Classes of groups in lattice framework

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Joint research with
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And not only these. In the weak congruence lattice of a group, also quotient subgroups are represented by their weak congruence lattices.
Hence all concrete objects and structural properties related to subgroups and their series, normal and quotient subgroups, can be identified and investigated in the mentioned weak congruence lattices.

Our first results related to weak congruence lattices of groups:

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- G. Czédli, B. Šešelja, A. Tepavčević, *Semidistributive elements in lattices; application to groups and rings*, Algebra Univers. 58 (2008) 349–355.
- G. Czédli, M. Erné, B. Šešelja, A. Tepavčević, *Characteristic triangles of closure operators with applications in general algebra*, Algebra Univers. 62 (2009) 399–418.

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- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice characterization of finite nilpotent groups*, Algebra Univers. 2021 82(3) 1–14.
- J. Jovanović, B. Šešelja, A. Tepavčević, *Lattices with normal elements* Algebra Univers. 2022 83(1) 1–28.
- M.Z. Grulović, J. Jovanović, B. Šešelja, A. Tepavčević, *Lattice Characterization Of Some Classes Of Groups By Series Of Subgroups*, International Journal of Algebra and Computation, 2023 33(02) 211–235.
- J. Jovanović, B. Šešelja, A. Tepavčević, *On the Uniqueness of Lattice Characterization of Groups*, Axioms 2023 27 12(2) 125.
- J. Jovanović, B. Šešelja, A. Tepavčević, *Nilpotent groups in lattice framework* (under review).

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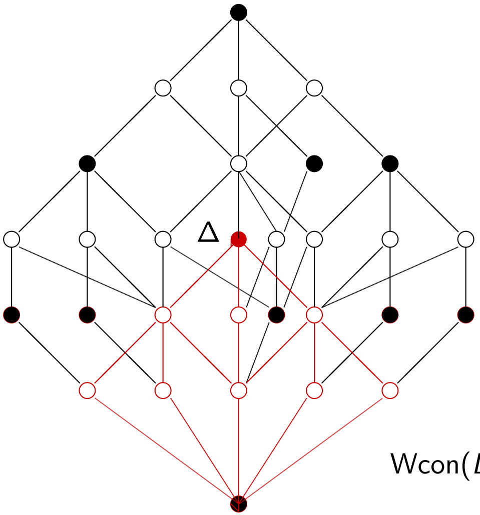
Congruence lattices of all subgroups are interval sublattices of $\mathrm{Wcon}(G)$.



$\mathrm{Wcon}(D_8)$

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T_a is a lattice under the order from L, it is closed under meets in L, but not necessarily under joins.

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By $n \blacktriangleleft b$ we denote that n *is normal in* $\downarrow b$; the sign is filled in, in order to indicate the difference with the normality among groups.

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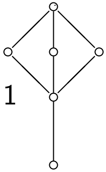
Figure 1  Q

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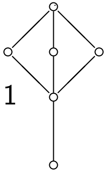

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We say that L an ***A*-lattice** if it is a modular lattice with normal elements determined by a in which $\downarrow a$ does not have an interval-sublattice which is isomorphic with the lattice Q in Fig. 1; Q represents the subgroup lattice of the quaternion group, which is uniquely determined by its subgroup lattice.

Theorem

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Corollary

If H *is a subgroup of a group* G, *then* $H \lhd G$, *if and only if the principal filter* $\uparrow(H^2)$ *in* $\mathrm{Wcon}(G)$ *is a lattice with normal elements determined by* $H^2 \vee \Delta$, *as the weak congruence lattice of* G/H. *Analogously, for subgroups* H, K *of* G, $H \lhd K$ *if and only if the interval* $[H^2, K^2]$ *in* $\mathrm{Wcon}(G)$ *is a lattice with normal elements determined with* $H^2 \vee \Delta_K$, *as the weak congruence lattice of* K/H.

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A group G is **nilpotent** if it has a central series. The smallest k so that G has a central series of length k is the **nilpotency class** of G, which is said to be **nilpotent of class** k.

Theorem

A group G is nilpotent if and only if the lattice $\mathrm{Wcon}(G)$ *has a finite series of intervals*

$$[\{e\}^2, H_1^2], [H_1^2, H_2^2], \ldots, [H_i^2, H_{i+1}^2], \ldots, [H_k^2, G^2],$$

so that for every $i \in \{0, 1, \ldots, k\}$ the following holds:

(a) $\Delta_{H_i} \blacktriangleleft \Delta$;

(b) *in the sublattice $[H_i^2, G^2]$ as a lattice with normal elements with the designated element $H_i^2 \vee \Delta$, for every $\delta \in \mathrm{C}([H_i^2, H_i^2 \vee \Delta])$, the interval $[H_i^2, \overline{H_i^2 \vee \Delta_{H_{i+1}} \vee \delta}]$ is an A-lattice with the designated element $H_i^2 \vee \Delta_{H_{i+1}} \vee \delta$.*

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Members of the subgroup systems are called **terms**.
If $\mathcal{C}$ is a subgroup system in G and $H, K \in \mathcal{C}$, then terms H and K form a **jump** in $\mathcal{C}$ if $H < K$ and there is no term $M \in \mathcal{C}$ such that $H < M < K$; In this case we say that $H < K$ *is a jump in* $\mathcal{C}$.

If $\mathcal{C}$ is a system of subgroups of a group G and $H \in \mathcal{C}$, then

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A complete subgroup system $\mathcal{C}$ of subgroups of a group G is called a **subnormal system** for G, if H is normal in H^u for all $H \in \mathcal{C}$, $H \neq G$ (or if H^b is normal in H for all $H \in \mathcal{C}$, $H \neq \{e\}$).

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$$H^u := \bigcap\{K \mid K \in \mathcal{C}, H < K\}; \quad H^b := \bigvee\{K \mid K \in \mathcal{C}, K < H\}.$$

In a complete subgroup system C both H^u and H^b are in C for every H in C.
If $H < H^u$, then (H, H^u) is a jump and similarly, if $H^b < H$, then (H^b, H) is a jump.
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Subnormal terms are terms H_α and $H_{\alpha+1}$, such that H_α is normal in $H_{\alpha+1}$.
A subnormal system $\mathcal{C}$ is **well-ordered ascending** if $H^u \neq H$ for all $H \in \mathcal{C}$, $H \neq G$; $\mathcal{C}$ is said to be **well-ordered descending** if $H^b \neq H$ for all $H \neq \{e\}$, or $H < H^u$ for all $H \in \mathcal{C}$, $H \neq G$ (and $H^b < H$ for all $H \neq \{e\}$).

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A subnormal system $\mathcal{C}$ is **solvable** if its factors are abelian groups.

Kurosh-Cernikov classes of groups

Kurosh-Cernikov classes of groups

SN-groups; each of them containing a solvable subnormal subgroup system;
SN^*-groups; they have a well-ordered ascending solvable subnormal subgroup system;
$\overline{SN}$-groups; every composition subgroup system in such a group is solvable;
SI-groups, they have a solvable normal subgroup system;
SI^*- groups; in each group of the class there exist a well-ordered ascending solvable normal subgroup system; these groups are also known as hyperabelian.
$\overline{SI}$-groups; any principal subgroup system in a group of this class is solvable;
Z-groups; groups having a central subgroup system;

ZA-groups; groups with a well-ordered ascending central subgroup system;
ZD-groups; groups with a well-ordered descending central subgroup system;
$\overline{Z}$-groups: each principal subgroup system of a group in this class is central;
$\widetilde{N}$-groups; through any subgroup of a group in this class there passes a subnormal subgroup system;
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In the finite case, the conditions given for classes SN, SN^*, SI, SI^*, $\overline{SI}$ and $\overline{SN}$ are equivalent to solvability, and the conditions for the remaining classes of groups, that are, Z, ZA, ZD, $\overline{Z}$, $\widetilde{N}$ and N, to nilpotency.

Kurosh-Cernikov classes of groups; characterizations

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Theorem

A group G is an SN-group if and only if in the lattice $\mathrm{Wcon}(G)$ *there is a chain $\mathcal{C}_w$ as a complete sublattice of $\downarrow\Delta$, such that the following hold:*
(i) $\{(e,e)\}, \Delta \in \mathcal{C}_w$;
(ii) for every $\Delta_H \in \mathcal{C}_w$ such that $\Delta_H < \Delta_{H^u}$, the interval $[H^2, (H^u)^2]$ is an A-lattice determined by $H^2 \vee \Delta_{H^u}$.
G is an SN-group if and only if in the lattice* $\mathrm{Wcon}(G)$ *there is a chain $\mathcal{C}_w$ satisfying the conditions above with an additional condition, $\Delta_H < \Delta_{H^u}$, for every $\Delta_H \in \mathcal{C}_w$ such that $\Delta_H < \Delta$.*

Theorem

A group G is an $\overline{SN}$-group if and only if in $\mathrm{Wcon}(G)$, for each maximal subnormal chain $\mathcal{C}_w$ in $\downarrow\Delta$ and for every jump $\Delta_H < (\Delta_H)^*$ in $\mathcal{C}_w$, the interval $[H^2, (H^u)^2]$ is an A-sublattice in $\mathrm{Wcon}(G)$ determined by $H^2 \vee \Delta_{H^u}$.

In other words, every maximal subnormal chain $\mathcal{C}_w$ in $\downarrow\Delta$ generates in $\mathrm{Wcon}(G)$ a chain of intervals $[H^2, (H^u)^2]$ which are A-lattices determined by $H^2 \vee \Delta_{H^u}$, where $\Delta_H < (\Delta_H)^*$ are jumps in $\mathcal{C}_w$.

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Theorem

A group G is an SI-group if and only if in $\mathrm{Wcon}(G)$ there is a normal chain $\mathcal{C}_w \subseteq \downarrow\Delta$, such that for every jump $\Delta_H < (\Delta_H)^*$ in $\mathcal{C}_w$, the interval $[H^2, (H^u)^2]$ is an A-lattice determined by $H^2 \vee \Delta_{H^u}$.
G is an SI^*-group if and only if in $\mathrm{Wcon}(G)$ there is a normal chain $\mathcal{C}_w \subseteq \downarrow\Delta$ satisfying the condition above and also an additional condition, $\Delta_H < \Delta_{H^u}$ for every $\Delta_H \in \mathcal{C}_w$, $\Delta_H \neq \Delta$.

Theorem

A group G is an $\overline{SI}$-group if and only if in $\mathrm{Wcon}(G)$ the following holds: for every maximal normal chain $\mathcal{C}_w$ in $\downarrow\Delta$, if $\Delta_H \in \mathcal{C}_w$ and $\Delta_H < \Delta_{H^u}$, then $[H^2, (H^u)^2]$ is an A-lattice determined by $H^2 \vee \Delta_{H^u}$.

Theorem

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Theorem

A group G is a Z-group if and only if in $\mathrm{Wcon}(G)$ there exits a normal chain $\mathcal{C}_w$ in $\downarrow\Delta$ such that for every $\Delta_H \in \mathcal{C}_w$ and for every $\delta \in \mathrm{C}([H^2, H^2 \vee \Delta])$, the interval $[H^2, \overline{H^2 \vee \Delta_{H^u} \vee \delta}]$ is an A-lattice determined by $H^2 \vee \Delta_{H^u} \vee \delta$.

Theorem

A group G is a ZA-group if and only if in $\mathrm{Wcon}(G)$ there exists a normal chain $\mathcal{C}_w$ in $\downarrow\Delta$ such that for every $\Delta_H \in \mathcal{C}_w$, $\Delta_H \neq \Delta$, the following holds:
(i) $\Delta_H < \Delta_{H^u}$;
(ii) for every $\delta \in \mathrm{C}([H^2, H^2 \vee \Delta])$, *the interval* $[H^2, \overline{H^2 \vee \Delta_{H^u} \vee \delta}]$ *is an A-lattice determined by* $H^2 \vee \Delta_{H^u} \vee \delta$.

Theorem

A group G is a ZA-group if and only if in $\mathrm{Wcon}(G)$ *there exists a normal chain* $\mathcal{C}_w$ *in* $\downarrow\Delta$ *such that for every* $\Delta_H \in \mathcal{C}_w$, $\Delta_H \neq \Delta$, *the following holds:*
(*i*) $\Delta_H < \Delta_{H^u}$;
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Theorem

A group G is a ZD-group if and only if in $\mathrm{Wcon}(G)$ *there exists a normal chain* $\mathcal{C}_w$ *in* $\downarrow\Delta$ *such that for every* $\Delta_H \in \mathcal{C}_w$,
$\Delta_H \neq \{(e, e)\}$, *the following holds:*
(*i*) $\Delta_{H^b} < \Delta_H$;
(*ii*) *for every* $\delta \in \mathrm{C}([(H^b)^2, (H^b)^2 \vee \Delta])$, *the interval*
$[(H^b)^2, \overline{(H^b)^2 \vee \Delta_H \vee \delta}]$ *is an A-lattice determined by*
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Theorem

A group G is a $\overline{Z}$*-group if and only if in the lattice* $\mathrm{Wcon}(G)$ *the following holds: for every maximal normal chain* $\mathcal{C}_w$ *in* $\downarrow\Delta$*, for every* $\Delta_H \in \mathcal{C}_w$ *such that* $\Delta_H \neq \Delta$*, and for every* $\delta \in \mathrm{C}([H^2, H^2 \vee \Delta])$*, the interval* $[H^2, \overline{H^2 \vee \Delta_{H^u} \vee \delta}]$ *is an A-lattice determined by* $H^2 \vee \Delta_{H^u} \vee \delta$*.*

Theorem

A group G is a $\overline{Z}$-group if and only if in the lattice $\mathrm{Wcon}(G)$ the following holds: for every maximal normal chain $\mathcal{C}_w$ in $\downarrow\Delta$, for every $\Delta_H \in \mathcal{C}_w$ such that $\Delta_H \neq \Delta$, and for every $\delta \in \mathrm{C}([H^2, H^2 \vee \Delta])$, the interval $[H^2, \overline{H^2 \vee \Delta_{H^u} \vee \delta}]$ is an A-lattice determined by $H^2 \vee \Delta_{H^u} \vee \delta$.

Theorem

A group G is an $\widetilde{\mathrm{N}}$-group if and only if in the lattice $\mathrm{Wcon}(G)$ for every $\Delta_H \in \downarrow\Delta$ there exists a subnormal chain $\mathcal{C}_w$ in $\downarrow\Delta$ such that $\Delta_H \in \mathcal{C}_w$.
A group G is an N-group if and only if in the lattice $\mathrm{Wcon}(G)$ for every $\Delta_H \in \downarrow\Delta$ there exists a subnormal chain $\mathcal{C}_w$ in $\downarrow\Delta$ such that $\Delta_H \in \mathcal{C}_w$, and $\Delta_K < \Delta_{K^*}$ for every $\Delta_K \in \mathcal{C}_w$, $\Delta_K \neq \Delta$.

Group theoretical classes

Group theoretical classes

A **group theoretical class** or briefly **class of groups** $\mathfrak{P}$ is a class in the set-theoretic sense, consisting of groups, with the following two properties:
(*a*) If a group G belongs to $\mathfrak{P}$ and $G_1 \cong G$, then also G_1 belongs to $\mathfrak{P}$;
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The group theoretical classes are ordered by inclusion: if $\mathfrak{P}$ and $\mathfrak{Q}$ are classes of groups then $\mathfrak{P} \subseteq \mathfrak{Q}$ means that the class $\mathfrak{P}$ is a subclass of the class $\mathfrak{Q}$.

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By $\mathfrak{I}$ we denote the class of *trivial, one-element groups* and by $\mathfrak{D}$, the class of *all groups*.

A group G is an **extension of a group A by a group B** if A is a normal subgroup of G and the quotient group G/A is isomorphic with B.

If $\mathfrak{P}$ and $\mathfrak{Q}$ are classes of groups then $\mathfrak{P}\mathfrak{Q}$ is an **extension class**, defined as follows: a group G belongs to $\mathfrak{P}\mathfrak{Q}$ if there is a normal subgroup N of G such that $N \in \mathfrak{P}$ and $G/N \in \mathfrak{Q}$. If $G \in \mathfrak{P}\mathfrak{Q}$, then G is said to be an **$\mathfrak{P}$-by-$\mathfrak{Q}$ group**.

An **operation** c on the class of all group theoretical classes assigns to each class of groups $\mathfrak{X}$ a class of groups $c\mathfrak{X}$ so that the following holds:

1. $\mathfrak{X} \leqslant c\mathfrak{X}$, in particular $c\mathfrak{I} = \mathfrak{I}$, and
2. if $\mathfrak{X} \leqslant \mathfrak{Y}$ then $c\mathfrak{X} \leqslant c\mathfrak{Y}$.

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Identity closure I, leaves every class of groups unchanged, and *Universal* closure U: the set of images consists of a single class $\mathfrak{D}$ - the class of all groups.

Operation S. $\mathfrak{X} = \mathrm{S}\mathfrak{X}$ means: if G is a group in the class $\mathfrak{X}$, then every subgroup of G is also an $\mathfrak{X}$-group.

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Operations D, $\mathrm{D_0}$. A class $\mathfrak{X}$ is D-*closed* ($\mathrm{D_0}$-*closed*) if the direct product of any collection (any pair) of $\mathfrak{X}$-groups is an $\mathfrak{X}$-group.

Operations N, N_0. A class $\mathfrak{X}$ is N-*closed* (N_0-*closed*) if the direct product of any collection (any pair) of normal $\mathfrak{X}$-subgroups of a group in $\mathfrak{X}$ is an an $\mathfrak{X}$-group $\mathfrak{X}$-group.

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Operation R. For a class $\mathfrak{X}$ of groups, $\mathfrak{X} = \mathrm{R}\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming subcartesian products, i.e.: if $N_i \lhd G$ and $G/N_i \in \mathfrak{X}$, $i \in I$, then $G/\bigcap_{i \in I} N_i \in \mathfrak{X}$. The groups in the class $\mathrm{R}\mathfrak{X}$ are said to be *residually* $\mathfrak{X}$*-groups*.

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Equivalently, if $\mathfrak{X}$ is a class of groups, then a group G is said to be *residually $\mathfrak{X}$-group* if for each $g \in G$, $g \neq e$, there is a normal subgroup N_g of G, such that $g \notin N_g$ and the quotient group G/N_g belongs to the class $\mathfrak{X}$.

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Operation R. For a class $\mathfrak{X}$ of groups, $\mathfrak{X} = R\mathfrak{X}$ means that $\mathfrak{X}$ is closed with respect to forming subcartesian products, i.e.: if $N_i \triangleleft G$ and $G/N_i \in \mathfrak{X}$, $i \in I$, then $G/\bigcap_{i\in I} N_i \in \mathfrak{X}$. The groups in the class $R\mathfrak{X}$ are said to be *residually $\mathfrak{X}$-groups*.

Equivalently, if $\mathfrak{X}$ is a class of groups, then a group G is said to be *residually $\mathfrak{X}$-group* if for each $g \in G$, $g \neq e$, there is a normal subgroup N_g of G, such that $g \notin N_g$ and the quotient group G/N_g belongs to the class $\mathfrak{X}$.

Operation R_0. This is the *finite residual closure operator* and it is defined so that $\mathfrak{X} = R_0\mathfrak{X}$ if and only if from $G/N_1, G/N_2 \in \mathfrak{X}$, it follows that $G/N_1 \cap G/N_2 \in \mathfrak{X}$.

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Birkhoff's famous result:

A class of algebras of the same type is an equational class if and only if it is a variety.

An *equational class* is the class of all algebraic structures of a given type satisfying a given set of identities; a *variety* is a class of algebras of the same type, closed under forming subalgebras, homomorphic images and direct products.
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More concretely, for groups we have:

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More concretely, for groups we have:

Proposition

Every equational class $\mathfrak{P}$ of groups is closed with respect to forming subgroups, homomorphic images and subcartesian products of groups in $\mathfrak{P}$.

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Proposition

A class of groups which is closed with respect to forming homomorphic images and subcartesian products of groups in the class is a variety.

L-classes of groups

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A class $\mathfrak{P}$ of groups is an **L-class** if the lattice Wcon(G) of every group $G \in \mathfrak{P}$ satisfies lattice theoretic properties $\mathrm{L}_{\mathfrak{P}}$.

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To investigate L-classes of group we use the following.

Proposition

A group G is a semidirect product of its subgroups H and K if and only if the following holds in the lattice $\mathrm{Wcon}(G)$:
$\Delta_H \blacktriangleleft \Delta$; $\Delta_H \vee \Delta_K = \Delta$ *and* $\Delta_H \wedge \Delta_K = \{(e, e)\}$.

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Proposition

Let G be a group. If $\Delta_A \blacktriangleleft \Delta$ in the lattice $\mathrm{Wcon}(G)$, then G is the extension of the subgroup A (by the group G/A).

Proposition

If $\mathfrak{P}$ and $\mathfrak{Q}$ are L-classes, then a group G is a $\mathfrak{P}$-by-$\mathfrak{Q}$ group (it belongs to the class $\mathfrak{P}\mathfrak{Q}$) if and only if in the lattice $\mathrm{Wcon}(G)$ *there exists some $\Delta_N \blacktriangleleft \Delta$ such that sublattices $\downarrow N^2$ and $\uparrow N^2$ fulfil lattice theoretic properties $L_{\mathfrak{P}}$ and $L_{\mathfrak{Q}}$, respectively.*

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Proposition

Let $\mathfrak{P}$ be an L-class of groups. A group G is a residually $\mathfrak{P}$-group (it belongs to the class $\mathbf{R}\mathfrak{P}$) if and only if the lattice $\mathrm{Wcon}(G)$ *fulfils:*
$(*)$ *For each $\Delta_X \in \mathrm{C}(\downarrow\Delta)$, $\Delta_X \neq \{(e,e)\}$, there is $\Delta_N \blacktriangleleft \Delta$, such that $\Delta_N \wedge \Delta_X < \Delta_X$ and the interval $[N^2, G^2]$, as the lattice with normal elements determined by $N^2 \vee \Delta$, satisfies the lattice theoretic properties $L_{\mathfrak{P}}$.*

Corollary

If $\mathfrak{P}$ is an L-class, so is the class of residually $\mathfrak{P}$-groups $\mathbf{R}\mathfrak{P}$.

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Proposition

An L-class $\mathfrak{P}$ of groups is S-closed (S_n-closed) if and only if for every group G in $\mathfrak{P}$, the following holds:
For every $\Delta_A \in \downarrow\Delta$ ($\Delta_A \in \downarrow\Delta$ and $\Delta_A \blacktriangleleft \Delta$), the ideal $\downarrow A^2$ as a lattice with normal elements determined by Δ_A, fulfils $L_{\mathfrak{P}}$, i.e., the lattice theoretic properties defining the class $\mathfrak{P}$ as an L-class.

Corollary

If $\mathfrak{P}$ is an L-class, so is the class of residually $\mathfrak{P}$-groups $\mathbf{R}\mathfrak{P}$.

Proposition

An L*-class $\mathfrak{P}$ of groups is* S*-closed (*S_n*-closed) if and only if for every group G in $\mathfrak{P}$, the following holds:*
For every $\Delta_A \in {\downarrow}\Delta$ ($\Delta_A \in {\downarrow}\Delta$ and $\Delta_A \blacktriangleleft \Delta$), the ideal ${\downarrow}A^2$ as a lattice with normal elements determined by Δ_A, fulfils $L_{\mathfrak{P}}$, i.e., the lattice theoretic properties defining the class $\mathfrak{P}$ as an L*-class.*

Proposition

An L*-class $\mathfrak{P}$ of groups is* H*-closed if and only if for every group G in $\mathfrak{P}$, the following holds:*
For every $\Delta_N \in {\downarrow}\Delta$ such that $\Delta_N \blacktriangleleft \Delta$, the interval $[N^2, G^2]$ as a lattice with normal elements determined by $N^2 \vee \Delta$, fulfils $L_{\mathfrak{P}}$, i.e., the lattice theoretic properties defining the class $\mathfrak{P}$ as an L*-class.*

Some known classes of groups that we proved to be L-classes:

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Empty class of groups	finite groups
Dedekind groups	abelian groups
Hamiltonian groups	nilpotent groups
solvable groups	supersolvable groups
cyclic groups	metabelian groups
perfect groups	metacyclic groups
T-groups	T^*-groups
hypercyclic groups	polycyclic groups
cocyclic groups	finite symmetric groups
simple groups	semisimple groups
fully simple groups	absolutely simple groups
strictly simple groups	perfect groups
SN-groups	SN^*-groups
$\overline{SN}$-groups	SI-groups
$\overline{SI}$-groups	SI^*- (hyperabelian) groups
Z-groups	ZA-groups
ZD-groups	$\overline{Z}$-groups
$\widetilde{N}$-groups	N-groups

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The class of groups determined by the property that the diagonal of each group G in the class is a neutral element in the lattice $\mathrm{Wcon}(G)$ *is the class of Dedekind groups.*

Theorem

The class of finite Dedekind groups is an L-class with respect to the property that the diagonal of each group G in this class is a distributive element of the lattice $\mathrm{Wcon}(G)$.

The class of groups is the ***CIP*-class** if the diagonal of every group in the class is a distributive element in the lattice $\mathrm{Wcon}(G)$.

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Proposition

A torsion-free group $\mathcal{G}$ is a non-abelian and has the CIP if and only if:
(1) $\mathcal{G}$ has a unique minimal normal subgroup N; this N is non-abelian and G/N is a torsion Dedekind group;
(2) $\langle x\rangle \cap \langle y\rangle \neq \mathbf{1}$ for every pair of elements of G;
(3) $(H \cap K)N = HN \cap KN$ for all subgroups H, K of $\mathcal{G}$.

Proposition

If G is a CIP group that is not a Dedekind group then G has a factor N with the following properties:

1. N is torsion free.

2. If $a, b \in N \setminus \{1\}$ then $\langle a \rangle \cap \langle b \rangle \neq \{1\}$.

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It was Obraztsov (J. of Algebra, 1998), who finally proved the existence of a such a group. This result shows that *in the CIP-class there are also non-Dedekind groups.*

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Theorem

A group G belongs to the class $\mathfrak{K}$ defined above if and only if G satisfies the Congruence Extension Property (the CEP).

We say that the above defined class is the *CEP*-**class**.

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A class of finite nilpotent groups is determined by the lower semi-modular weak congruence lattices of its members.

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Dealing with the dual case, we define a class of groups to be the *wSM*-class (*w* for weak congruence lattice, *SM* for semi-modular), if the weak congruence lattice of each member has a finite length and is (upper) semi-modular. The class is not empty, e.g., Tarski monster groups belongs to it; and of course, finite Dedekind groups, since their weak congruence lattices are modular.

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Proposition

The wSM-class of groups is a subclass of the CEP-class.

As a negative example for lattice properties determining L-classes of groups, we have that *a weak congruence lattice of any group* G *could not be complemented*.

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Δ' should belong to the class $[\{(e,e)\}]_{\varphi_\Delta}$; however, this is a one-element class consisting of the bottom $\{(e,e)\}$ of the lattice.

As a negative example for lattice properties determining L-classes of groups, we have that *a weak congruence lattice of any group G could not be complemented*.
Indeed, the diagonal Δ which determines $\mathrm{Wcon}(G)$ as a lattice with normal elements has no complement $\Delta^{'}$:
$\Delta^{'}$ should belong to the class $[\{(e,e)\}]_{\varphi_\Delta}$; however, this is a one-element class consisting of the bottom $\{(e,e)\}$ of the lattice.
Consequently, a class could not be determined by the lattice property of e.g., being boolean.

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Theorem

An L-*class* $\mathfrak{P}$ *of groups is a variety if and only if the following hold:*
(i) if G is a $\mathfrak{P}$-group, then in the lattice $\mathrm{Wcon}(G)$ *for every $\Delta_H \in \downarrow\Delta$, such that $\Delta_H \blacktriangleleft \Delta$, the interval $[H^2, G^2]$, which is a lattice with normal element determined by $H^2 \vee \Delta$, satisfies $L_{\mathfrak{P}}$, i.e., the lattice properties determining the class $\mathfrak{P}$ and*
(ii) every group G, such that the lattice $\mathrm{Wcon}(G)$ *satisfies $(*)$, belongs to $\mathfrak{P}$.*

Corollary

If a class of groups $\mathfrak{P}$ is an L*-class with respect to a set of lattice identities $L_{\mathfrak{P}}$, then $\mathfrak{P}$ is a variety of groups if and only if the weak congruence lattices of its members satisfy the property* $(*)$.

Thanks for watching!