

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Fact: In groups, loops, the center is the classical center.

Fact: In monoids, $[1]_{\zeta M} = \{a : ax = xa, [ax = ay \rightarrow \text{x=y}] \textit{forall}\text{x,y} \in M\}$.

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In general, it seems rather difficult to describe the center!

For example, in quandles [Bonatto-S 2019]

$$\zeta_Q = \{(a, b) : \mathrm{Dis}(Q)_a = \mathrm{Dis}(Q)_b \text{ and } L_a L_b^{-1} \in Z(\mathrm{Dis}(Q))\}$$

Inverse semigroups

... $(S, \cdot, ')$ where $(S, \cdot)$ is a semigroup and $aa'a = a$, $a'aa' = a'$ $\forall a$

... partial bijections on a set X form an inverse semigroup

... (Cayley-like repre.) every inverse semigroup embeds into a semigroup of partial bijections [Wagner, Preston 1950s]

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What is the center of inverse semigroup?

Indeed, $\{a : ax = xa \text{ for all } x\}$ is a wrong guess. (It is not even a normal subsemigroup.)

Center of a group / loop, revisited

Groups:

$$\begin{aligned} Z(G) &= \{a : ax = xa \ \forall x\} \\ &= \{a : \phi_x(a) = a \ \forall x\} \\ &= Ker(\Phi) \end{aligned}$$

$$\begin{aligned} \Phi : & G \to \mathrm{Aut}(G) \\ & x \mapsto [\phi_x : y \mapsto x^{-1}yx] \end{aligned}$$

Loops:

$$\begin{aligned} Z(Q) &= \{a : ax = xa, a(xy) = (ax)y, \dots \ \forall x, y\} \\ &= \{a : \phi(a) = a \ \forall \phi \in \mathrm{Inn}(Q)\} \end{aligned}$$

... what is more important, *commutativity* OR *conjugacy/inner mappings* ?

Center of an inverse semigroup

[Kinyon-S.]

$$\zeta_S = \mathcal{H} \cap \{(a,b) : axb = bxa \ \forall x\} = Ker(\Phi) = Ker(\Psi)$$

$$Z(S) = \{a : axa'a = aa'xa \ \forall x\}$$

$$\Phi : S \to \mathrm{PAut}(S), \quad x \mapsto [\phi_a : a'Sa \to aSa', y \mapsto xyx']$$

$$\Psi : S \to \mathrm{Trans}(S), \quad x \mapsto [\psi_a : S \to Sa, y \mapsto xyx']$$

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Sad corollary: Solvable/nilpotent inverse semigroups are groups.

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