S-preclones
and the Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

Peter Jipsen
Erkko Lehtonen
Reinhard Pöschel

Chapman University, Orange, CA (USA)
Khalifa University, Abu Dhabi (United Arab Emirates)
Technische Universität Dresden (Germany)

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Outline

S-preclones

S-relational clones

The Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

The lattice ${}^S\mathcal{L}_A$ of S-preclones

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Motivating example

some history:

Nov. 2021 PALS talk by P. Jipsen on partially ordered algebras (and po-clones): operations which in each argument are *order-preserving* **or** *order-reversing* (for some given order on the base set).

Questions: how to characterize such "po-clones"?

R.P.: characterization via invariant relations?

Analogies to many-sorted algebras
(results of E. Lehtonen/ R. Pöschel/ T. Waldhauser),

Let P be a property for unary functions $g \in A^A$.

"motivating example": $P = +$: order-preserving
$P = -$: order-reversing

An n-ary *operation* $f(x_1, \ldots, x_n)$ *has property* P *in an argument*, say x_1, $:\iff$ each translation $x_1 \mapsto f(x_1, c_2, \ldots, c_n)$ has this property P (for all constants $c_2, \ldots, c_n \in A$).

How to handle composition? order-reversing composed with order-reversing is order-preserving! Formalization: Collect the properties in a monoid $S = (\{+, -\}, \cdot)$, here a group

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S-operations

S finite monoid with unit element e.

n-ary *S-operation*: operation f together with its *signum*

$$f : A^n \to A \text{ with } \operatorname{sgn}(f) = (s_1, \ldots, s_n) \in S^n,$$

i.e., the i-th argument of f gets a label (signum) $s_i \in S$ ($i = 1, \ldots, n$).

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***S*-preclone** := set $F \subseteq {}^S\mathrm{Op}(A)$ of *S*-operations closed under:

(1) $\mathrm{id}_A \in F$, $\mathrm{id}_A(x) = x$, $\mathrm{sgn}(\mathrm{id}_A) := (e)$,

(2) permutation of arguments (operations ζ, τ),

(3) identification of arguments *with the same signum s* (Δ^s),

(4) adding fictitious arguments of (arbitrary) signum $s \in S$,
e.g., $(\nabla^s f)(x_1, x_2, \dots, x_{n+1}) := f(x_2, \dots, x_{n+1})$, where
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$\mathrm{sgn}(f) = (s_1, \dots, s_n)$ and $\mathrm{sgn}(g) = (s'_1, \dots, s'_m)$. Then

$$(f \circ g)(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n-1}) := f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})$$

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(3) identification of arguments *with the same signum s* (Δ^s),

(4) adding fictitious arguments of (arbitrary) signum $s \in S$, e.g., $(\nabla^s f)(x_1, x_2, \dots, x_{n+1}) := f(x_2, \dots, x_{n+1})$, where $\mathrm{sgn}(\nabla^s f) = (s, s_1, \dots, s_n)$ for $\mathrm{sgn}(f) = (s_1, \dots, s_n)$,

(5) "linearized" composition
$\mathrm{sgn}(f) = (s_1, \dots, s_n)$ and $\mathrm{sgn}(g) = (s'_1, \dots, s'_m)$. Then

$$(f \circ g)(x_1, \dots, x_m, x_{m+1}, \dots, x_{m+n-1}) := f(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})$$

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"motivating" Example

$S = \{+, -\}$ two-element group with unit element $+$. $(S \cong \{+1, -1\})$

$\leq$ order relation on (finite) base set A.

$F \subseteq {}^S\mathrm{Op}(A) :=$ set of all S-operations $f(x_1, \ldots, x_n)$ $(\mathrm{sgn}(f) = (s_1, \ldots, s_n) \in S^n)$ such that argument x_i is order-preserving if $s_i = +$, otherwise order-reversing (signum $-$)

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Outline

S-preclones

S-relational clones

The Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

The lattice ${}^S\mathcal{L}_A$ of S-preclones

S-relations

m-ary *S-relation*: $\varrho = (\varrho_s)_{s\in S}$ with $\varrho_s \subseteq A^m$

notation also
$\varrho = (\varrho_s, \varrho_{s'}, \ldots, \varrho_{s''})$ (for $S = \{s, s', \ldots, s''\}$) or
$\varrho = (r_1, \ldots, r_t)$ with $\boldsymbol{\lambda}_\varrho = (s_1, \ldots, s_t)$ s.t. $\varrho_s = \{r_i \mid s_i = s\}$

Example: $S = \{+, -\}$, $A = \{0, 1\}$,
$\varrho = (\varrho_s)_{s\in S}$ with $\varrho_+ := \leq$, $\varrho_- := \geq$, i.e.,

$$\varrho = (\leq, \geq) = \left(\begin{smallmatrix} 0 & 0 & 1 & & 0 & 1 & 1 \\ 0 & 1 & 1 & & 0 & 0 & 1 \end{smallmatrix}\right) = (r_1, r_2, r_3\,, \; r_4, r_5, r_6)$$

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(2) permutation of rows (consider the elements $r \in \varrho_s$ as columns)

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(4) Cartesian product: $\varrho \times \varrho' = (\varrho_s)_{s\in S} \times (\varrho'_s)_{s\in S} := (\varrho_s \times \varrho'_s)_{s\in S}$

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$\sqcap^t \varrho = ((\sqcap^t \varrho)_s)_{s\in S} := (\bigcap\{\varrho_{s'} \mid s't = s\})_{s\in S}$

${}^S[Q]$:= S-relational clone generated by $Q \subseteq {}^S\mathrm{Rel}(A)$.

S-relational clones

S-relational clone := set $Q \subseteq {}^S\mathrm{Rel}(A)$ of S-relations closed under:

(1) $\delta^S := (\Delta_A)_{s\in S} \in Q \quad (\Delta_A := \{(x,x) \mid x \in A\}$ diagonal)

(2) permutation of rows (consider the elements $r \in \varrho_s$ as columns)

(3) deleting of rows (projection on selected rows)

(4) Cartesian product: $\varrho \times \varrho' = (\varrho_s)_{s\in S} \times (\varrho'_s)_{s\in S} := (\varrho_s \times \varrho'_s)_{s\in S}$

(5) intersection: $\varrho \wedge \varrho' = (\varrho_s)_{s\in S} \wedge (\varrho'_s)_{s\in S} := (\varrho_s \wedge \varrho'_s)_{s\in S}$

(6) index translation by $t \in S$: $\mu_t(\varrho) := (\varrho_{st})_{s\in S}$

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Outline

S-preclones

S-relational clones

The Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

The lattice ${}^S\mathcal{L}_A$ of S-preclones

S-preservation $\overset{S}{\triangleright}$

classical notion of preservation: $f \triangleright \varrho :\Longleftrightarrow f(\varrho, \ldots, \varrho) \subseteq \varrho$

The "S-version":
$f \in {}^S\mathrm{Op}(A)$ with $\mathrm{sgn}(f) = (s_1, \ldots, s_n)$, $\varrho = (\varrho_s)_{s \in S} \in {}^S\mathrm{Rel}^{(m)}(A)$

$$f \overset{S}{\triangleright} (\varrho_s)_{s \in S} :\Longleftrightarrow \forall s \in S\colon f(\varrho_{s_1 s}, \ldots, \varrho_{s_n s}) \subseteq \varrho_s.$$

$f \overset{S}{\triangleright} \varrho$: f *S-preserves* ϱ, f is an *S-polymorphism* of ϱ, ϱ is *(S-)invariant* for f

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f(& a_{11} & a_{12} & \ldots & a_{1n} &) = & \bullet \\
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 & & & & & & \\
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The Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

$\overset{S}{\triangleright}$ induces a Galois connection with the operators

$${}^S\mathrm{Pol}\, Q := \{f \in {}^S\mathrm{Op}(A) \mid \forall \varrho \in Q \colon f \overset{S}{\triangleright} \varrho\} \quad (S\text{-polymorphisms}),$$

$${}^S\mathrm{Inv}\, F := \{\varrho \in {}^S\mathrm{Rel}(A) \mid \forall f \in F \colon f \overset{S}{\triangleright} \varrho\} \quad (\text{invariant } S\text{-relations}).$$

for $F \subseteq {}^S\mathrm{Op}(A)$ and $Q \subseteq {}^S\mathrm{Rel}(A)$.

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for $F \subseteq {}^S\mathrm{Op}(A)$ and $Q \subseteq {}^S\mathrm{Rel}(A)$.

once more: "motivating" Example

$(A, \leq)$ poset, $S = \{+, -\}$ (group).

Example: $F \subseteq {}^S\mathrm{Op}(A) :=$ set of all S-operations $f(x_1, \ldots, x_n)$ ($\mathrm{sgn}(f) = (s_1, \ldots, s_n) \in S^n$) such that argument x_i is order-preserving if $s_i = +$, otherwise order-reversing (signum $-$)

e.g., $A = \{0, 1\}$, $0 < 1$,
$f(x_1, x_2) = \neg x_1 \wedge x_2$, $\mathrm{sgn}(f) = (s_1, s_2) = (-, +)$,
$g(x_1, x_2) = x_1 \vee \neg x_2$, $\mathrm{sgn}(g) = (s_1', s_2') = (+, -)$.

Proposition: *F is an S-preclone.*

Then we have:

$$F = {}^S\mathrm{Pol}\,\varrho \text{ for the } S\text{-relation } \varrho = (\varrho_+, \varrho_-) := (\leq, \geq).$$

Example: $A = \{0, 1\}$,
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with $\lambda_\varrho = (+, +, +\,,\ -, -, -)$

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The Galois closures

Let A be a finite set

Theorem
Let S be a finite monoid. Then, for $F \subseteq {}^S\mathrm{Op}(A)$, we have

$$ {}^S\langle F\rangle = {}^S\mathrm{Pol}\,{}^S\mathrm{Inv}\,F, $$

i.e., the Galois closure is the S-preclone generated by F.

Theorem
Let S be a finite monoid. Then, for $Q \subseteq {}^S\mathrm{Rel}(A)$, we have

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Concerning the proofs

${}^S\langle F\rangle \subseteq {}^S\mathrm{Pol}\,{}^S\mathrm{Inv}\,F$ and ${}^S[Q] \subseteq {}^S\mathrm{Inv}\,{}^S\mathrm{Pol}\,Q$ straightforward

for S being a group:
generalization of the proofs for the "classical" Galois connection Pol – Inv

for arbitrary (finite) monoids S:
more complicated (proof was completed few months ago)

[JipLP2023]: arXiv http://arxiv.org/abs/2306.00493

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Outline

S-preclones

S-relational clones

The Galois connection ${}^S\mathrm{Pol} - {}^S\mathrm{Inv}$

The lattice ${}^S\mathcal{L}_A$ of S-preclones

The lattices ${}^S\mathcal{L}_A$ and ${}^S\mathcal{L}^*_A$

${}^S\mathcal{L}_A :=$ lattice of all S-preclones on A (w.r.t. $\subseteq$)

${}^S\mathcal{L}^*_A :=$ lattice of all S-relational clones on A $\quad ({}^S\mathcal{L}^*_A \cong_d {}^S\mathcal{L}_A)$

${}^S J_A = S$-projections $= {}^S\langle \mathrm{id}_A \rangle$ $\quad$ ${}^S D_A = S$-diagonals $= {}^S[\delta]$

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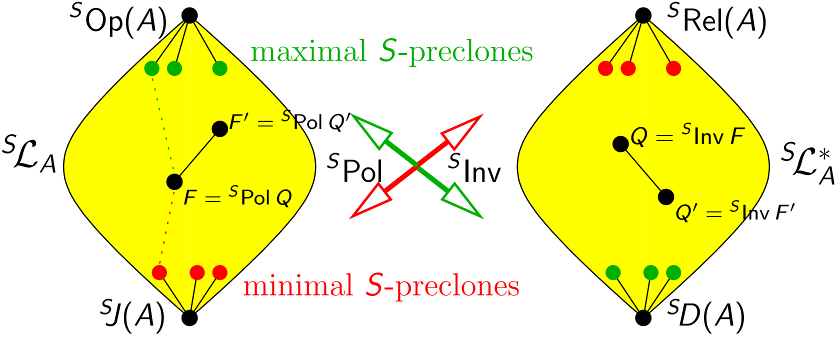

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Some properties

Each S-preclone is contained in a maximal one (coatom) and contains a minimal one (atom):

${}^S\mathcal{L}_A$ is atomic and coatomic

There are finitely many atoms and coatoms.

${}^S\mathrm{Op}(A)$ is finitely generated (by at most binary S-operations),

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${}^S\mathrm{Op}(A)$ is finitely generated (by at most binary S-operations),
e.g., for $A = \{0, 1, \ldots, k-1\}$ we have ${}^S\langle\{m^{(e,e)}\} \cup \{\mathrm{id}^s \mid s \in S\}\rangle = {}^S\mathrm{Op}(A)$,
where $m^{(e,e)}$ is the binary S-operation defined by the $m(x, y) := \max(x, y) \oplus 1$ (known as Sheffer function, $\oplus$ addition modulo k) with $\mathrm{sgn}(m) = (e, e)$.

${}^S\mathrm{Rel}(A)$ finitely generated (by at most ternary S-relations),
e.g., $|A| \geq 3$: ${}^S[(\Delta, \nabla, \ldots, \nabla), (\leq, \leq, \ldots, \leq), (\neq, \neq, \ldots, \neq)] = {}^S\mathrm{Rel}(A)$. Here $(\sigma, \sigma', \ldots, \sigma')$ denotes the relation $\varrho \in {}^S\mathrm{Rel}(A)$ with $\varrho_e = \sigma$ and $\varrho_s = \sigma'$ for $s \in S \setminus \{e\}$. ($\nabla = \nabla_A = A^2$, $\Delta = \Delta_A = \{(x, x) \mid x \in A\}$)
(For $|A| = 2$ a ternary S-relation is needed)

Example: Boolean $\pm$-preclones

$\pm := S := \{+, -\}$ (two-element group $\cong \{+1, -1\}$)

notation for

S-preclone, ${}^S\mathcal{L}_A$, ${}^S\langle F\rangle$, ${}^S[Q]$, ${}^S\mathrm{Pol}$, ${}^S\mathrm{Inv}$:
$\pm$-preclone, ${}^\pm\mathcal{L}_A$, ${}^\pm\langle F\rangle$, ${}^\pm[Q]$, ${}^\pm\mathrm{Pol}$, ${}^\pm\mathrm{Inv}$

$A := \{0, 1\}$:
$\pm$-preclone = *Boolean $\pm$-preclone*
${}^\pm\mathcal{L}_2$ lattice of Boolean $\pm$-preclones

Recall:
$\mathcal{L}_2$, the Post lattice of Boolean clones, is countable and has 5 maximal and 7 minimal clones.

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The maximal Boolean $\pm$-preclones

Theorem

There are nine maximal Boolean $\pm$-preclones listed below. Each such preclone is of the form $F = {}^{\pm}\mathrm{Pol}\,\varrho$ for some $\pm$-relation $\varrho = (\varrho_+, \varrho_-)$:

(a) ${}^{\pm}\mathrm{Pol}(\sigma, \sigma)$ *with* $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ *where* $\mathrm{Pol}\,\sigma_i$ *is maximal in* $\mathcal{L}_2$ *(0-preserving, 1-preserving, monotone, self-dual, linear operations)*
$\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0,0),(0,1),(1,1)\}$,
$\sigma_3 = \{(0,1),(1,0)\}$, $\sigma_4 = \{(x,y,z,u) \in A^4 \mid x+y+z+u = 0\}$.

(b) ${}^{\pm}\mathrm{Pol}(\leq, \geq)$ *our motivating example! all $\pm$-operations where each $+$argument is order-preserving and each $-$argument is order-reversing.*

(c) ${}^{\pm}\mathrm{Pol}(A, \emptyset) =$ *all functions with positive or mixed signum.*

(d) ${}^{\pm}\mathrm{Pol}(A^2, \Delta_A) =$ *all Boolean $\pm$-operations, where each negative argument is fictitious (including all negative constants).*

(e) ${}^{\pm}\mathrm{Pol}(\{0\}, \{1\})$.

The maximal Boolean $\pm$-preclones

Theorem

There are nine maximal Boolean $\pm$-preclones listed below. Each such preclone is of the form $F = {}^{\pm}\mathrm{Pol}\,\varrho$ for some $\pm$-relation $\varrho = (\varrho_+, \varrho_-)$:

(a) ${}^{\pm}\mathrm{Pol}(\sigma,\sigma)$ *with* $\sigma \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ *where* $\mathrm{Pol}\,\sigma_i$ *is maximal in* $\mathcal{L}_2$ *(0-preserving, 1-preserving, monotone, self-dual, linear operations)*
$\sigma_0 = \{0\}$, $\sigma_1 = \{1\}$, $\sigma_2 = \leq = \{(0,0),(0,1),(1,1)\}$,
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(b) ${}^{\pm}\mathrm{Pol}(\leq,\geq)$ *our motivating example! all $\pm$-operations where each $+$argument is order-preserving and each $-$argument is order-reversing.*

(c) ${}^{\pm}\mathrm{Pol}(A,\emptyset) =$ *all functions with positive or mixed signum.*

(d) ${}^{\pm}\mathrm{Pol}(A^2,\Delta_A) =$ *all Boolean $\pm$-operations, where each negative argument is fictitious (including all negative constants).*

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Theorem

There are twenty three minimal Boolean $\pm$-preclones. Each such $\pm$-preclone is of the form ${}^{\pm}\langle f \rangle$ with one $\pm$-operation f as generator:

(A) ${}^{\pm}\langle h_0 \rangle, {}^{\pm}\langle h_1 \rangle, {}^{\pm}\langle h_y \rangle$ *where* $h_i(x,y,z,u) = \begin{cases} x & \text{if } x = y \text{ or } z = u, \\ i & \text{otherwise,} \end{cases}$
where the generators have signum $\lambda = (+,+,-,-)$, (#3)

(B) ${}^{\pm}\langle (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \rangle, {}^{\pm}\langle x + y + z \rangle$
where the generators have signum $\lambda = (+,+,+,-)$,
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(C) ${}^{\pm}\langle x \wedge y \rangle, {}^{\pm}\langle x \vee y \rangle, {}^{\pm}\langle x \vee (y \wedge z) \rangle, {}^{\pm}\langle x \wedge (y \vee z) \rangle,$
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Further research

Some open problems that we hope to solve in the future:

Is the lattice of Boolean $\pm$-preclones countable?

Classify the maximal S-preclones for $|S| \geq 2$ and $|A| \geq 2$.

Further research:

Can the notions of S-preclone and S-relational clone be extended to the setting where the monoid S of signa is only assumed to be a semigroup?

Take an "interesting" result about **clones** or **relational clones** or **universal algebras** and ask for an analogous result for S-**preclones** or S-**relational clones** or S-**algebras** (i.e., $(A, (f_i)_{i \in I})$ with fundamental operations $f_i \in {}^S\mathrm{Op}(A)$ for a fixed finite monoid S).

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Thank you for your ATTENTION!