More on tense operators

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Outline

Motivation

- Several years ago we studied tense operators on sup-lattices.
- One of our goals of our research was to generalize the concept and take other ordered structures to obtain analogical results.
- We have decided to work with the following three categories parametrized by a unital commutative quantale V:
  1. unital V-modules,
  2. unital V-F-sup-semilattices,
  3. V-frames.

More exactly, we have replaced the notions of sup-semilattices by unital V-modules (the category of them denoted as V-$\mathbb{S}$, the category of F-sup-semilatitices by unital V-F-sup-semilattices (the category of them denoted as $V - F{-}\mathbb{S}_{\leq}$) and the category of frames by V-frames (the category of them denoted as $V - \mathbb{J}$).

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Quantales

Definition 1

A *quantale* is a pair $V = (V, \otimes)$, where V is a $\bigvee$-semilattice and $\otimes$ is a binary operation on V satisfying:

(V1) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$ for all $a, b, c \in V$ (associativity).

(V2) $a \otimes (\bigvee S) = \bigvee_{s \in S} (a \otimes s)$ for every $S \subseteq V$ and every $a \in V$.

(V3) $(\bigvee S) \otimes a = \bigvee_{s \in S} (s \otimes a)$ for every $S \subseteq V$ and every $a \in V$.

A quantale $V = (V, \otimes)$ is called *unital* if there exists an element $e \in V$ such that for every $a \in V$ the equalities $a \otimes e = a$ and $e \otimes a = a$ hold. $V = (V, \otimes)$ is called *commutative* if $\otimes$ is commutative.

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V-modules

Definition 2

Given a unital quantale V, a unital *left V-module* is a pair $(A, *)$ such that A is a $\bigvee$-semilattice and $* : V \times A \longrightarrow A$ is a map satisfying:

(M1) $v * (\bigvee S) = \bigvee_{s \in S}(v * s)$ for every $S \subseteq A$ and every $v \in V$.

(M2) $(\bigvee T) * a = \bigvee_{t \in t}(t * a)$ for every $T \subseteq V$ and every $a \in A$.

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Given two left *V-modules* $(A, *)$, $(B, *)$, a map $f : A \longrightarrow B$ is called a *left-V module homomorphism* provided that it preserves all joins and $f(v * a) = v * f(a)$ for every $a \in A$ and every $v \in V$.

Whenever we mention a quantale or a V-module, we would mean by that a unital commutative quantale and a unital V-module, respectively.

Definition 3

Given a quantale V a V-relation r from set X to set Y is a map $r : X \times Y \longrightarrow V$.

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Definition 4

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Note that the definition of V-frame is a generalization of the standard notion of a time frame.

Definition 5

Given a quantale V and two V-frames T and S, a map $f : T \longrightarrow S$ is called a V-frame homomorphism if it satisfyies $r(i,j) \leq s((f(i), f(j)))$ for every $i, j \in T$.

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V-F-semilattices

Definition 6

Given a unital quantale V, a V-F-semilattice is a pair (G, F) where G is a unital V-module and F is a join preserving map $F : G \longrightarrow G$ satisfying $v * (F(a)) = F(v * a)$.

Definition 7

Given a unital quantale V and two unital V-F-semilattices (G_1, F) and V-H-semilattice (G_2, H) (where G_1 and G_2 are unital V-modules, a module homomorphism $f : G_1 \longrightarrow G_2$ is called a homomorphism between (G_1, F) and (G_2, H) if it satisfies $H(f(a)) \leq f(F(a))$ for any $a \in G_1$.

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Tense product

Definition 8

Let $(A, *)$ be a unital V-module and $J = (T, r)$ be a V-frame. Let us define a unital V-F-sup-semilattice A^J as (A^T, F^J) where

$$(F^J(x))(i) = \bigvee\{r(i,k) * x(k) \mid k \in T\}$$

The operation on the V-module A^T, denoted as $*^T$ is defined for any pair (v, x) as $v * x(t)$ for every t from the from the V-frame.

Similarly, the join is defined component-wise.

The construction above is mentioned as a definition, but it contains a theorem within. One can show that (A^T, F^J) is indeed a V-F-sup-semilattice.

Note that the definition above is a generalization of our previous definition using different categories:

Tense product in 2-valued setting

Definition 9

Let $\mathsf{L} = (L, \bigvee)$ *be a sup-semilattice and* $\mathsf{J} = (T, S)$ *a frame. Let us define an* V*-*F*-sup-semilattice* L^{J} *as* $\mathsf{L}^{\mathsf{J}} = (\mathsf{L}^T, F^{\mathsf{J}})$*, where*

$$(F^{\mathsf{J}}(x))(i) = \bigvee\{x(k) \mid (i, k) \in S\}$$

for all $x \in L^T$*.* F^{J} *will be called an* operator on L^T constructed by means of the frame J*.*

This follows from the fact that a V-frame is just a standard frame if V is a trivial, two element quantale (maps from $T \times T$ to $\{0, 1\}$ are just standard relations on T and r is a 'belonging to the relation function, whether it maps a pair to zero or one').

Functor theorems in enriched posets

Theorem 10

Let A_1 *and* A_2 *be* V*-modules, let* $f : \mathsf{A}_1 \longrightarrow \mathsf{A}_2$ *be a homomorphism, and let* $\mathsf{J} = (T, r)$ *be a* V*-frame. Then there exists a homomorphism* $f^{\mathsf{J}} : \mathsf{A}_1^{\mathsf{J}} \longrightarrow \mathsf{A}_2^{\mathsf{J}}$ *in the category of unital* V*-*F*-semilattices such that, for every* $x \in A_1^T$ *and every* $i \in T$*, it holds*

$$(f^{\mathsf{J}}(x))(i) = f(x(i)).$$

Moreover $(-)^J$ *is a functor from* V*-*$\mathbb{S}$ *to to* $V - F{-}\mathbb{S}_{\leq}$.

Functor theorems in enriched posets

Theorem 11

Let J_1 and J_2 be V-frames, let $t\colon \mathrm{J}_1 \longrightarrow \mathrm{J}_2$ be a homomorphism of V-frames, and let A be a unital V-module. Then there exists a lax morphism $\mathrm{A}^t\colon \mathrm{A}^{\mathrm{J}_2} \longrightarrow \mathrm{A}^{\mathrm{J}_1}$ of unital V-F-sup-semilattices such that, for every $x \in A^{T_2}$ and every $i \in T_1$, it holds

$$(\mathrm{A}^t(x))(i) = x(t(i)).$$

Moreover, A^- is a contravariant functor from $V - \mathbb{J}$ to $V - F{-}\mathbb{S}_{\leq}$.

Definition 12

Let V be a unital quantale, A be a V-module and $\mathrm{J} = (T, r)$ a V-frame. Then, for arbitrary $x \in A$ and $i \in T$, we define $x_{ir}(j) = r(i,j) * x$ and $x_{i=}$ by $x_{i=}(j) = \begin{cases} x & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$

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Functor theorems in enriched posets

Definition 13

Let V be a quantale, $\mathrm{H} = (\mathrm{A}, F)$ an V-F-sup-semilattice, and $\mathrm{J} = (T, r)$ a V-frame. We put

$$[\mathrm{J}, \mathrm{H}] = \{(x_{ir} \vee F(x)_{i=}, F(x)_{i=}) \mid x \in A, i \in T\}.$$

We then define a V-module $\mathrm{J} \otimes \mathrm{H}$ as follows:

$$\mathrm{A}^{T}_{j[\mathrm{J},\mathrm{H}]},$$

where $j[\mathrm{J}, \mathrm{H}]$ is a surjective homomorphism of V-modules such that $j[\mathrm{J}, \mathrm{H}](x_{ir} \vee F(x)_{i=}) = j[\mathrm{J}, \mathrm{H}](F(x)_{i=})$ for all $x \in A, i \in T$.

Functor theorems in enriched posets

Definition 14

Let $\mathrm{J}_1 = (T_1, r)$ *and* $\mathrm{J}_2 = (T_2, s)$ *be* V*-frames,* $f : T_1 \longrightarrow T_2$ *a* V*-frame homomorphism, and* $(\mathrm{A}, *)$ *a* V*-module.*
We define a forward operator $f^{\rightarrow} : A^{T_1} \longrightarrow A^{T_2}$ *evaluated on* $k \in T_2$ *for any* $x \in A^{T_1}$ *as follows:*

$$(f^{\rightarrow}(x))(k) = \bigvee \{x(i) \mid f(i) = k\}$$

where $k \in T_2$*.*

Functor theorems in enriched posets

Theorem 15

Let $f\colon \mathrm{J}_1 \longrightarrow \mathrm{J}_2$ *be a homomorphism of* V*-frames* $\mathrm{J}_1 = (T_1, r_1)$ *and* $\mathrm{J}_2 = (T_2, r_2)$, *and* $\mathrm{H} = (\mathrm{A}, F)$ *an* V*-*F*-sup-semilattice. Then there exists a unique morphism* $f \otimes \mathrm{H}\colon \mathrm{J}_1 \otimes \mathrm{H} \to \mathrm{J}_2 \otimes \mathrm{H}$ *of* V*-modules such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathrm{A}^{T_1} & \xrightarrow{\mathrm{n}(j[\mathrm{J}_1, \mathrm{H}])} & \mathrm{J}_1 \otimes \mathrm{H} \\
{\scriptstyle f^{\to}}\downarrow & & \downarrow{\scriptstyle f \otimes \mathrm{H}} \\
\mathrm{A}^{T_2} & \xrightarrow{\mathrm{n}(j[\mathrm{J}_2, \mathrm{H}])} & \mathrm{J}_2 \otimes \mathrm{H}
\end{array}
$$

Moreover, $(-) \otimes \mathrm{H}$ *is a functor from* V*-*$\mathbb{J}$ *to* V*-*$\mathbb{S}$.

Functor theorems in enriched posets

Theorem 16

Let $\mathsf{H}_1 = (\mathsf{G}_1, F_1), \mathsf{H}_2 = (\mathsf{G}_2, F_2)$ *be* V*-*F*-sup-semilattices,* $f\colon \mathsf{H}_1 \to \mathsf{H}_2$ *a lax morphism of* V*-*F*-sup-semi-lattices and* $\mathsf{J} = (T, r)$ *a* V*-frame. Then there is a unique morphism* $\mathsf{J} \otimes f\colon \mathsf{J} \otimes \mathsf{H}_1 \to \mathsf{J} \otimes \mathsf{H}_2$ *of* V*-modules such that the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{G}_1^T & \xrightarrow{\mathrm{n}(j[\mathsf{J},\mathsf{H}_1])} & \mathsf{J}\otimes \mathsf{H}_1 \\ \Big\downarrow f^{\mathsf{J}} & & \Big\downarrow \mathsf{J}\otimes f \\ \mathsf{G}_2^T & \xrightarrow{\mathrm{n}(j[\mathsf{J},\mathsf{H}_2])} & \mathsf{J}\otimes \mathsf{H}_2 \end{array}$$

Moreover, $\mathsf{J} \otimes (-)$ *is a functor from the category of* $V - F{-}\mathbb{S}_{\leq}$ *to* V*-*$\mathbb{S}$*.*

Functor theorems in enriched posets

Definition 17

Let V be unital quantale and $(A, *)$ a V-module. Let us define $a \to b \in V$ as follows:

$$a \to b = \bigvee \{v \in V; v * a \leq b\}$$

Let H *be a V-F-sup-semilattice and let* A *be a V-module.*
Let us define a V-frame J[H, A] *as a pair $(T_{[H,A]}, r_{[H,A]})$, where $T_{[H,A]}$ are V-module morphisms from* H *to* A *and $r_{[H,A]}$ is defined as $r_{[H,A]}(\alpha, \beta) = \bigwedge_{x \in H} \beta(x) \to \alpha(F(x))$.*

Functor theorems in enriched posets

Theorem 18

Let $\mathrm{A}_1, \mathrm{A}_2$ *be* V*-modules,* $\mathrm{H} = (\mathrm{G}, F)$ *a* V*-*F*-sup-semilattice, and let* $f\colon \mathrm{A}_1 \to \mathrm{A}_2$ *be a morphism of* V*-modules. Then there exists a homomorphism* $\mathrm{J}[\mathrm{H}, f]\colon \mathrm{J}[\mathrm{H}, \mathrm{A}_1] \to \mathrm{J}[\mathrm{H}, \mathrm{A}_2]$ *of* V*-frames such that*

$$(\mathrm{J}[\mathrm{H}, f](\alpha))(x) = f(\alpha(x))$$

for all $\alpha \in T_{[\mathrm{H},\mathrm{L}_1]}$ *and all* $x \in G$.

Moreover, $\mathrm{J}[\mathrm{H}, (-)]$ *is a functor from* V*-*$\mathbb{S}$ *to* V*-*$\mathbb{J}$.

Functor theorems in enriched posets

Theorem 19

Let $\mathsf{H}_1 = (\mathsf{G}_1, F_1), \mathsf{H}_2 = (\mathsf{G}_2, F_2)$ *be* V*-*F*-sup-semilattices,* A *a* V*-frame and* $f\colon \mathsf{H}_1 \to \mathsf{H}_2$ *a lax morphism of* V*-*F*-sup-semilattices. Then there exists a homomorphism* $\mathsf{J}[f, \mathsf{A}]\colon \mathsf{J}[\mathsf{H}_2, \mathsf{A}] \to \mathsf{J}[\mathsf{H}_1, \mathsf{A}]$ *of* V*-frames such that*

$$(\mathsf{J}[f, \mathsf{A}](\alpha))(x) = \alpha(f(x)) = (\alpha \circ f)(x)$$

for all $\alpha \in T_{[\mathsf{H}_2, \mathsf{A}]}$ *and all* $x \in G_1$.

Moreover, $\mathsf{J}[(-), \mathsf{A}]$ *is a contravariant functor from* $V - F{-}\mathbb{S}_{\leq}s$ *to the* V*-*$\mathbb{J}$.

Example

$Q = \{0, a, b, c, 1\}$ is a quantale (see Eklund Nr. 5.2.13)

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	a	a
b	0	a	b	c	1
c	0	a	1	1	1
1	0	a	1	1	1

$\bigvee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

Example

$V = \{0, b, 1\}$ is a subquantale of the quantale Q.

$*$	0	b	1
0	0	0	0
b	0	b	1
1	0	1	1

$\bigvee$	0	b	1
0	0	b	1
b	b	b	1
1	1	1	1

$\wedge$	0	b	1
0	0	0	0
b	0	b	b
1	0	b	1

Example

Put $\mathrm{G} = (\{0, a, b, c, 1\}, \bigvee)$. Then G is a V-module.

$*$	0	a	b	c	1
0	0	0	0	0	0
b	0	a	b	c	1
1	0	a	1	1	1

$\bigvee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

Example

We now put $F(x) = a * x$ for all $x \in G$. Then F preserves arbitrary joins and
$F(u*x) = a*(u*x) = (a*u)*x = (u*a)*x = u*(a*x) = u*F(x)$
for all $u \in \{0, b, c\}$ and $x \in G$.
Let $\mathrm{L} = (\{0,1\}, \bigvee\}$ be a V-module where $0 < 1$.

Let us define a frame $\mathrm{J[H, L]} = (\mathbb{S}(\mathrm{G, L}), r)$ where r is the map from Definition 17.
Clearly, $\mathbb{S}(\mathrm{G, L})$ potentially has 8 elements, which we will denote f_i, where $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ and their description is given by the following table:

	0	a	b	c	1
f_1	0	0	0	0	0
f_2	0	0	0	1	1
f_3	0	0	1	0	1
f_4	0	1	0	0	1
f_5	0	1	1	0	1
f_6	0	1	0	1	1
f_7	0	0	1	1	1
f_8	0	1	1	1	1

Example

Since every of these potential morphism also have to satisfy $f(v * y) = v * f(y)$ for every $v \in V$ and every $y \in G$ we can show f_1, f_7 and f_8 are the only morphisms that actually satisfy this property. For any other f_i there is an element $y \in \{a, b, c\}$ such that $f_i(y) = 0$. But then $f_i(1 * y) = 1$. Yet $f_i(y) = 0$, and we obtain $1 * f_i(y) = 0$. So we get a contradiction.

By one of the previous theorems, there exists a lax morphism $\mu_{\mathrm{H}} \colon \mathrm{H} \longrightarrow \mathrm{L}^{\mathrm{J}[\mathrm{H},\mathrm{L}]}$ of V-F-sup-semilattices defined for arbitrary $x \in G$ and $f_i \in \mathbb{S}(\mathrm{G}, \mathrm{L})$ by

$$(\mu_{\mathrm{H}}(x))(f_i) = f_i(x).$$

Example

Let us now describe the map r. For all $i \in \{1, 7, 8\}$ it holds that:

$$r(f_i, f_1) = \bigwedge_{x \in G} f_1(x) \to f_i(F(x)) = 1$$

since $f_1(x) = 0$ and therefore it holds for all $x \in$G.
Let $i \in \{1, 7, 8\}$ and $j \in \{7, 8\}$ it holds that:

$$r(f_i, f_j) = \bigwedge_{x \in G} f_j(x) \to f_i(F(x)) = 0$$

since $f_8(x) = 1$ for all x other than 0 and $f_7 = 1$ for all x other than 0 or a. The map r is given by the following table:

r	f_1	f_7	f_8
f_1	1	0	0
f_7	1	0	0
f_8	1	0	0

By the previous, there exists a lax morphism $\mu_{\mathrm{H}}\colon \mathrm{H} \longrightarrow \mathrm{L}^{\mathrm{J}[\mathrm{H},\mathrm{L}]}$ of V-F-sup-semilattices defined for arbitrary $x \in G$ and $f_i \in \mathbb{S}(\mathrm{G}, \mathrm{L})$ by

$$(\mu_{\mathrm{H}}(x))(f_i) = f_i(x).$$

Let us now compute μ_H on elements of G. It holds that:

$$(\mu_{\mathrm{H}}(x))(f_1) = f_1(x) = 0$$

for all $x \in G$,
and
$(\mu_{\mathrm{H}}(x))(f_8) = f_8(x) = 0$ if $x = 0$ and $(\mu_{\mathrm{H}}(x))(f_8) = f_8(x) = 1$ otherwise.
and
$(\mu_{\mathrm{H}}(x))(f_7) = f_7(x) = 0$ if $x = 0, a$ and $(\mu_{\mathrm{H}}(x))(f_7) = f_7(x) = 1$ otherwise.

$*$	f_1	f_7	f_8
$\mu_H(0)$	0	0	0
$\mu_H(a)$	0	0	1
$\mu_H(b)$	0	1	1
$\mu_H(c)$	0	1	1
$\mu_H(1)$	0	1	1

We see that the morphism is not injective and so it is not an embedding.

First adjoint situation

Let $\mathsf{J} = (T, r)$ be a V-frame. Then:

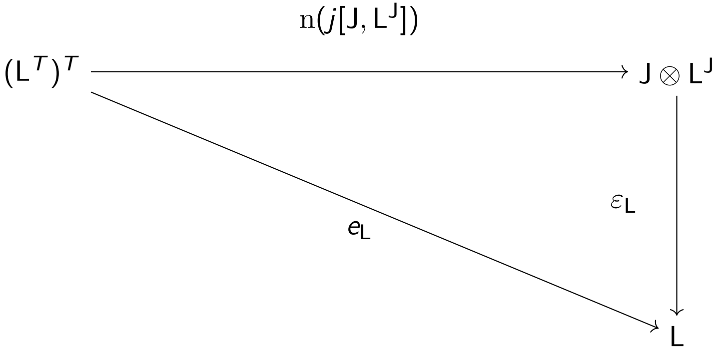
(a) For an arbitrary V-F-sup-semilattice $\mathsf{H} = (\mathsf{G}, F)$ there exists a lax morphism $\eta_{\mathsf{H}} \colon \mathsf{H} \to (\mathsf{J} \otimes \mathsf{H})^{\mathsf{J}}$ of V-F-sup-semilattices defined in such a way that

$$(\eta_{\mathsf{H}}(x))(i) = \mathrm{n}(j[\mathsf{J}, \mathsf{H}])(x_{i=}).$$

Moreover, $\eta = (\eta_{\mathsf{H}} \colon \mathsf{H} \to (\mathsf{J} \otimes \mathsf{H})^{\mathsf{J}})_{\mathsf{H} \in V-F-\mathbb{S}_{\leq}}$ is a natural transformation.

First adjoint situation

- For an arbitrary V-module L there exists a unique morphism $\varepsilon_{\mathsf{L}}\colon \mathsf{J}\otimes \mathsf{L}^{\mathsf{J}} \to \mathsf{L}$ of V-modules such that the following diagram commutes:



where $e_{\mathsf{L}}\colon (\mathsf{L}^{T})^{T} \to \mathsf{L}$ is defined by $e_{\mathsf{L}}(\bar{x}) = \bigvee_{i\in T}(\bar{x}(i))(i)$ for any $\bar{x} \in (\mathsf{L}^{T})^{T}$.

First adjoint situation

Moreover, $\varepsilon = (\varepsilon_{\mathsf{L}} \colon \mathsf{J} \otimes \mathsf{L}^{\mathsf{J}} \to \mathsf{L})_{\mathsf{L} \in \mathbb{S}}$ is a natural transformation.

(a) There exists an adjoint situation $(\eta, \varepsilon) \colon (\mathsf{J} \otimes -) \dashv (-^{\mathsf{J}}) \colon \mathbb{S} \to$ the category of V-F-sup-semilattices.

Second adjoint situation

Let $\mathsf{H} = (\mathsf{G}, F)$ be an V-F-sup-semilattice. Then:

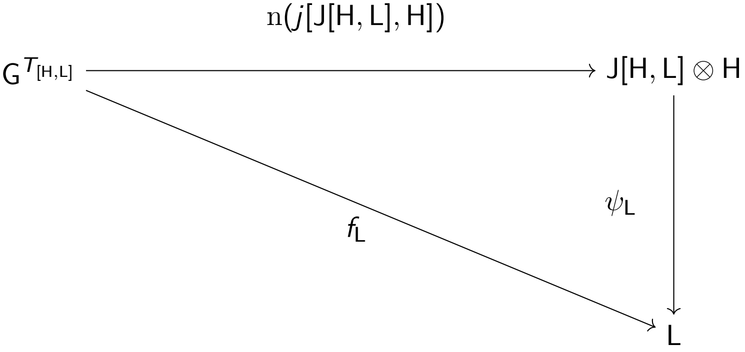
ⓐ For an arbitrary V-frame $\mathsf{J} = (T, r)$, there exists a unique homomorphism of V-frames
$\varphi_{\mathsf{J}} \colon \mathsf{J} \to \mathsf{J}[\mathsf{H}, \mathsf{J} \otimes \mathsf{H}]$ defined for arbitrary $x \in G$ and $i \in T$ in such a way that

$$(\varphi_{\mathsf{J}}(i))(x) = \mathrm{n}(j[\mathsf{J}, \mathsf{H}])(x_{i=}).$$

Moreover, $\varphi = (\varphi_{\mathsf{J}} \colon \mathsf{J} \to \mathsf{J}[\mathsf{H}, \mathsf{J} \otimes \mathsf{H}])_{\mathsf{J} \in \mathbb{V}-J}$ is a natural transformation.

Second adjoint situation

- For an arbitrary V-module L there exists a unique morphism $\psi_{\mathsf{L}} \colon \mathsf{J}[\mathsf{H},\mathsf{L}] \otimes \mathsf{H} \to \mathsf{L}$ of V-modules such that the following diagram commutes:



where $f_{\mathsf{L}} \colon \mathsf{G}^{T_{[\mathsf{H},\mathsf{L}]}} \to \mathsf{L}$ is defined by $f_{\mathsf{L}}(x) = \bigvee_{\alpha \in \mathsf{J}[\mathsf{H},\mathsf{L}]} \alpha(x(\alpha))$ for any $x \in G^{T_{[\mathsf{H},\mathsf{L}]}}$.

Second adjoint situation

Moreover, $\psi = (\psi_{\mathsf{L}} \colon \mathsf{J}[\mathsf{H},\mathsf{L}] \otimes \mathsf{H} \to \mathsf{L})_{\mathsf{L} \in \mathbb{V}-S}$ is a natural transformation.

a. There exists an adjoint situation $(\varphi,\psi)\colon (- \otimes \mathsf{H}) \dashv \mathsf{J}[\mathsf{H},-])\colon \mathbb{V} - S \to \mathbb{V} - J$.

Third adjoint situation

Let L be a V-module. Then the following holds:

a. For an arbitrary V-frame $\mathrm{J} = (T, r)$, there exists a unique homomorphism of V-frames $\nu_{\mathrm{J}} \colon \mathrm{J} \to \mathrm{J}[\mathrm{L}^{\mathrm{J}}, \mathrm{L}]$ defined for arbitrary $x \in L^T$ and $i \in T$ in such a way that

$$(\nu_{\mathrm{J}}(i))(x) = x(i).$$

Moreover, $\nu = (\nu_{\mathrm{J}} \colon \mathrm{J} \to \mathrm{J}[\mathrm{L}^{\mathrm{J}}, \mathrm{L}])_{\mathrm{J} \in \mathbb{V}-J}$ is a natural transformation.

b. For an arbitrary V-F-sup-semilattice $\mathrm{H} = (\mathrm{G}, F)$ there exists a lax morphism $\mu_{\mathrm{H}} \colon \mathrm{H} \to \mathrm{L}^{\mathrm{J}[\mathrm{H},\mathrm{L}]}$ of V-F-sup-semilattices defined for arbitrary $x \in G$ and $\alpha \in T_{\mathrm{J}[\mathrm{H},\mathrm{L}]}$ by

$$(\mu_{\mathrm{H}}(x))(\alpha) = \alpha(x).$$

Third adjoint situation

Let L be a V-module. Then the following holds:

(a) For an arbitrary V-frame $\mathsf{J} = (T, r)$, there exists a unique homomorphism of V-frames $\nu_{\mathsf{J}} \colon \mathsf{J} \to \mathsf{J}[\mathsf{L}^{\mathsf{J}}, \mathsf{L}]$ defined for arbitrary $x \in L^T$ and $i \in T$ in such a way that

$$(\nu_{\mathsf{J}}(i))(x) = x(i).$$

Moreover, $\nu = (\nu_{\mathsf{J}} \colon \mathsf{J} \to \mathsf{J}[\mathsf{L}^{\mathsf{J}}, \mathsf{L}])_{\mathsf{J} \in \mathbb{V}-J}$ is a natural transformation.

(b) For an arbitrary V-F-sup-semilattice $\mathsf{H} = (\mathsf{G}, F)$ there exists a lax morphism
$\mu_{\mathsf{H}} \colon \mathsf{H} \to \mathsf{L}^{\mathsf{J}[\mathsf{H},\mathsf{L}]}$ of V-F-sup-semilattices defined for arbitrary $x \in G$ and $\alpha \in T_{\mathsf{J}[\mathsf{H},\mathsf{L}]}$
by

$$(\mu_{\mathsf{H}}(x))(\alpha) = \alpha(x).$$

Third adjoint situation

Moreover, $\mu = (\mu_{\mathsf{H}}\colon \mathsf{H} \to \mathsf{L}^{\mathsf{J}[\mathsf{H},\mathsf{L}]})_{\mathsf{H}\in V-F-\mathbb{S}_{\leq}}$ is a natural transformation.

(c) There exists an adjoint situation $(\nu, \mu)\colon \mathsf{J}[-,\mathsf{L}]) \dashv \mathsf{L}^{-}\colon \mathbb{V} - J \to V - F{-}\mathbb{S}_{\leq}{}^{op}$.

Thank you for your attention!