Abstract commutator theory in concrete classes

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Part I:

Solvability and nilpotence: the beginnings and motivation

Part II:

Abelianness and Centrality: examples and module representation

Part III:

The commutator in specific varieties

Galois theory

Theorem (Galois 1830s)

A polynomial $f \in \mathbb{Q}[x]$ is solvable in radicals if and only if the group $Gal(f)$ is solvable.

A group G is solvable if there are $N_i \trianglelefteq G$ such that

$$1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$$

and N_{i+1}/N_i is an abelian group.

A polynomial f is solvable in radicals if there are field extensions

$$\mathbb{Q} = F_0 \leq F_1 \leq \ldots \leq F_k \geq S$$

such that F_{i+1} is a splitting field over F_i for some $x^n - a$.
Here S is the splitting field for f.

$Gal(f) = Gal(S/\mathbb{Q}) = \{\mathbb{Q}\text{-automorphisms of } S\}$

Galois theory

Theorem (Galois 1830s)

A polynomial $f \in \mathbb{Q}[x]$ is solvable in radicals if and only if the group $Gal(f)$ is solvable.

Proof idea:

$$\mathbb{Q} = F_0 \leq F_1 \leq \ldots \leq F_k \geq S$$

$$\Downarrow \text{[Galois correspondence]}$$

$$Gal(F_k/F_0) \geq Gal(F_k/F_1) \geq \ldots \geq Gal(F_k/F_k) = 1$$

and $Gal(F_k/F_0) \twoheadrightarrow Gal(f)$.

Crucial fact: $Gal(x^n - a)$ is metabelian, i.e., solvable of length ≤ 2.

Solvable groups

Galois theory lead to development of abstract group theory, including solvable groups.

- [Hall 1928] Solvable groups have Hall π-subgroups, and all of them are cunjugate (for any set π of primes).
- [Burnside 1904] Groups of order $p^k q^l$ are solvable.
- [Feit-Thompson 1962] Groups of odd order are solvable.

Solvability: alternative definitions (derived series)

A group G is solvable if

- there are $N_i \trianglelefteq G$ such that $1 = N_0 \leq N_1 \leq \ldots \leq N_k = G$ and N_{i+1}/N_i is an abelian group.
- there is k such that $G_{(k)} = 0$ where $G_0 = G$, $G_{(i+1)} = [G_{(i)}, G_{(i)}]$.

Here $[A, B]$ is the subgroup generated by all elementwise commutators

$$[a, b] = a^{-1}b^{-1}ab = (ba)^{-1}(ab)$$

where $a \in A$, $b \in B$.

Solvability and nilpotence

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A group G is nilpotent if

- there are $N_i \trianglelefteq G$ such that $1 = N_0 \leq N_1 \leq ... \leq N_k = G$ and $N_{i+1}/N_i \leq Z(G/N_i)$.
- there is k such that $G^{(k)} = 0$ where $G_0 = G$, $G^{(i+1)} = [G, G^{(i)}]$.

Here $Z(G) = \{a : ax = xa \text{ for all } x \in G\}$.

Nilpotent groups

- Finite nilpotent groups are direct products of their Sylow subgroups, i.e., direct products of p-groups.
- Groups of order p^k are nilpotent.

Solvability and nilpotence are natural properties!

Equation solving and Identity checking

Fix a finite group G.

INPUT: a polynomial $t(x_1, ..., x_n)$ over G

Pol-SAT(G): $\exists\ a_1, ..., a_n \in G$ such that $t(a_1, ..., a_n) = 1$?

Pol-ID(G): $\forall\ a_1, ..., a_n \in G$ we have $t(a_1, ..., a_n) = 1$?

Theorem (Burris-Lawrence 1994, Goldman-Russel 1999)

If G is nilpotent, then Pol-SAT(G) and Pol-ID(G) are tractable.

If G is not solvable, then Pol-SAT(G) is NP-complete and Pol-ID(G) is coNP-complete.

Solvable, non-nilpotent: open (partial results by Horváth, Földvári, ...)

Generalize! Elementwise commutators

groups: $[a, b] = (ba)^{-1}(ab)$

Lie algebras: $[a, b] = ab - ba$

Define $[A, B]$ to be the subobject generated by all $[a, b]$, $a \in A$, $b \in B$.
Works well.

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loops: $[a, b] = (ba)\backslash(ab), \quad [a, b, c] = (ab \cdot c)\backslash(a \cdot bc)$

A normal subloop $N \trianglelefteq Q$ is an abelian group iff $[a, b] = [a, b, c] = 1$ for all $a, b, c \in N$.

The center of Q is defined as
$\{a \in Q : \ [a, x] = [a, x, y] = [x, a, y] = [x, y, a] = 1 \text{ for all } x, y \in Q\}$.

This defines a meaningful concept nilpotence and solvability (literally the same definition as for groups).
(Spoiler: Solvability is different from the universal algebraic one.)

Generalize! A more systematic approach

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Several approaches:

- syntactic: associativity, commutativity (see semigroup theory)
- universal algebraic approach (Smith commutator)
- categorical approach (Huq commutator, Higgins commutator, etc.)

I will talk about the universal algebraic approach.

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Principal idea: abelian = "module-like"

... abelian groups are precisely those groups/loops that are polynomially equivalent to a module

The commutator

Let A be an algebra, α, β, δ congruences of A.

We say that α centralizes β modulo δ, shortly, $C(\alpha, \beta; \delta)$, if

for every term $t(x, y_1, \ldots, y_n)$ and every $a \stackrel{\alpha}{\equiv} b$, $u_i \stackrel{\beta}{\equiv} v_i$

$$t(a, \bar{u}) \stackrel{\delta}{\equiv} t(a, \bar{v}) \;\Rightarrow\; t(b, \bar{u}) \stackrel{\delta}{\equiv} t(b, \bar{v})$$

The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

The commutator has good properties for congruence modular varieties (for instance, $[\alpha, \beta] = [\beta, \alpha]$).

see the [Freese, McKenzie] book

Nilpotence, solvability

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We say that α centralizes β modulo δ, shortly, $C(\alpha, \beta; \delta)$, if for every term $t(x, y_1, \dots, y_n)$ and every $a \overset{\alpha}{\equiv} b$, $u_i \overset{\beta}{\equiv} v_i$

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The *commutator* $[\alpha, \beta]$ is the smallest δ such that $C(\alpha, \beta; \delta)$.

A congruence α of A is called

- *abelian* if $C(\alpha, \alpha; 0_A)$, i.e., if $[\alpha, \alpha] = 0_A$.
- *central* if $C(\alpha, 1_A; 0_A)$, i.e., if $[\alpha, 1_A] = 0_A$.

An algebra A is called *solvable*, resp. *nilpotent*, if there are congruences α_i such that

$$0_A = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_k = 1_A$$

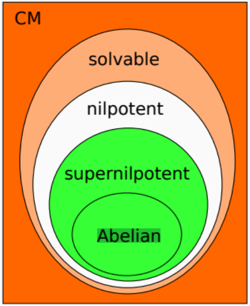
and α_{i+1}/α_i is an *abelian*, resp. *central* congruence of A/α_i, for all i.

Solvability and nilpotence are natural properties!

Equation solving and Identity checking

CEQV **in congruence modular varieties**

A... from congruence modular variety:



- **A** Abelian $\leftrightarrow$ module. CEQV(**A**) $\in$ P
- **A** k-supernilpotent. CEQV(**A**) $\in$ P
  (Aichinger, Mudrinski '10)
- **A nilpotent, not supernilpotent...?**
- **A** solvable, non-nilpotent
  $\exists\theta$: CEQV($\mathbf{A}/\theta$) $\in$ coNP-c
  (Idziak, Krzaczkowski '18)
- **A** non-solvable: CEQV(**A**) $\in$ coNP-c
  (Idziak, Krzaczkowski '18)

For CSAT the picture is similar (modulo products with DL algebras).

11

slide by Michael Kompatscher

Supernilpotence

Recall:

- Groups of prime power order are nilpotent.
- Finite nilpotent groups are direct products groups of prime power order.

Both properties fail in general, even under a Mal'tsev term (for instance, in loops).

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Supernilpotence:

Idea: bounded arity of nontrivial absorbing polynomials

Theorem: An finite algebra with a Mal'tsev term is supernilpotent iff it is a direct product of nilpotent algebras of prime power order.

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