Tolerances on posets

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- Tolerances on posets need not form a lattice
- Quotients of posets by tolerances
- Quotient tolerances
- Analogies of the Isomorphism Theorem

Tolerances

Tolerances

Definition 1

A tolerance T on a universal algebra $\mathbf{A} = (A, F)$ is a reflexive and symmetric binary relation on A that is compatible with all fundamental operations on $\mathbf{A}$. Let $\mathrm{Tol}\,\mathbf{A}$ denote the set of all tolerances on $\mathbf{A}$. $(\mathrm{Tol}\,\mathbf{A}, \subseteq)$ forms a complete lattice. A block of T is a maximal subset B of A satisfying $B^2 \subseteq T$. Let A/T denote the set of all blocks of T.

The congruences on $\mathbf{A}$ are exactly the transitive tolerances on $\mathbf{A}$. The set A is the union of all blocks of T. Different blocks of T may overlap.

Tolerances on lattices

Tolerances on lattices

Lemma 2

(G. Czédli 1982) *Let* $\mathbf{L} = (L, \vee, \wedge)$ *be a lattice,* $T \in \operatorname{Tol}\mathbf{L}$ *and* $B_1, B_2 \in L/T$*. Then there exists a unique block of* T *including* $\{b_1 \vee b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ *and a unique block of* T *including* $\{b_1 \wedge b_2 \mid b_1 \in B_1, b_2 \in B_2\}$*.*

Definition 3

(G. Czédli 1982) *For every lattice* $\mathbf{L} = (L, \vee, \wedge)$ *and every* $T \in \operatorname{Tol}\mathbf{L}$ *let* $\mathbf{L}/T$ *denote the algebra* $(L/T, \vee, \wedge)$ *of type* $(2,2)$ *where for all* $B_1, B_2 \in L/T$ $B_1 \vee B_2$ *is the unique block of* T *including* $\{b_1 \vee b_2 \mid b_1 \in B_1, b_2 \in B_2\}$ *and* $B_1 \wedge B_2$ *is the unique block of* T *including* $\{b_1 \wedge b_2 \mid b_1 \in B_1, b_2 \in B_2\}$*.*

Theorem 4

(G. Czédli 1982) *For every lattice* $\mathbf{L}$ *and every* $T \in \operatorname{Tol}\mathbf{L}$ *the algebra* $\mathbf{L}/T$ *is a lattice.*

Tolerances on lattices, continued

Theorem 5

(G. Czédli 1982) *Every lattice can be embedded into the quotient lattice of a distributive lattice by a suitable tolerance. Every finite lattice is isomorphic to the quotient lattice of a finite distributive lattice by a suitable tolerance.*

Definition 6

A poset is said to be of finite length if the cardinalities of its chains are bounded by a fixed integer.

Tolerances on lattices, continued

Theorem 7

(J. Grygiel and S. Radeleczki 2013) *On the set of tolerances on a lattice* $\mathbf{L}$ *of finite length a partial order relation* $\leq$ *can be introduced in such a way that for* $S, T \in \operatorname{Tol}\mathbf{L}$ *with* $S \leq T$ *there can be defined* $T/S \in \operatorname{Tol}(\mathbf{L}/S)$ *such that the Isomorphism Theorem for tolerances*

$$(\mathbf{L}/S)/(T/S) \cong \mathbf{L}/T$$

holds.

Tolerances on posets

Tolerances on posets

Definition 8

A *tolerance on a poset* $\mathbf{P} = (P, \leq)$ *is a reflexive and symmetric binary relation* T *on* P *satisfying the following conditions:*

(i) *If* $(x, y), (z, u) \in T$ *and* $x \vee z$ *and* $y \vee u$ *exist then* $(x \vee z, y \vee u) \in T$,

(ii) *if* $(x, y), (z, u) \in T$ *and* $x \wedge z$ *and* $y \wedge u$ *exist then* $(x \wedge z, y \wedge u) \in T$,

(iii) *if* $x, y, z \in P$ *and* $(x, y), (y, z) \in T \neq P^2$ *then there exist* $u, v \in P$ *with* $u \leq x, y, z \leq v$ *and* $(u, y), (y, v) \in T$,

(iv) *if* $(x, y) \in T \neq P^2$ *then there exists some* $(z, u) \in T$ *with both* $z \leq x, y \leq u$ *and* $(v, z), (v, u) \in T$ *for all* $v \in P$ *with* $(v, x), (v, y) \in T$.

Let $\operatorname{Tol}\mathbf{P}$ *denote the set of all tolerances on* $\mathbf{P}$*. The* *tolerances* $\{(x, x) \mid x \in P\}$ *and* P^2 *are called* *trivial*. *A* *block* *of* T *is a maximal subset* B *of* P *satisfying* $B^2 \subseteq T$*. Let* P/T *denote the set of all blocks of* T*. A* *congruence* *on* $\mathbf{P}$ *is a transitive tolerance on* $\mathbf{P}$*. Let* $\operatorname{Con}\mathbf{P}$ *denote the set of all congruences on* $\mathbf{P}$.

Tolerances on posets, continued

Conditions (iii) and (iv) are quite natural since they are satisfied by every tolerance on a lattice. In condition (iii) one can take $u := x \wedge y \wedge z$ and $v := x \vee y \vee z$, and in condition (iv) one can take $z := x \wedge y$ and $u := x \vee y$.

Tolerances on posets, continued

Lemma 9

Let $\mathbf{P} = (P, \leq)$ *be a poset,* $T \in \operatorname{Tol}\mathbf{P}$, $a, b \in P$ *with* $a \leq b$ *and* $B \in P/T$. *Then the following holds:*

(i) *If* $(a, b) \in T$ *then* $[a, b]^2 \subseteq T$,

(ii) *if* B *has bottom element* a *and top element* b *then* $B = [a, b]$.

Definition 10

Let $(P, \leq)$ *be a poset and* $A \subseteq P$. *Then* A *is called*

- *directed if for every* $x, y \in A$ *there exist* $z, u \in A$ *with* $z \leq x, y \leq u$,
- *convex if for all* $x, y \in A$ *with* $x \leq y$ *we have* $[x, y] \subseteq A$.

Properties of blocks of tolerances on posets

Theorem 11

Every block of a non-trivial tolerance on a poset is directed and convex.

Definition 12

A poset is said to be of finite height if it does not contain an infinite chain.

Corollary 13

Every block of a non-trivial tolerance on a poset of finite height, especially every block of a non-trivial tolerance on a finite poset is an interval of the form $[a, b]$ with $a \leq b$.

Tolerances on relatively complemented posets

Relatively complemented posets

Definition 14

A poset $(P, \leq)$ is called relatively complemented if for all $x, y, z \in P$ with $x \leq y \leq z$ there exists some $u \in P$ with $y \vee u = z$ and $y \wedge u = x$.

Theorem 15

Any tolerance on a relatively complemented poset is a congruence.

A bounded relatively complemented poset

Example 16

The poset depicted in Figure 1 is relatively complemented, but not a lattice:

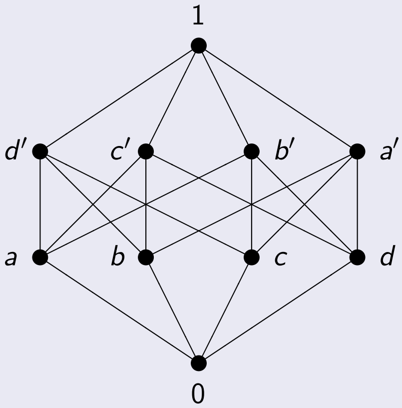


Fig. 1

A relatively complemented non-directed poset

Example 17

The poset **P** *visualized in Figure 2 is relatively complemented, but not directed:*

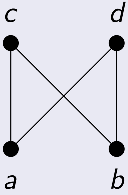


Fig. 2

Example, continued

Example 18

P *has the following congruences:*

$$\begin{aligned}
C_1 &= \{a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d\}^2, \\
C_2 &= \{a\}^2 \cup \{c\}^2 \cup \{b, d\}^2, \\
C_3 &= \{a\}^2 \cup \{d\}^2 \cup \{b, c\}^2, \\
C_4 &= \{b\}^2 \cup \{c\}^2 \cup \{a, d\}^2, \\
C_5 &= \{b\}^2 \cup \{d\}^2 \cup \{a, c\}^2, \\
C_6 &= \{a, c\}^2 \cup \{b, d\}^2, \\
C_7 &= \{a, d\}^2 \cup \{b, c\}^2, \\
C_8 &= \{a, b, c, d\}^2.
\end{aligned}$$

Example, continued

Example 19

The poset $(\text{Con}\,\mathbf{P}, \subseteq)$ *is depicted in Figure 3:*

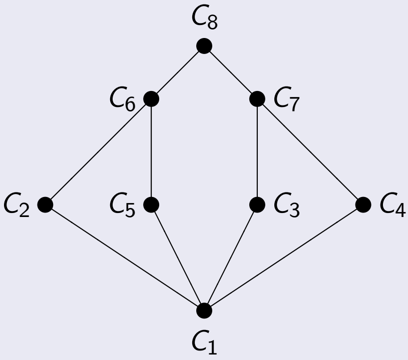


Fig. 3

Tolerances on posets need not form a lattice

Tolerances on posets need not form a lattice

The following two examples show that the intersection of two tolerances need not be a tolerance and that the posets $(\mathrm{Tol}\,\mathbf{P}, \subseteq)$ and $(\mathrm{Con}\,\mathbf{P}, \subseteq)$ need not form lattices.

Example

Example 20

Consider the poset **P** *depicted in Figure 4:*

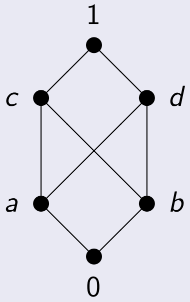


Fig. 4

Example, continued

Example 21

Then

$$T_1 = [0, c]^2 \cup [b, 1]^2,$$
$$T_2 = [0, d]^2 \cup [a, 1]^2$$

are tolerances, but not congruences on **P** *and*

$$T_1 \cap T_2 = \{0, a, b\}^2 \cup \{a, c\}^2 \cup \{b, d\}^2 \cup \{c, d, 1\}^2,$$

is not a tolerance on **P** *since* $\{0, a, b\}$ *and* $\{c, d, 1\}$ *are not directed.*

Example

Example 22

Consider the poset **P** *visualized in Figure 5:*

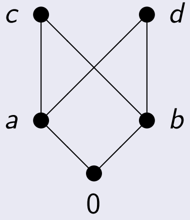


Fig. 5

Example, continued

Example 23

Then

$$T_1 = \{0, a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d\}^2,$$
$$T_2 = \{0, b\}^2 \cup \{a\}^2 \cup \{c\}^2 \cup \{d\}^2,$$
$$T_3 = [0, c]^2 \cup \{d\}^2,$$
$$T_4 = [0, d]^2 \cup \{c\}^2$$

are congruences on $\mathbf{P}$ *and* T_3 *and* T_4 *are minimal upper bounds of* $\{T_1, T_2\}$ *in* $(\operatorname{Tol}\mathbf{P}, \subseteq)$ *and hence also in* $(\operatorname{Con}\mathbf{P}, \subseteq)$. *Therefore* $T_1 \vee T_2$ *does not exist neither in* $(\operatorname{Tol}\mathbf{P}, \subseteq)$ *nor in* $(\operatorname{Con}\mathbf{P}, \subseteq)$ *showing that neither* $(\operatorname{Tol}\mathbf{P}, \subseteq)$ *nor* $(\operatorname{Con}\mathbf{P}, \subseteq)$ *forms a lattice.*

Quotients of posets by tolerances

Ordering of blocks

Definition 24

Let $\mathbf{P} = (P, \leq)$ *be a poset,* $T \in \operatorname{Tol}\mathbf{P}$ *and* $B_1, B_2 \in P/T$. *We define* $B_1 \sqsubseteq B_2$ *if*

- *for every* $b_1 \in B_1$ *there exists some* $b_2' \in B_2$ *with* $b_1 \leq b_2'$ *and*
- *for every* $b_2 \in B_2$ *there exists some* $b_1' \in B_1$ *with* $b_1' \leq b_2$.

It is easy to see that if B_1 and B_2 are intervals of the form $[a, b]$ and $[c, d]$, respectively, then $B_1 \sqsubseteq B_2$ if and only if $a \leq c$ and $b \leq d$.

Theorem 25

Let $\mathbf{L}$ *be a lattice and* $T \in \operatorname{Tol}\mathbf{L}$. *Then the relation* $\sqsubseteq$ *is the partial order relation induced by* $\mathbf{L}/T$.

Theorem 26

Let $\mathbf{P}$ *be a poset and* $T \in \operatorname{Tol}\mathbf{P}$. *Then* $\mathbf{P}/T := (P/T, \sqsubseteq)$ *is again a poset.*

Example of a quotient poset

Example 27

Consider the poset **P** *depicted in Figure 4 and the following tolerances on* **P***:*

$$T_1 = [0, c]^2 \cup [b, 1]^2 = B_1^2 \cup B_2^2,$$
$$T_2 = \{0, a\}^2 \cup \{b, c\}^2 \cup \{d, 1\}^2 = C_1^2 \cup C_2^2 \cup C_3^2.$$

Example of a quotient poset, continued

Example 28

Then $\mathbf{P}/T_i$ $(i = 1, 2)$ *are visualized in Figure 6:*

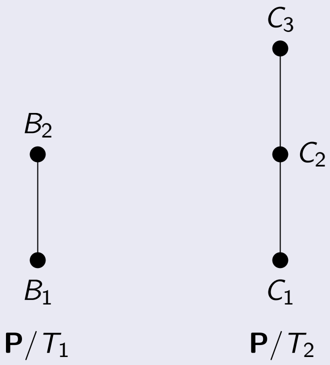


Fig. 6

Quotient tolerances

Quotient tolerances

Definition 29

Let $\mathbf{P}$ *be a poset and* $S, T \in \operatorname{Tol}\mathbf{P}$. *We say that* $S \leq T$ *if the following conditions hold:*

- *Every block of* S *is included in exactly one block of* T,
- *every block of* T *is a union of blocks of* S.

Note that the first condition implies $S \subseteq T$. It is easy to see that $\leq$ is reflexive and antisymmetric.

Definition 30

For every poset $\mathbf{P} = (P, \leq)$ *and any* $S, T \in \operatorname{Tol}\mathbf{P}$ *with* $S \leq T$ *we define* $T/S := \{(B_1, B_2) \in (P/S)^2 \mid$ *there exists some* $B_3 \in P/T$ *with* $B_1, B_2 \subseteq B_3\}$.

Analogies of the Isomorphism Theorem

Analogies of the Isomorphism Theorem

Lemma 31

Let $\mathbf{P}$ *be a poset and* $S, T \in \operatorname{Tol} \mathbf{P}$ *with* $S \leq T$. *Then* T/S *is reflexive and symmetric.*

Theorem 32

Let $\mathbf{P} = (P, \leq)$ *be a poset and* $S, T \in \operatorname{Tol} \mathbf{P}$ *with* $S \leq T$. *Further assume that* $(T/S, \sqsubseteq)$ *satisfies* (i) – (iv) *of Definition 8. Then*

(i) $T/S \in \operatorname{Tol}(\mathbf{P}/S)$,

(ii) $|P/T| \leq |(P/S)/(T/S)|$.

Example for $\mathbf{P}/T \cong (\mathbf{P}/S)/(T/S)$

The following example demonstrates that sometimes we have $|P/T| = |(P/S)/(T/S)|$.

Example 33

Let $\mathbf{P}$ *denote the poset visualized in Figure 5 and put*

$$S = \{0, a\}^2 \cup \{b, c\}^2 \cup \{d\}^2 = B_1^2 \cup B_2^2 \cup B_3^2,$$
$$T = [0, c]^2 \cup \{d\}^2 = C_1^2 \cup C_2^2.$$

Example, continued

Example 34

Then $S, T \in \operatorname{Tol}\mathbf{P}$, $S \leq T$ *and* $\mathbf{P}/S$ *and* $\mathbf{P}/T$ *look as follows:*

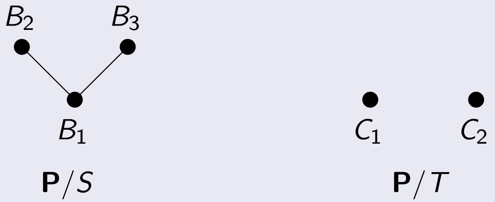


Fig. 7

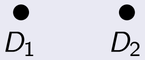
Example, continued

Example 35

Further,

$$T/S = \{B_1, B_2\}^2 \cup \{B_3\}^2 = D_1^2 \cup D_2^2 \in \mathrm{Tol}(\mathbf{P}/S),$$

and $(\mathbf{P}/S)/(T/S)$ *looks as follows:*



$(\mathbf{P}/S)/(T/S)$

Fig. 8

This shows $\mathbf{P}/T \cong (\mathbf{P}/S)/(T/S)$.

The case $|P/T| = |(P/S)/(T/S)|$

Theorem 36

Let $\mathbf{P} = (P, \leq)$ *be a poset and* $S, T \in \operatorname{Con}\mathbf{P}$ *with* $S \leq T$ *and assume* $T/S \in \operatorname{Tol}(\mathbf{P}/S)$. *Then there exists a bijective order-preserving mapping from* $(\mathbf{P}/S)/(T/S)$ *onto* $\mathbf{P}/T$.

If we apply the proof of Theorem 36 to Example 33 we obtain $f(D_i) = C_i$ for $i = 1, 2$. That the inverse of the bijective order-preserving mapping mentioned in Theorem 36 need not be order-preserving is demonstrated by the following example.

Final example

Example 37

Let **P** *be the poset depicted in Figure 4 and put*

$$S = \{0, a\}^2 \cup \{b\}^2 \cup \{c\}^2 \cup \{d, 1\}^2 = B_1^2 \cup B_2^2 \cup B_3^2 \cup B_4^2,$$
$$T = \{0, a\}^2 \cup \{b, c\}^2 \cup \{d, 1\}^2 = C_1^2 \cup C_2^2 \cup C_3^2.$$

Final example, continued

Example 38

Then $S, T \in \operatorname{Con} \mathbf{P}$, $S \leq T$ *and* $\mathbf{P}/S$ *and* $\mathbf{P}/T$ *look as follows:*

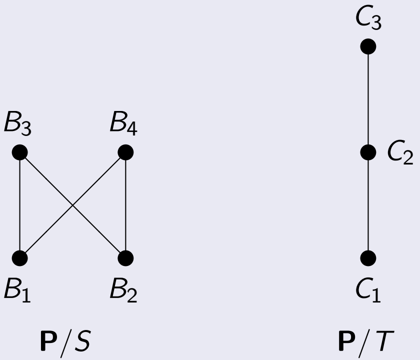


Fig. 9

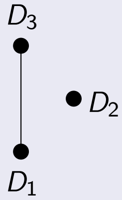
Final example, continued

Example 39

Further,

$$T/S = \{B_1\}^2 \cup \{B_2, B_3\}^2 \cup \{B_4\}^2 = D_1^2 \cup D_2^2 \cup D_3^2 \in \mathrm{Tol}(\mathbf{P}/S),$$

and $(\mathbf{P}/S)/(T/S)$ *looks as follows:*



$(\mathbf{P}/S)/(T/S)$

Fig. 10

Final example, continued

Example 40

The mapping f from the proof of Theorem 36 maps D_i onto C_i for $i = 1, 2, 3$. Since $C_1 \sqsubseteq C_2$, but $f^{-1}(C_1) = D_1 \not\sqsubseteq D_2 = f^{-1}(C_2)$, the mapping f^{-1} is not order-preserving. Even more, there does not exist a bijective order-preserving mapping from $\mathbf{P}/T$ to $(\mathbf{P}/S)/(T/S)$. Hence, the Isomorphism Theorem for tolerances on posets does not hold in general even in the case when the tolerances in question are congruences.

References I

[1] I. Chajda, Algebraic Theory of Tolerance Relations: Palacký Univ. Press, Olomouc 1991. ISBN 80-7067-042-8.

[2] I. Chajda, G. Czédli and R. Halaš, Independent joins of tolerance factorable varieties. Algebra Universalis **69** (2013), 83–92.

[3] I. Chajda, G. Czédli, R. Halaš and P. Lipparini, Tolerances as images of congruences in varieties defined by linear identities. Algebra Universalis **69** (2013), 167–169.

[4] I. Chajda and H. Länger, Filters and congruences in sectionally pseudocomplemented lattices and posets. Soft Computing **25** (2021), 8827–837.

[5] I. Chajda and H. Länger, Tolerances on posets. Miskolc Math. Notes **24** (2023), 725–736.

References II

[6] I. Chajda, J. Niederle and B. Zelinka, On existence conditions for compatible tolerances. Czechoslovak Math. J. **26** (1976), 304–311.

[7] I. Chajda and B. Zelinka, Tolerance relation on lattices. Časopis Pěst. Mat. **99** (1974), 394–399.

[8] G. Czédli, Factor lattices by tolerances. Acta Sci. Math. (Szeged) **44** (1982), 35–42.

[9] J. Grygiel and S. Radelecki, On the tolerance lattice of tolerance factors. Acta Math. Hungar. **141** (2013), 220–237.

Thank you for your attention!