On Congruences of Weakly Dicomplemented Lattices

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Contents

1. Preliminaries on Weakly Dicomplemented Lattices

2. The Largest Numbers of Congruences of Finite (Dual) Weakly (Di)Complemented Lattices

Presenting results from:

L. Kwuida, C. Mureșan, On Nontrivial Weak Dicomplementations and the Lattice Congruences that Preserve Them, *Order* **40** (2), 423–453, 2023.

WEAKLY DICOMPLEMENTED LATTICES:

- abstractions of CONCEPT ALGEBRAS
- introduced by Rudolf Wille in:

R. Wille, *Boolean Concept Logic*. In B. Ganter & G.W. Mineau (Eds.) ICCS 2000, *Conceptual Structures: Logical, Linguistic, and Computational Issues*, Springer LNAI **1867** (2000), 317–331.

Notation

- $\mathbb{WCL}$:= the variety of *weakly complemented lattices*
- $\mathbb{DWCL}$:= the variety of *dual weakly complemented lattices*
- $\mathbb{WDL}$:= the variety of *weakly dicomplemented lattices*

Notation

$\mathbb{V}$: variety; $A \in \mathbb{V}$. Then:
$\mathrm{Con}_{\mathbb{V}}(A)$:= the lattice of the congruences of A (w.r.t. the type of $\mathbb{V}$).

Definition

- $(L, \wedge, \vee, 0, 1)$: bounded lattice
- ${}^{\Delta}, {}^{\nabla} : L \to L$, order–reversing

$(L, {}^{\Delta}) := (L, \wedge, \vee, {}^{\Delta}, 0, 1) \in \mathbb{WCL}$ and ${}^{\Delta}$: *weak complementation* on L iff, for all $x, y \in L$:

- $x^{\Delta\Delta} \leq x$ and
- $(x \wedge y) \vee (x \wedge y^{\Delta}) = x$

$(L, {}^{\nabla}) := (L, \wedge, \vee, {}^{\nabla}, 0, 1) \in \mathbb{DWCL}$ and ${}^{\nabla}$: *dual weak complementation* on L iff, for all $x, y \in L$:

- $x \leq x^{\nabla\nabla}$ and
- $(x \vee y) \wedge (x \vee y^{\nabla}) = x$

$(L, {}^{\Delta}, {}^{\nabla}) := (L, \wedge, \vee, {}^{\Delta}, {}^{\nabla}, 0, 1) \in \mathbb{WDL}$ and $({}^{\Delta}, {}^{\nabla})$: *weak dicomplementation* on L iff:

- $(L, {}^{\Delta}) \in \mathbb{WCL}$ and
- $(L, {}^{\nabla}) \in \mathbb{DWCL}$

Example

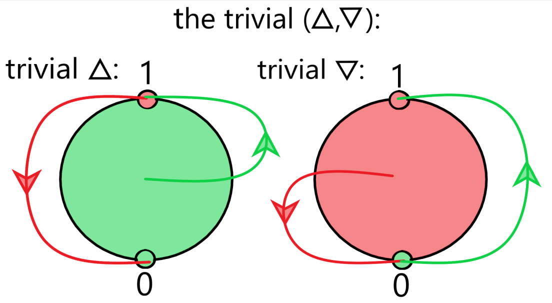
If $(B, \wedge, \vee, \bar{\ }, 0, 1)$: Boolean algebra, then $(B, \bar{\ }, \bar{\ }) \in \mathbb{WDL}$.

Notation

For any $n \in \mathbb{N}^*$: $\mathcal{C}_n :=$ the n–element chain.

Example

If L: bounded lattice, then $(L,^{\Delta},^{\nabla}) \in \mathbb{WDL}$, where:



- If $1 \in \mathrm{Ji}(L)$ (in particular if $L = K \oplus \mathcal{C}_n$ for some bounded lattice K and some $n \in \mathbb{N} \setminus \{0, 1\}$), then the only $^{\Delta}$ on L is the trivial one.
- If $0 \in \mathrm{Mi}(L)$ (in particular if $L = \mathcal{C}_n \oplus K$ for some...), then the only $^{\nabla}$ on L is the trivial one.

Example

- L: complete lattice
- $J, M \subseteq L$, J: join–dense and M: meet–dense in L
- ${}^{\Delta J}, {}^{\nabla M} : L \to L$, for all $x \in L$:

$$x^{\Delta J} = \bigvee (J \setminus (x]) \text{ and } x^{\nabla M} = \bigwedge (M \setminus [x))$$

Then:

- $(L, {}^{\Delta J}, {}^{\nabla M}) \cong \mathcal{B}(J, M, \leq) \in \mathbb{WDL}$: the weakly dicomplemented lattice of the formal concepts of the context $(J, M, \leq)$ (SEE LEONARD KWUIDA'S TALK FROM A COUPLE OF DAYS AGO) and
- $({}^{\Delta J}, {}^{\nabla M})$: *representable* weak dicomplementation on L.

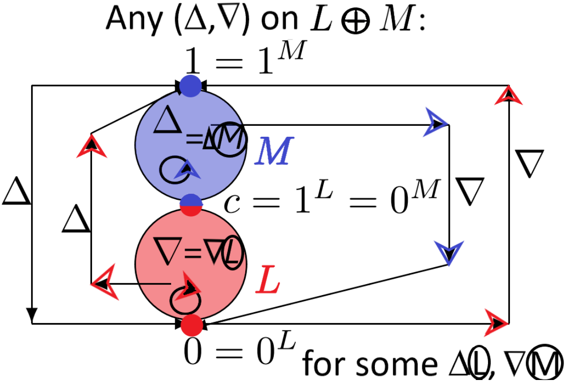
Note that $({}^{\Delta L}, {}^{\nabla L})$ is the trivial weak dicomplementation on L.

Weak dicomplementations on ordinal/glued sums

Now let:

- L, M: bounded lattices with
- $|L| > 1$ and $|M| > 1$.

Then:



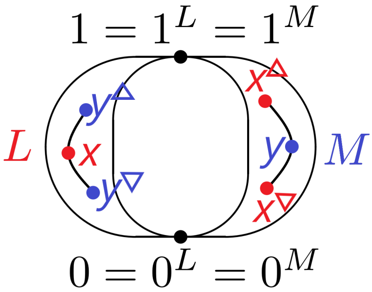
Weak dicomplementations on horizontal sums

Now assume:

- $|L| > 2$ and $|M| > 2$.

Then:

For any (Δ,∇) on $L \boxplus M$:

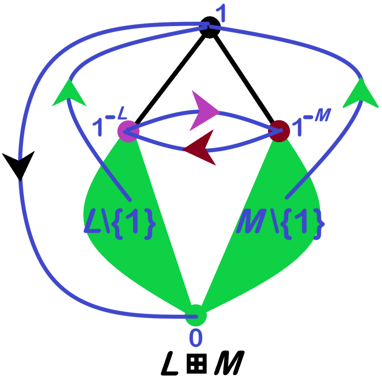


For all $x \in L \setminus \{0,1\}$ and all $y \in M \setminus \{0,1\}$:

- $x^\nabla \leq y \leq x^\Delta$ and
- $y^\nabla \leq x \leq y^\Delta$.

Hence the only ${}^{\Delta}$ on $L \boxplus M$ are:

- the trivial one $(^{\Delta(L \boxplus M)}$ when L and M are complete),
- and, if (and only if) $1 \in \mathrm{Sji}(L) \cap \mathrm{Sji}(M)$, then also the following $(^{\Delta(L \boxplus M) \setminus \{1\}}$ when L and M are complete):



Dually for ${}^{\nabla}$: a (single) nontrivial one exists iff $0 \in \mathrm{Smi}(L) \cap \mathrm{Smi}(M)$, namely the dual of the ${}^{\Delta}$ above: ${}^{\nabla(L \boxplus M) \setminus \{0\}}$ when L and M are complete.

Theorem

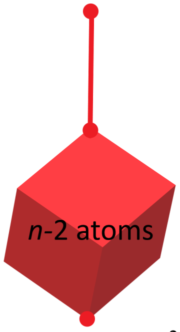
For any $n \in \mathbb{N}^*$, any lattice L with $|L| = n$ and any weak complementation $^\Delta$ on L, we have:

- $|\mathrm{Con}_{\mathbb{WCL}}(L,^\Delta)| \leq 2^{n-2} + 1$;

① $|\mathrm{Con}_{\mathbb{WCL}}(L,^\Delta)| = 2^{n-2} + 1$ iff $\mathrm{Con}_{\mathbb{WCL}}(L,^\Delta) \cong \mathcal{C}_2^{n-2} \oplus \mathcal{C}_2$ iff $n \geq 2$ and $L \cong \mathcal{C}_n$;

$2^{n-2} + 1$

$n \geq 2$



The next largest number of congruences of an *n*-element $L \in \mathbb{WCL}$ (then $\mathbb{WDL}$).

Condition on $n \in \mathbb{N}^*$ for this congruence number to exist.

Shape of L with this number of congruences.

Shape of its congruence lattice.

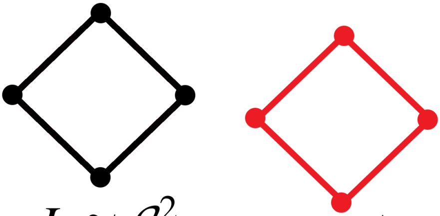
$L \cong \mathcal{C}_n$

$\mathrm{Con}_{\mathbb{WCL}}(L,^\Delta) \cong \mathcal{C}_2^{n-2} \oplus \mathcal{C}_2$

② $\mid \mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \mid = 2^{n-2}$ iff $n = 4$ and $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2$ iff $L \cong \mathcal{C}_2^2$ and $^{\Delta} =^{\Delta L \setminus \{1\}}$ is the Boolean complementation;

$$2^{n-2}$$

$$n = 4$$

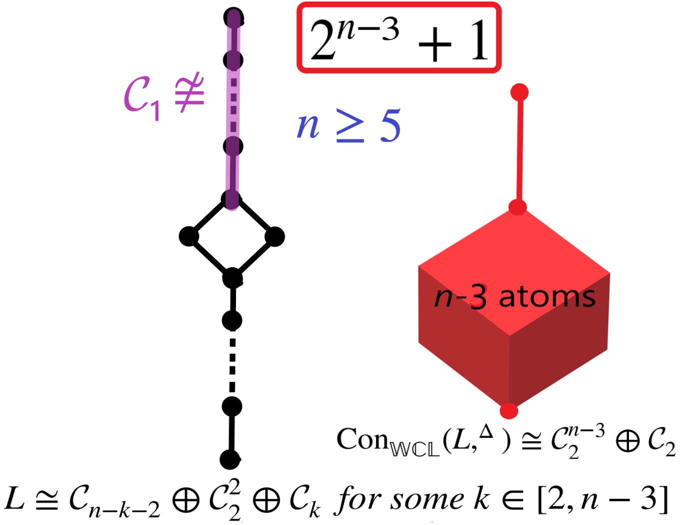


$L \cong \mathcal{C}_2^2$ $\quad \mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2$

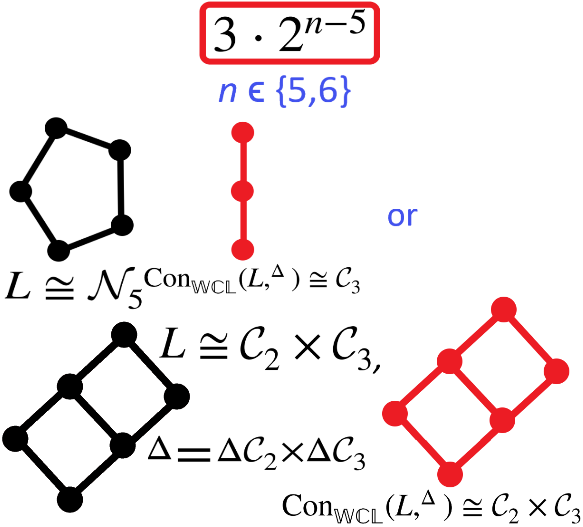
Δ Boolean

- if $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| < 2^{n-2}$, then $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-3}+1$;

③ $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-3}+1$ iff $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^{n-3} \oplus \mathcal{C}_2$ iff $n \geq 5$ and $L \cong \mathcal{C}_{n-k-2} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_k$ for some $k \in [2, n-3]$;

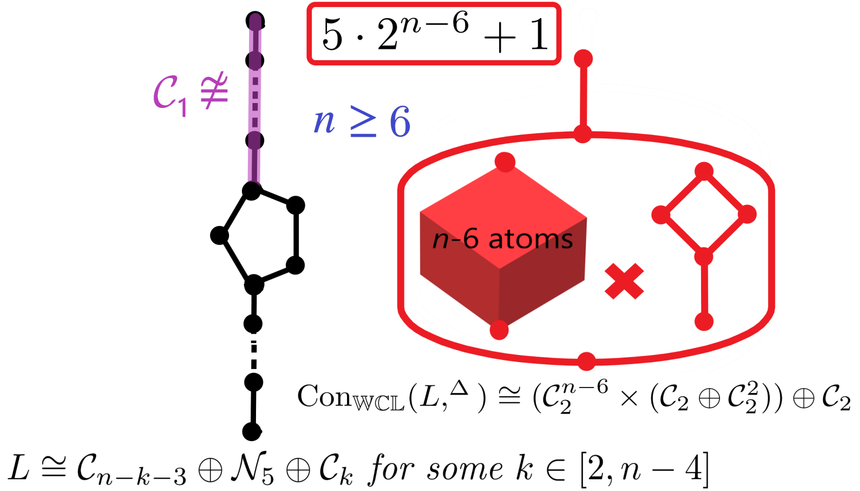


④ $|\,\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\,| = 3\cdot 2^{n-5}$ iff $n=5$ and $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_3$ or $n=6$ and $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2\times\mathcal{C}_3$ iff $L\cong\mathcal{N}_5$ or $L\cong\mathcal{C}_2\times\mathcal{C}_3$ and $^{\Delta} = ^{\Delta\mathcal{C}_2\times\Delta\mathcal{C}_3}$ is the direct product of the trivial weak complementations on the chains $\mathcal{C}_2$ and $\mathcal{C}_3$;



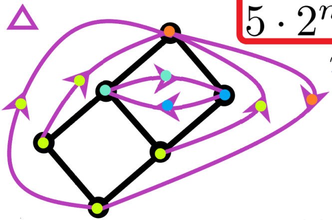
• if $L \ncong \mathcal{N}_5$, $(L,^{\Delta}) \ncong_{\mathbb{WCL}} (\mathcal{C}_2,^{\Delta\mathcal{C}_2}) \times (\mathcal{C}_3,^{\Delta\mathcal{C}_3})$ and $|\,\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\,| \leq 2^{n-3}$, then $|\,\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\,| \leq 5 \cdot 2^{n-6} + 1$;

⑤ $|\,\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\,| = 5 \cdot 2^{n-6} + 1$ iff $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong (\mathcal{C}_2^{n-6} \times (\mathcal{C}_2 \oplus \mathcal{C}_2^2)) \oplus \mathcal{C}_2$ iff $n \geq 6$ and $L \cong \mathcal{C}_{n-k-3} \oplus \mathcal{N}_5 \oplus \mathcal{C}_k$ for some $k \in [2, n-4]$;



⑥ $|\,\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})\,| = 5\cdot 2^{n-6}$ iff $n=6$ and either $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$ or $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$ iff one of the following holds:

- $L \cong \mathcal{C}_2 \times \mathcal{C}_3$ and $^{\Delta} = ^{\Delta L\setminus\{1\}}$, case in which $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$;
- $L \cong \mathcal{C}_3 \boxplus \mathcal{C}_5$ or $L \cong \mathcal{C}_4 \boxplus \mathcal{C}_4$ and $^{\Delta} = ^{\Delta L}$ is trivial, case in which $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$;



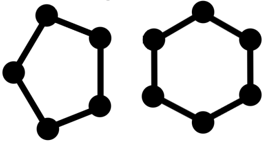
$5\cdot 2^{n-6}$

$n=6$

$\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2 \oplus \mathcal{C}_2^2$

$L \cong \mathcal{C}_2 \times \mathcal{C}_3$ *and* $^{\Delta} = ^{\Delta L\setminus\{1\}}$

or



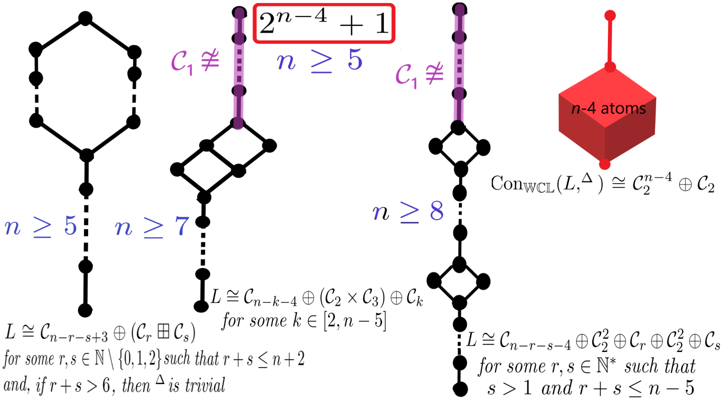
$L \cong \mathcal{C}_3 \boxplus \mathcal{C}_5$ *or* $L \cong \mathcal{C}_4 \boxplus \mathcal{C}_4$ *and* $^{\Delta}$ *is trivial*

$\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^2 \oplus \mathcal{C}_2$

- if $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| < 5 \cdot 2^{n-6}$, then $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| \leq 2^{n-4}+1$;

(7) $|\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta})| = 2^{n-4}+1$ iff $n \geq 5$ and $\mathrm{Con}_{\mathbb{WCL}}(L,^{\Delta}) \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$ iff one of the following holds:

- $n \geq 5$, $L \cong \mathcal{C}_{n-r-s+3} \oplus (\mathcal{C}_r \boxplus \mathcal{C}_s)$ for some $r, s \in \mathbb{N} \setminus \{0,1,2\}$ such that $r+s \leq n+2$ and, if $r+s > 6$ (that is if $L \ncong \mathcal{C}_{n-3} \oplus \mathcal{C}_2^2$), then $^{\Delta} = ^{\Delta L}$ is trivial;
- $n \geq 7$ and $L \cong \mathcal{C}_{n-k-4} \oplus (\mathcal{C}_2 \times \mathcal{C}_3) \oplus \mathcal{C}_k$ for some $k \in [2, n-5]$;
- $n \geq 8$ and $L \cong \mathcal{C}_{n-r-s-4} \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_r \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_s$ for some $r, s \in \mathbb{N}^*$ such that $s > 1$ and $r+s \leq n-5$.



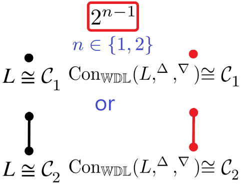
Dually in $\mathbb{DWCL}$.

Corollary

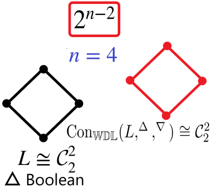
For any $n \in \mathbb{N}^*$, any lattice L with $\mid L \mid = n$ and any weak dicomplementation $(^{\Delta}, ^{\nabla})$ on L, we have:

- $\mid \mathrm{Con}_{\mathbb{WDL}}(L, ^{\Delta}, ^{\nabla}) \mid \leq 2^{n-1}$;

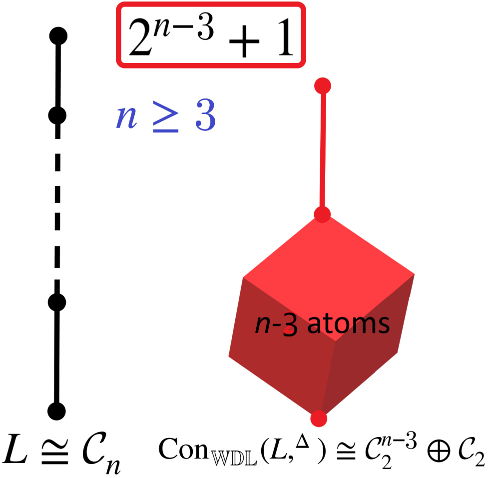
① $\mid \mathrm{Con}_{\mathbb{WDL}}(L, ^{\Delta}, ^{\nabla}) \mid = 2^{n-1}$ iff $n \in \{1, 2\}$;



② $\mid \mathrm{Con}_{\mathbb{WDL}}(L, ^{\Delta}, ^{\nabla}) \mid = 2^{n-2}$ iff $n = 4$ and $\mathrm{Con}_{\mathbb{WDL}}(L, ^{\Delta}, ^{\nabla}) \cong \mathcal{C}_2^2$ iff $L \cong \mathcal{C}_2^2$ and $^{\Delta} = ^{\nabla}$ is the Boolean complementation;



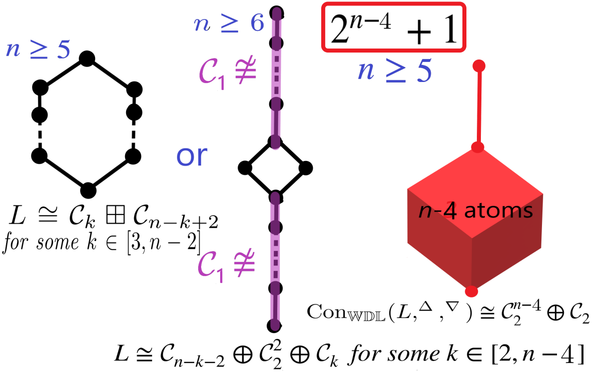
• if $L \ncong \mathcal{C}_2^2$ or its weak dicomplementation is not Boolean, then:
$|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| < 2^{n-1}$ iff $|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| \leq 2^{n-3}+1$;
③ $|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| = 2^{n-3}+1$ iff $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla}) \cong \mathcal{C}_2^{n-3} \oplus \mathcal{C}_2$ iff $n \geq 3$ and $L \cong \mathcal{C}_n$;



- if $|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| \leq 2^{n-3}$, then $|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| \leq 2^{n-4}+1$;

④ $|\,\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla})\,| = 2^{n-4}+1$ iff $\mathrm{Con}_{\mathbb{WDL}}(L,^{\Delta},^{\nabla}) \cong \mathcal{C}_2^{n-4} \oplus \mathcal{C}_2$ iff one of the following holds:

- $n \geq 5$, $L \cong \mathcal{C}_k \boxplus \mathcal{C}_{n-k+2}$ for some $k \in [3, n-2]$ and $(^{\Delta},^{\nabla})$ is the trivial weak dicomplementation on L;
- $n \geq 6$ and $L \cong \mathcal{C}_k \oplus \mathcal{C}_2^2 \oplus \mathcal{C}_{n-k-2}$ for some $k \in [2, n-4]$.



THANK YOU FOR YOUR ATTENTION !