Monotone deviations in completely normal lattices

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Motivation: Abelian ℓ-groups

Abelian lattice-ordered group (ℓ-group) is an algebra $(G, +, 0, -, \vee, \wedge)$ such that

(i) $(G, +, 0, -)$ is an Abelian group;

(ii) $(G, \vee, \wedge)$ is a lattice;

(iii) $x \leq y$ implies $x + z \leq y + z$ for every $x, y, z \in G$.

Abelian ℓ-groups form a variety generated by $\mathbb{Z}$ (with natural ordering).

Congruences and ideals

Congruences on abelian ℓ-groups correspond to ℓ-ideals (=convex ℓ-subgroups), these form a distributive algebraic lattice $\operatorname{Id} G$.
Compact congruences correspond to compact (finitely generated) ℓ-ideals. They are principal and have the form

$$\langle a \rangle = \{x \in G \mid (-na) \leq x \leq na \text{ for some } n \in \omega\},$$

for $a \geq 0$.
Compact ℓ-ideals form a sublattice $\operatorname{Id}_c G$ of $\operatorname{Id} G$. The lattice $\operatorname{Id} G$ is determined by $\operatorname{Id}_c G$ uniquely.

Problem. Which distributive lattices are isomorphic to $\operatorname{Id}_c G$ for some Abelian ℓ-group G?

Complete normality

Known for > 50 years:

Theorem

Every lattice $\mathrm{Id}_c\, G$ *is distributive and completely normal, i.e. satisfies*

$$(\forall a, b)(\exists x, y)(b \leq a \vee y,\ a \leq b \vee x,\ \text{and } x \wedge y = 0).$$

Equivalently: the ordered set of all prime ideals of the lattice $\mathrm{Id}_c\, G$ is a root system. (poset in which $\uparrow x$ is a chain for every x)
Intuitively: $x = a \setminus b$, $y = b \setminus a$ (similar to dual relative pseudocomplements)

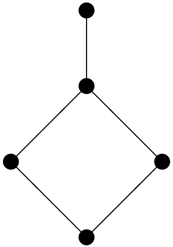
Proof: For $a, b \geq 0$,

$$\langle a \rangle \setminus \langle b \rangle = \langle (a - b)^+ \rangle,$$

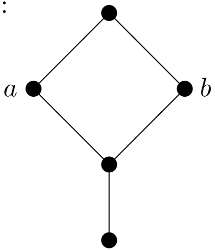
where $x^+ = x \vee 0$.

Good and bad

good:



bad:



Countable case

Theorem

Every (at most) countable completely normal distributive lattice is isomorphic to $\operatorname{Id}_c G$ *for some Abelian* ℓ*-group* G.

On the uncountable level, CN is not sufficient. Every lattice of the form $\operatorname{Id}_c G$

- has countably based differences (Cignoli, Gluschankof and Lucas 1999, Iberkleid, Martinez and McGovern 2011);
- is Cevian (Wehrung 2020).

All these properties are still not sufficient (MP 2021). Are there other necessary conditions?

CN revisited

A distributive lattice L with 0 is completely normal, if it admits a binary operation $\setminus$ such that

(i) $y \vee (x \setminus y) \geq x$ for every $x, y \in L$;

(ii) $(x \setminus y) \wedge (y \setminus x) = 0$ for every $x, y \in L$.

Such an operation will be called *a deviation*. It will be called *monotone*, if $x_1 \leq y_1$ and $x_2 \geq y_2$ imply $x_1 \setminus x_2 \leq y_1 \setminus y_2$.

Monotone deviations - the first result

The monotonicity does not follow from the definition of complete normality. However, it is not difficult to show that on every *countable* completely normal lattice the difference operation $\setminus$ can be taken monotone.

Theorem

(MP 2022) *There exists a completely normal lattice of cardinality* $\aleph_1$*, which does not admit monotone difference.*

Construction: the free CN lattice generated by the uncountable chain $\aleph_1$ and one incomparable element.

Now, the question was:
Does $\mathrm{Id}_c\, G$ admit monotone difference for every Abelian ℓ-group G?

The formula

$$\langle a \rangle \setminus \langle b \rangle = \langle (a-b)^+ \rangle$$

suggests that the difference on $\mathrm{Id}_c\, G$ should be monotone. The problem is that the formula depends on the choice of the generators of the ideals $\langle a \rangle$ and $\langle b \rangle$.
However, if there exists an order-preserving map $\varphi \colon \mathrm{Id}_c\, G \to G^+$ such that $K = \langle \varphi(K) \rangle$ for every $K \in \mathrm{Id}_c\, G$, then the deviation

$$I \setminus J = \langle (\varphi(I) - \varphi(J))^+ \rangle$$

is monotone. Such a map φ is called *a Belluce section*.

Ideal lattices of free Abelian ℓ-groups

Let J be a set and $E = \mathbb{Q}^J$. For every vector $\boldsymbol{u} \in E$ with finite support let

$$[\boldsymbol{u}] = \{\boldsymbol{x} \in E \mid \sum_{j \in J} u_j x_j > 0\}$$

be the open halfspace defined by $\boldsymbol{u}$. Let $\mathrm{Op}(E)$ be the closure of the family of all open halfspaces under finite intersections and unions.

Baker-Beynon-Madden duality: The lattice $\mathrm{Op}(E)$ is isomorphic to $\mathrm{Id}_c\, F(J)$, where $F(J)$ is the Abelian ℓ-group freely generated by J.

Theorem

For every set J, the lattice $\mathrm{Op}(E)$ admits a monotone deviation.

A generalization to finitely separable lattices

Theorem

Every finitely separable completely normal distributive 0- lattice admits a monotone difference.

Theorem

Every lattice $\mathrm{Op}(E)$ is finitely separable.

But what are these finitely separable lattices?

Finitely separable lattices

A poset M is *finitely separable* if there are functions A and B with domain M such that each $A(z)$ is a finite set of upper bounds of z, each $B(z)$ is a finite set of lower bounds of z, and for all $x, y \in M$, $x \leq y$ implies $A(x) \cap B(y) \neq \emptyset$.

This concept is due to Freese and Nation (1978), who found it as one of conditions that characterize projective lattices. In 2015, Freese and Nation proved the following strong result.

Theorem

A lattice L is finitely separable iff every surjective lattice homomorphism $f : K \to L$ has an isotone section.

Section: a map $g : L \to K$ such that fg is the identity on L.

One open problem

For an Abelian ℓ-group G, the assignment $a \mapsto \langle a \rangle$ is a surjective lattice homomorphism $G^+ \to \mathrm{Id}_c\, G$. So, if $\mathrm{Id}_c\, G$ is finitely separable (as in the case of free Abelian ℓ-groups), then G has a Belluce section. This is, formally, a stronger property than having a monotone deviation.
But is it really stronger?

Problem. *Is there an Abelian ℓ-group G which has a monotone deviation, but not a Belluce Section?*

An ℓ-group without monotone difference

Let $W \subseteq \mathbb{Q}^{\omega_1}$ consisting of those $\boldsymbol{w} = (w_\alpha \mid \alpha \in \omega_1)$ that satisfy

- $w_\alpha \geq 0$ for every α;
- $w_\alpha \leq 2w_\beta$ whenever $0 < \alpha < \beta$.

Let $\pi_\alpha : W \to \mathbb{Q}$ be the projections and let F be the closure of the set $\{\pi_\alpha \mid \alpha < \omega_1\}$ under the pointwise addition, subtraction, join and meet.
So, F is a subalgebra of $\mathbb{Q}^W$. In fact F is freely generated by $\{\pi_\alpha \mid \alpha < \omega_1\}$ with the generators satisfying $0 \leq \pi_\alpha$ for every α and $\pi_\alpha \leq 2\pi_\beta$ for every $0 < \alpha < \beta$.
Notice that, for $0 < \alpha < \beta$,

$$\pi_\alpha \not\leq \pi_\beta, \quad \text{while} \quad \langle \pi_\alpha \rangle \leq \langle \pi_\beta \rangle = \langle 2\pi_\beta \rangle.$$

An ℓ-group without monotone difference

Theorem

$\mathrm{Id}_c\, F$ does not admit monotone difference.

The proof is based on the following freeset variation.

Lemma

Let $\Phi : \omega_1 \to [\omega_1]^{<\omega}$. Let $k > 0$ be a natural number. Then there are ordinals

$$\tau_0 < \alpha_0 < \tau_1 < \alpha_1 < \tau_2 < \cdots < \alpha_{k-1} < \tau_k = \omega_1$$

such that $\Phi(\alpha_i) \subseteq \tau_0 \cup (\tau_{i+1} \setminus \tau_i)$ for every i.

Another open problem

In the above proof we in fact prove that, for every deviation $\setminus$ on $\mathrm{Id}_c F$, there are $0 < \alpha < \beta$ such that

$$\langle \pi_0 \rangle \setminus \langle \pi_\beta \rangle \nleq \langle \pi_0 \rangle \setminus \langle \pi_\alpha \rangle.$$

This means that $\mathrm{Id}_c F$ does not have a deviation monotone in the second variable.

Problem. *Is there a completely normal distributive lattice which has a deviation monotone in one variable but not in both?*

Thanks

Thank you for attention.