A dagger kernel category of orthomodular lattices

Jan Paseka[1] Milan Lekár[1]

[1]Department of Mathematics and Statistics, Faculty of Science
Masaryk University, Brno, Czech Republic
e-mail: paseka@math.muni.cz

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Outline

Dagger kernel categories

- Dagger kernel categories in which morphisms $f\colon X \to Y$ can be reversed to obtain $f^*\colon Y \to X$ have been introduced by Heunen and Jacobs in [HeJa] as a simple setting in which one can study categorical quantum logic.
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Aim of the talk

- This talk shows that the category of orthomodular lattices **OMLatLin** where morphisms are mappings having adjoints is a dagger kernel category.
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Ortholattices

Definition 1

A meet semi-lattice $(X, \wedge, 1)$ is called an *ortholattice* if it comes equipped with a function $(-)^{\perp}\colon X \to X$ satisfying:

- $x^{\perp\perp} = x$;
- $x \leq y$ implies $y^{\perp} \leq x^{\perp}$;
- $x \wedge x^{\perp} = 1^{\perp}$.

One can then define a bottom element as $0 = 1 \wedge 1^{\perp} = 1^{\perp}$ and join by $x \vee y = (x^{\perp} \wedge y^{\perp})^{\perp}$, satisfying $x \vee x^{\perp} = 1$.

We write $x \perp y$ if and only if $x \leq y^{\perp}$.

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Orthomodular lattices

Definition 2

An ortholattice $(X, \wedge, 1)$ is called *orthomodular* if it satisfies (one of) the three equivalent conditions:

- $x \leq y$ implies $y = x \vee (x^{\perp} \wedge y)$;
- $x \leq y$ implies $x = y \wedge (y^{\perp} \vee x)$;
- $x \leq y$ and $x^{\perp} \wedge y = 0$ implies $x = y$.

Example 3 (Our guiding example)

The collection $\mathcal{C}(H)$ of closed subspaces of a Hilbert space H is the prototypical example of an orthomodular lattice such that $\wedge = \cap$ and $P^{\perp}$ is the orthogonal complement of a closed subspace P of H.

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Dagger categories

Definition 4

A *dagger* on a category $\mathcal{C}$ is a functor $^\star \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ that is involutive ($f^{**} = f$) and the identity on objects.
We will call f^* the *adjoint morphism* of f or simply the *adjoint* of f.
A category equipped with a dagger is called a *dagger category*.

Let $\mathcal{C}$ be a dagger category. A morphism $f \colon A \to B$ is called

1. a *dagger monomorphism* if $f^* \circ f = \mathrm{id}_A$, and
2. f is called a *dagger isomorphism* if $f^* \circ f = \mathrm{id}_A$ and $f \circ f^* = \mathrm{id}_B$.

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Examples of dagger categories

Example 5

Let $\mathrm{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathcal{C}$ be a category with objects K^n, $n \in \mathbb{N}$ and morphisms $A\colon \mathrm{K}^n \to \mathrm{K}^m$ where A is a matrix over K and type $m \times n$. Composition of matrices of suitable type is their multiplication.

- If $\mathrm{K} = \mathbb{R}$ then $A^* = A^T$.
- If $\mathrm{K} = \mathbb{C}$ then $A^* = \overline{A}^T$.

Then $\mathcal{C}$ is a dagger category.

Example 6

Let **Rel** be a category of relations with objects sets and morphisms relations $R\colon A \to B$. Composition of relations is the classical composition $S \circ R = \{(a, c) \in A \times C \mid (\exists b \in B)\, R(a, b) \wedge S(b, c)\}$. We put $R^* = R^{-1}$. Then **Rel** is a dagger category.

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Linear maps

We now introduce a new way of organising orthomodular lattices into a dagger category.

Definition 7

The category **OMLatLin** has orthomodular lattices as objects. A morphism $f\colon X \to Y$ in **OMLatLin** is a function $f\colon X \to Y$ between the underlying sets such that there is a function $h\colon Y \to X$ and, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp h(y).$$

We say that h is an *adjoint* of a *linear map* f. It is clear that adjointness is a symmetric property: if a map f possesses an adjoint h, then f is also an adjoint of h.

Linear maps

Moreover, a map $f\colon X \to X$ is called *self-adjoint* if f is an adjoint of itself.
The identity morphism on X is the self-adjoint identity map $\mathrm{id}\colon X \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by usual composition of maps.

Example 8 (Our guiding example - continuation)

Let $f\colon H_1 \to H_2$ be a bounded linear map between Hilbert spaces and let $f^\star$ be the usual adjoint of f given by $\langle f(x), y\rangle = \langle x, f^\star(y)\rangle$.

Then the induced map $C(H_1) \to C(H_2)$, $\langle S\rangle \mapsto \langle f(S)\rangle$ has the adjoint $C(H_2) \to C(H_1)$, $\langle T\rangle \mapsto \langle f^\star(T)\rangle$.

Properties of the category **OMLatLin**

Lemma 9

Let $f : X \to Y$ be a map between orthomodular lattices and assume that f possesses the adjoint $h \colon Y \to X$. Then we have:

(i) *f possesses a right order-adjoint $\hat{h} \colon Y \to X$ such that $\hat{h} = {}^{\perp} \circ h \circ {}^{\perp}$.*

(ii) *f preserves arbitrary existing joins in X. In particular, f preserves finite joins and $f(0) = 0$.*

Moreover, we define the *kernel* and the *range* of f, respectively, by

$$\ker f = \{x \in X \colon f(x) = 0\},$$
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OMLatGal vs. **OMLatLin**

Remark 10

The categories **OMLatLin** *and* **OMLatGal** [Jac] *both have orthomodular lattices as objects.*

A morphism $X \to Y$ in **OMLatGal** *is a pair $\hat{f} = (f_\bullet, f^\bullet)$ of "antitone" functions $f_\bullet \colon X^{\mathrm{op}} \to Y$ and $f^\bullet \colon Y \to X^{\mathrm{op}}$ forming a Galois connection (or adjunction $f^\bullet \dashv f_\bullet$): $x \leq f^\bullet(y)$ iff $y \leq f_\bullet(x)$ for $x \in X$ and $y \in Y$.*

The identity morphism on X is the pair $(\perp, \perp)$ given by the self-adjoint map $\mathrm{id}^\bullet = \mathrm{id}_\bullet = (-)^\perp \colon X^{\mathrm{op}} \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ and dagger † are given by:

$$(g \circ f)_\bullet = g_\bullet \circ \perp \circ f_\bullet,\ (g \circ f)^\bullet = f^\bullet \circ \perp \circ g^\bullet,\ (f_\bullet, f^\bullet)^* = (f^\bullet, f_\bullet).$$

$f_\bullet$ preserves meets, as right adjoint, and thus sends joins in X (meets in X^{op}) to meets in Y, and dually, $f^\bullet$ preserves joins and sends joins in Y to meets in X.

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A morphism $X \to Y$ *in* **OMLatGal** *is a pair* $\hat{f} = (f_\bullet, f^\bullet)$ *of "antitone" functions* $f_\bullet \colon X^{\mathrm{op}} \to Y$ *and* $f^\bullet \colon Y \to X^{\mathrm{op}}$ *forming a Galois connection (or adjunction* $f^\bullet \dashv f_\bullet$*):* $x \leq f^\bullet(y)$ *iff* $y \leq f_\bullet(x)$ *for* $x \in X$ *and* $y \in Y$*.*

The identity morphism on X *is the pair* $(\bot, \bot)$ *given by the self-adjoint map* $\mathrm{id}^\bullet = \mathrm{id}_\bullet = (-)^\bot \colon X^{\mathrm{op}} \to X$*. Composition of* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *and dagger* † *are given by:*

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OMLatGal vs. **OMLatLin**

Theorem 11

OMLatGal *and* **OMLatLin** *are dagger isomorphic via functors* $\Lambda\colon$ **OMLatLin** $\to$ **OMLatGal** *and* $\Gamma\colon$ **OMLatGal** $\to$ **OMLatLin** *which are identities on objects and otherwise given by*

$$\Lambda(f) = (\perp \circ f, \perp \circ f^*) \text{ and } \Gamma(f_\bullet, f^\bullet) = \perp \circ f_\bullet.$$

Principal downsets in orthomodular lattices

Lemma 12

[Jac, Lemma 3.4] *Let X be an orthomodular lattice, with element $a \in X$. The (principal) downset $\downarrow a = \{u \in X \mid u \leq a\}$ is again an orthomodular lattice, with order, meets and joins as in X, but with its own orthocomplement $\perp_a$ given by $u^{\perp_a} = a \wedge u^{\perp}$, where $\perp$ is the orthocomplement from X.*

Definition 13

Let X be an orthomodular lattice. Then the map $\pi_a : X \to X$, $y \mapsto a \wedge (a^{\perp} \vee y)$ is called the *Sasaki projection* to $a \in X$.

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Properties of Sasaki projection

Lemma 14 ([LiVe])

Let X be an orthomodular lattice, and let $a \in X$. Then for each $y, z \in L$ we have

(a) $y \leq a$ *if and only if* $\pi_a(y) = y$;

(b) $\pi_a(\pi_a(y^{\perp})^{\perp})) \leq y$;

(c) $\pi_a(y) = 0$ *if and only if* $y \leq a^{\perp}$;

(d) $\pi_a(y) \perp z$ *if and only if* $y \perp \pi_a(z)$.

Corollary 15

Let X be an orthomodular lattice, and let $a \in X$. Then π_a is self-adjoint and idempotent.

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Let X be an orthomodular lattice, and let $a \in X$. Then for each $y, z \in L$ we have

(a) $y \leq a$ *if and only if* $\pi_a(y) = y$;

(b) $\pi_a(\pi_a(y^{\perp})^{\perp})) \leq y$;

(c) $\pi_a(y) = 0$ *if and only if* $y \leq a^{\perp}$;

(d) $\pi_a(y) \perp z$ *if and only if* $y \perp \pi_a(z)$.

Corollary 15

Let X be an orthomodular lattice, and let $a \in X$. Then π_a is self-adjoint and idempotent.

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Dagger monomorphisms in **OMLatLin**

Lemma 16

Let X be an orthomodular lattice, with element $a \in X$. There is a dagger monomorphism $\downarrow a \rightarrowtail X$ in **OMLatLin**, *for which we also write a, with*

$$a(u) = u \quad \text{and} \quad a^*(x) = \pi_a(x).$$

Lemma 17

Let $f\colon X \to Y$ be a morphism of orthomodular lattices. Then $\ker f = \downarrow f^*(1)^{\perp}$ is an orthomodular lattice.

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Zero object $\underline{0}$ in **OMLatLin**

We show that **OMLatLin** has a *zero object* $\underline{0}$; this means that there is, for any orthomodular lattice X, a unique morphism $\underline{0} \to X$ and hence also a unique morphism $X \to \underline{0}$.

The zero object $\underline{0}$ will be one-element orthomodular lattice $\{0\}$. Let us show that $\underline{0}$ is indeed an initial object in **OMLatLin**.

Let X be an arbitrary orthomodular lattice. The only function $f\colon \underline{0} \to X$ is $f(0) = 0$. Since we may identify $\underline{0}$ with $\downarrow 0$ we have that f is is a dagger monomorphism and it has an adjoint $f^*\colon X \to \underline{0}$ defined by $f^*(x) = \pi_0(x) = 0$.

Dagger kernels in **OMLatLin**

Definition 18

1. For a morphism $f\colon A \to B$ in a category with zero morphisms, we say that a morphism $k\colon K \to A$ is a *kernel* of f if $fk = 0_{K,B}$, and if $m\colon M \to A$ satisfies $fm = 0_{M,B}$ then there is a unique morphism $u\colon M \to K$ such that $ku = m$.
   We sometimes write $\ker f$ for k or K.
2. For a morphism $f\colon A \to B$ in a dagger category with zero morphisms, we say that a morphism $k\colon K \to A$ is a *weak dagger kernel* of f if $fk = 0_{K,B}$, and if $m\colon M \to A$ satisfies $fm = 0_{M,B}$ then $kk^*m = m$.
3. A *dagger kernel category* is a dagger category with a zero object, hence zero morphisms, where each morphism f has a weak dagger kernel k (called *dagger kernel*) that additionally satisfies $k^*k = 1_K$.

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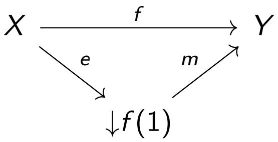
Dagger kernels in **OMLatLin**

Theorem 19

The category **OMLatLin** *is a dagger kernel category. The dagger kernel of a morphism* $f\colon X \to Y$ *is* $k\colon {\downarrow}k \to X$*, where* $k = f^*(1)^\perp \in X$*, like in Lemma 17.*

Corollary 20

Every morphism $f\colon X \to Y$ *in* **OMLatLin** *has a factorisation me where* $m = f(1)\colon {\downarrow}f(1) \to Y$ *and* $e = f|^{\downarrow f(1)}\colon X \to {\downarrow}f(1)$*.*



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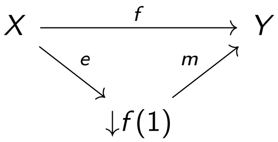

Dagger biproducts in **OMLatLin**

Definition 21

By a *dagger biproduct* of objects A, B in a dagger category $\mathcal{C}$ with a zero object, we mean a coproduct $A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$ such that ι_A, ι_B are dagger monomorphisms and $\iota_B{}^\star \circ \iota_A = 0_{A,B}$.
The dagger biproduct of an arbitrary set of objects is defined in the expected way.

Proposition 22

The category **OMLatLin** *has arbitrary finite dagger biproducts* $\bigoplus$. *Explicitly,* $\bigoplus_{i \in I} X_i$ *is the cartesian product of orthomodular lattices* X_i, $i \in I$, I *finite.*
The coprojections $\kappa_j \colon X_j \to \bigoplus_{i \in I} X_i$ *are defined by* $(\kappa_j)(x) = x_{j=}$ *with* $x_{j=}(i) = \begin{cases} x & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$ *and* $(\kappa_j)^*((x_i)_{i \in I}) = x_j$. *The dual product structure is given by* $p_j = (\kappa_j)^*$.

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Free objects on a finite set in **OMLatLin**

Proposition 23

A free object on a finite set A in **OMLatLin** *is isomorphic to the finite Boolean algebra $\mathcal{P}A$.*

Final remarks

This talk presented a new way of organising orthomodular lattices into a dagger category.

References

C. Heunen, M. Karvonen.
Limits in dagger categories,
Theory Appl. Categ. **34** 468–513 (2019).

C. Heunen, B. Jacobs.
Quantum Logic in Dagger Kernel Categories,
Electr. Notes Theor. Comput. Sci. **270** (2) 79–103 (2011).

B. Jacobs.
Orthomodular lattices, Foulis Semigroups and Dagger Kernel Categories,
Logical Methods in Computer Science, June 18 **6** (2:1) 1–26 (2010).

B. Lindenhovius, T. Vetterlein,
A characterisation of orthomodular spaces by Sasaki maps,
Int. J. Theor. Phys., **62**, Article number: 59 (2023).

Thank you for your attention!