Quantum Suplattices

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SSAOS 2023
September 4, 2023

A very short summary

In the context of supremum-preserving functions as morphisms, complete lattices are called suplattices.

We introduce a noncommutative version of complete lattices, which we call quantum suplattices, which:

- are obtained via a scheme called discrete quantization;
- are algebras for monads that are quantum versions of the power set monad and the lower set monad;
- are not generalizations of ordinary suplattices;
- satisfy usual theorems for ordinary suplattices such as the existence of Galois connections and the Knaster-Tarski Theorem;
- Lead to possible quantum versions of topological spaces.

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Non-commutative mathematics

- A program to obtain natural models of quantum structures;
- Main idea: algebras of operators on a Hilbert space H can be used to construct 'non-commutative' generalizations of classical structures;
- Example: $X \mapsto C(X)$ yields a categorical duality between the categories of compact Hausdorff spaces and of commutative unital C*-algebras (Gelfand duality);
- Hence the dual of the category of unital C*-algebras can be regarded as the category of 'non-commutative' compact Hausdorff spaces.
- Quantization is the process of finding noncommutative versions of a mathematical structure.

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Background

- Duan, Severini, Winter: quantum graphs in quantum error correction;
- Kuperberg and Weaver: quantization of metric spaces; quantum hamming metric in quantum error correction
- Weaver: identification of quantum relations as underlying structure of quantum metric spaces and quantum graphs;
- Weaver: quantum posets;
- Kornell: quantum sets and their categorical properties;
- Kornell, L., Mislove: categorical structure of quantum posets;
- Kornell, L., Mislove: quantum cpos and their application in the semantics of quantum programming languages.

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Internalization

Internalization is the process of generalizing set-theoretic constructions that can be defined in terms of the categorical structure of **Set** or **Rel** to other categories that posses the same categorical structure needed for these constructions. Example: in any category with all finite products, a group G is an object equipped with morphisms $m : G \times G \to G$, $e : 1 \to G$, and $(-)^{-1} : G \to G$ such that:

- Unitality: $$\begin{array}{ccc} G\times 1 & \xrightarrow{\mathrm{id}_G\times e} & G\times G \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow[=]{} & G \end{array} \qquad \begin{array}{ccc} 1\times G & \xrightarrow{e\times\mathrm{id}_G} & G\times G \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle m} \\ G & \xrightarrow[=]{} & G \end{array}$$
- Associativity: $$\begin{array}{ccc} G\times G\times G & \xrightarrow{m\times\mathrm{id}_G} & G\times G \\ {\scriptstyle \mathrm{id}_G\times m}\downarrow & & \downarrow{\scriptstyle m} \\ G\times G & \xrightarrow[m]{} & G \end{array}$$
- Inverses: $$\begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}_G} & G\times G & \xrightarrow{\mathrm{id}_G\times(-)^{-1}} & G\times G \\ {\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle m} \\ 1 & & \xrightarrow[e]{} & & G \end{array} \qquad \begin{array}{ccccc} G & \xrightarrow{\mathrm{diag}_G} & G\times G & \xrightarrow{(-)^{-1}\times\mathrm{id}_G} & G\times G \\ {\scriptstyle !}\downarrow & & & & \downarrow{\scriptstyle m} \\ 1 & & \xrightarrow[e]{} & & G \end{array}$$

Groups in **Top** are topological groups, groups in **SmoothManifolds** are Lie groups.

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Quantization by internalization

- We employ a method of quantization by internalizing structures in a suitable category of C*-algebras whose objects are noncommutative generalizations of sets;
- In general, one can internalize functions in a category resembling **Rel**, whereas binary relations cannot always be internalized in a category resembling **Set**;
- Therefore, our category of operator algebras should be a noncommutative generalization of the category **Rel**;
- The dual of the category **WStar** of von Neumann algebras can be regarded the category of 'non-commutative' measure spaces.
- Weaver: quantum relations between von Neumann algebras are certain operator spaces generalizing measurable binary relations[1].

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Hereditarily atomic von Neumann algebras

- Hereditarily atomic von Neumann algebras are von Neumann algebras isomorphic to $\bigoplus_{i\in I} L(H_i)$ with H_i a finite-dimensional Hilbert space, and can be used as non-commutative generalizations of sets;
- The category **WRel** of von Neumann algebras and quantum relations is a quantaloid (**Sup**-enriched category) with a dagger;
- Its full subcategory $\mathbf{WRel}_{\mathrm{HA}}$ of hereditarily atomic von Neumann algebras is a dagger compact quantaloid[2] just like **Rel**.
- Discrete quantization is the process of internalizing mathematical structures in $\mathbf{WRel}_{\mathrm{HA}}$;
- Compare: fuzzification can be regarded as internalizing structures in V-**Rel** for a quantale V such as $[0,1]$;
- $\mathbf{WRel}_{\mathrm{HA}}$ is equivalent to a category **qRel** of quantum sets, which are essentially families of finite-dimensional Hilbert spaces called atoms;
- We have a fully faithful functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ preserving the dagger structure and the order between relations.

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Quantum sets and functions

A morphism $f : X \to Y$ in **Rel** is a function if and only if $f^\dagger \circ f \geq 1_X$ and $f \circ f^\dagger \leq 1_Y$.

Definition

A quantum function $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a quantum relation satisfying $F^\dagger \circ F \geq I_\mathcal{X}$ and $F \circ F^\dagger \leq I_\mathcal{Y}$. The category of quantum sets and functions is denoted by **qSet**.

- **qSet** is complete, cocomplete and symmetric monoidal closed[3];
- The assignment $\mathcal{X} \mapsto \ell^\infty(\mathcal{X}) := \bigoplus_{X \in \mathcal{X}} L(X)$ extends to a duality between **qSet** and the category $\mathbf{WStar}_{\mathrm{HA}}$ of hereditarily atomic von Neumann algebras and normal unital $*$-homomorphisms;
- $`(-)$ restricts to a fully faithful functor **Set** $\to$ **qSet**;
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A morphism $f : X \to Y$ in **Rel** is a function if and only if $f^\dagger \circ f \geq 1_X$ and $f \circ f^\dagger \leq 1_Y$.

Definition

A quantum function $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a quantum relation satisfying $F^\dagger \circ F \geq I_\mathcal{X}$ and $F \circ F^\dagger \leq I_\mathcal{Y}$. The category of quantum sets and functions is denoted by **qSet**.

- **qSet** is complete, cocomplete and symmetric monoidal closed[3];
- The assignment $\mathcal{X} \mapsto \ell^\infty(\mathcal{X}) := \bigoplus_{X \in \mathcal{X}} L(X)$ extends to a duality between **qSet** and the category $\mathbf{WStar}_{\mathrm{HA}}$ of hereditarily atomic von Neumann algebras and normal unital $*$-homomorphisms;
- '$(-)$ restricts to a fully faithful functor **Set** $\to$ **qSet**;
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Quantum posets

An order $\sqsubseteq$ on a set X is a binary relation such that $1_X \leq (\sqsubseteq)$, $(\sqsubseteq \circ \sqsubseteq) \leq (\sqsubseteq)$, and $(\sqsubseteq) \wedge (\sqsupseteq) = 1_X$, where $\sqsupseteq := \sqsubseteq^\dagger$.

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Proposition

The functors $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ *and* $`(-) : \mathbf{Set} \to \mathbf{qSet}$ *induce a fully faithful functor* $`(-) : \mathbf{Pos} \to \mathbf{qPos}$, $(X, \sqsubseteq) \mapsto (`X, `\sqsubseteq)$.

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Lower sets, suplattices and monotone relations

(1) Suplattices are the algebras of the lower set monad D on **Pos**;

(2) Suplattices are posets X such that the canonical embedding $X \to D(X)$, $x \mapsto \downarrow x$ has a lower Galois adjoint $\bigvee$.

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A monotone relation $r : X \to Y$ between posets is a binary relation such that $(x_1, y_1) \in r$ implies $(x_2, y_2) \in r$ for each $x_1 \leq x_2$ in X and each $y_1 \geq y_2$ in Y.

- Any monotone relation $r : X \to Y$ corresponds to a monotone function $X^{\mathrm{op}} \times Y \to 2$, so to a 2-enriched profunctor when X and Y are regarded as 2-enriched categories;
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Monotone relations between quantum posets

Definition

A monotone relation $R : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ between quantum posets is a binary relation $R : \mathcal{X} \to \mathcal{Y}$ such that $\succcurlyeq_{\mathcal{Y}} \circ R \leq R$ and $R \circ \succcurlyeq_{\mathcal{X}} \leq R$.

Theorem

The category **qRelPos** *of quantum posets and monotone relations is compact closed.*

Theorem

The embedding **qPos** $\to$ **qRelPos** *has a right adjoint; its induced monad* $\mathcal{D}$ *is called the quantum lower set monad.*

The existence of right adjoints of embeddings **Pos** $\to$ **RelPos**, **Set** $\to$ **Rel**, **qSet** $\to$ **qRel** and **qPos** $\to$ **qRelPos** can all be proven in one scheme involving the embedding of a symmetric monoidal closed category **S** into a compact closed category **R**.

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Galois connections between quantum posets

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The pointwise order $\sqsubseteq_{\mathcal{Y}}$ of functions $F, G : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ is a quantum set and $\mathcal{Y}$ is a quantum poset ordered by $\preccurlyeq$ is defined by $F \sqsubseteq_{\mathcal{Y}} G$ if and only if $F \leq \succcurlyeq \circ G$.

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A Galois connection between quantum posets $(\mathcal{X}, \preccurlyeq_{\mathcal{X}})$ and $(\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ consists of a pair of monotone maps $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{X}$ such that $I_{\mathcal{X}} \sqsubseteq_{\mathcal{X}} G \circ F$ and $F \circ G \sqsubseteq_{\mathcal{Y}} I_{\mathcal{Y}}$. F is called the lower Galois adjoint of G.

Also in the quantum case there is a concept of closure operators related to Galois connections:

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The unit of the **qPos**/**qRelPos** adjunction yields a canonical order embedding $\mathcal{X} \to \mathcal{D}(\mathcal{X})$; a quantum generalization of the order embedding $X \to D(X)$, $x \mapsto \downarrow x$ for ordinary posets X.

Definition

A quantum poset $(\mathcal{X}, \preccurlyeq_{\mathcal{X}})$ is called a quantum suplattice if the canonical order embedding $\mathcal{X} \to \mathcal{D}(\mathcal{X})$ has a lower Galois adjoint $\bigvee_{\mathcal{X}}$. A monotone map $F : (\mathcal{X}, \preccurlyeq_{\mathcal{X}}) \to (\mathcal{Y}, \preccurlyeq_{\mathcal{Y}})$ between quantum suplattices is called a sup-homomorphism if $F \circ \bigvee_{\mathcal{X}} = \bigvee_{\mathcal{Y}} \circ \mathcal{D}(F)$. The category of quantum suplattices and sup-homomorphisms is denoted by **qSup**.

Example

Let $\mathcal{X}$ be a quantum poset. Then $\mathcal{D}(\mathcal{X})$ is a quantum suplattice where $\bigvee_{\mathcal{D}(\mathcal{X})}$ is the multiplication $\mathcal{D}^2(\mathcal{X}) \to \mathcal{D}(\mathcal{X})$.

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qSup *is equivalent to the Eilenberg-Moore category of* $\mathcal{D}$.

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Suplattices are not quantum suplattices

Proposition

The fully faithful functor $`(-) : \mathbf{Pos} \to \mathbf{qPos}$ *does not restrict and corestrict to a functor* $\mathbf{Sup} \to \mathbf{qSup}$.

Counterexample

The 4-element Boolean algebra is not a quantum suplattice.

- If X is a poset with poset $D(X)$ of lower sets, then $`X$ is a quantum poset, and $`D(X)$ is a quantum poset which embeds into $\mathcal{D}(`X)$;
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Quantum versions of some theorems on suplattices

Theorem

The opposite $(\mathcal{X}, \succcurlyeq)$ of a quantum suplattice $(\mathcal{X}, \preccurlyeq)$ is a quantum suplattice.

Theorem

Let $F : \mathcal{X} \to \mathcal{Y}$ be a monotone map between quantum suplattices. Then F is a sup-homomorphism if and only if F is a lower Galois adjoint.

Let $F : \mathcal{X} \to \mathcal{X}$ be a monotone endomap on a quantum poset $\mathcal{X}$. A subset $\mathcal{Y} \subseteq \mathcal{X}$ with canonical embedding $J_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{X}$ is called a subset of fixpoints if $F \circ J_{\mathcal{Y}} = J_{\mathcal{Y}}$.

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Let $F : \mathcal{X} \to \mathcal{X}$ be a monotone endomap on a quantum suplattice $(\mathcal{X}, \preccurlyeq)$. Then the largest subset of fixpoints $\mathcal{Y}$ of $\mathcal{X}$ exists and is a quantum suplattice in its relative order $J_{\mathcal{Y}}^{\dagger} \circ \preccurlyeq \circ J_{\mathcal{Y}}$.

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Open problems

Conjecture (Quantum Cantor–Schröder–Bernstein)

Let $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{X}$ be injective functions between quantum sets $\mathcal{X}$ and $\mathcal{Y}$. Then there is a bijection $\mathcal{X} \cong \mathcal{Y}$.

In terms of operator algebras, this translates to

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Let $f : M \to N$ and $g : N \to M$ be surjective normal unital $*$-homomorphisms between hereditarily atomic von Neumann algebras M and N. Then there is a $*$-isomorphism $M \to N$.

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Towards quantum topology

- C*-algebras form the standard approach to quantum topology, but only generalize locally compact Hausdorff spaces.
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- Examples: the Alexandrov topology on a poset, the Scott topology on a cpo.
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First two candidate definitions

- The quantum power set $\mathcal{P}(\mathcal{X})$ of a quantum set $\mathcal{X}$ is given by $\mathcal{D}(\mathcal{X}, I_{\mathcal{X}}) := [\mathcal{X}^*, \mathbf{2}]_{\mathbf{qPOS}}$.
- Here, $\mathbf{2} = `2$, where 2 is the ordinary two-element chain.
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Definition (First candidate definition of a quantum topology)

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Examples of structures that should be quantum topologies

- The quantum Alexandrov topology on a quantum poset $(\mathcal{X}, \preccurlyeq)$ should be the internal hom $[(\mathbf{X}, \succcurlyeq)^*, \mathbf{2}]_{\mathbf{qPos}}$.
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The quantum ultrafilter monad

- There are two possible definitions of the quantum ultrafilter monad:
  - As the codensity monad of the embedding of $\mathbf{qSet}_{\mathrm{fin}}$ into $\mathbf{qSet}$;
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- How are the algebras of this monad related to unital C*-algebras?
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Definition (Third candidate definition of a quantum topological space)

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