An introduction to Triadic Concept Analysis

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joint work with
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  $(A, B) \leq (C, D) :\iff A \subseteq C \qquad (\iff D \subseteq B)$.
- $\underline{\mathfrak{B}}(G, M, I) := (\mathfrak{B}(G, M, I), \leq)$

The Basic Theorem on Concept Lattices

$\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right)$$

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A complete lattice L is isomorphic to a concept lattice $\underline{\mathfrak{B}}(G, M, I)$ iff there are mappings $\tilde{\gamma} : G \to L$ and $\tilde{\mu} : M \to L$ such that $\tilde{\gamma}(G)$ is $\vee$-dense in L, $\tilde{\mu}(M)$ is $\wedge$-dense in L and for all $g \in G$ and $m \in M$

$$gIm \iff \tilde{\gamma}(g) \leq \tilde{\mu}(m).$$

Examples

Finite lattices $L \cong \underline{\mathfrak{B}}(L, L, \leq) \cong \underline{\mathfrak{B}}(J(L), M(L), \leq)$.

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Remark

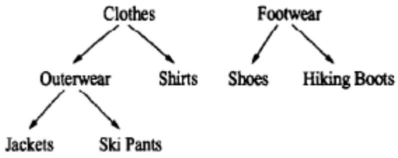
- The derivation $(',')$ is a Galois connection between $\mathcal{P}(G)$ and $\mathcal{P}(M)$.
- The operator $''$ is a closure operator on G (resp. M).
- $\mathrm{Ext}(\mathbb{K})$ denotes the set of extents of $\mathbb{K}$, and is a closure system on G.
- $\mathrm{Int}(\mathbb{K})$ denotes the set of intents of $\mathbb{K}$, and is a closure system on M.

An example of data analysis

Database $\mathcal{D}$

Transaction	Items Bought
100	Shirt
200	Jacket, Hiking Boots
300	Ski Pants, Hiking Boots
400	Shoes
500	Shoes
600	Jacket

Taxonomy $\mathcal{T}$



Frequent Itemsets

Itemset	Support
{ Jacket }	2
{ Outerwear }	3
{ Clothes }	4
{ Shoes }	2
{ Hiking Boots }	2
{ Footwear }	4
{ Outerwear, Hiking Boots }	2
{ Clothes, Hiking Boots }	2
{ Outerwear, Footwear }	2
{ Clothes, Footwear }	2

Rules

Rule	Support	Conf.
Outerwear ⇒ Hiking Boots	33%	66.6%
Outerwear ⇒ Footwear	33%	66.6%
Hiking Boots ⇒ Outerwear	33%	100%
Hiking Boots ⇒ Clothes	33%	100%

An example of data analysis (cont')

	Shirt	Jacket	Hiking Boots	Ski Pants	Shoes	Outerwear	Clothes	Footwear
100	×						×	
200		×	×			×	×	×
300			×	×		×	×	×
400					×			×
500					×			×
600		×				×	×	

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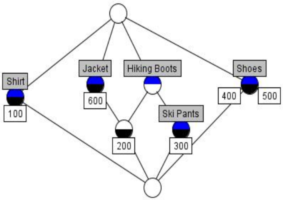

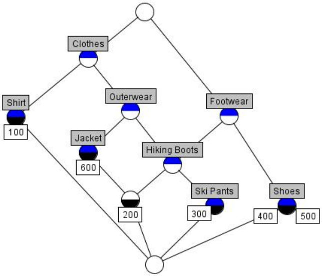

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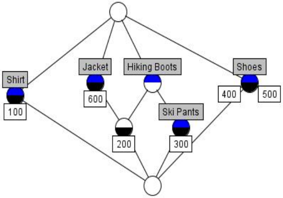

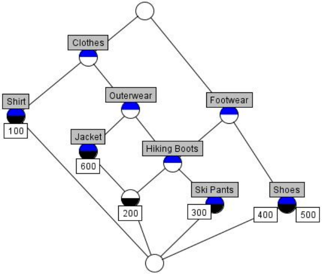

Any transaction with clothes and footwear has hiking boots.

$\{$Clothes, Footwear $\} \implies \{$ Outerwear, Hiking Boots$\}$.

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- From $\mathbb{K}$ we get three dyadic contexts:
  $\mathbb{K}^{(1)} := (K_1, K_2 \times K_3, \mathcal{R}^{(1)_\mathcal{R}})$, $\mathbb{K}^{(2)} := (K_2, K_1 \times K_3, \mathcal{R}^{(2)_\mathcal{R}})$ and
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Contexts in triadic setting (cont')

$\mathbb{K}$	P				F				S			
	a	b	c	d	a	b	c	d	a	b	c	d
1	x					x	x			x		x
2	x	x	x		x		x		x		x	x
3		x						x		x		
4				x		x		x	x		x	
5	x		x	x	x		x			x		x

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4				x		x		x	x		x	
5	x		x	x	x		x			x		x

$\mathbb{K}$	P	F	S
1	a	bc	bd
2	abc	ac	acd
3	b	d	b
4	d	bd	ac
5	acd	ac	bd

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- $(2, ac, PF)$ is not a triconcept and $(25, ac, PF)$ is a triconcept.

Concepts in triadic setting

- A **triadic concept** of $\mathbb{K}$ is a triple (A_1, A_2, A_3) with $A_i \subseteq K_i$ and $A_i = (A_j \times A_k)^{(i)_\mathcal{R}}$, $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$. We call A_1 extent, A_2 intent and A_3 modus.
- In dyadic FCA, concepts are maximal rectangles of I. In triadic setting, (tri)concepts are maximal cuboids of $\mathcal{R}$. i.e. $A_1 \times A_2 \times A_3 \subseteq \mathcal{R}$, maximal with respect to inclusion.
-

$\mathbb{K}$	P	F	S
1	a	bc	bd
2	abc	ac	acd
3	b	d	b
4	d	bd	ac
5	acd	ac	bd

- $2 \times ac \times PF \subsetneq 25 \times ac \times PF \subseteq \mathcal{R}$.
- $(2, ac, PF)$ is not a triconcept and $(25, ac, PF)$ is a triconcept.
- $\mathbb{K}$ has 24 concepts and $\mathbb{K}^{(i)}$ 16, 11, 8 concepts for $i = 1, 2, 3$ resp.

Concept hierarchy

- The set $\mathfrak{T}(\mathbb{K})$ of all triconcepts of $\mathbb{K}$ is quasi-ordered by set inclusion on each component: $(\mathfrak{T}(\mathbb{K}), \lesssim_1, \lesssim_2, \lesssim_3)$ with

$$(A_1, A_2, A_3) \lesssim_i (B_1, B_2, B_3) :\iff A_i \subseteq B_i$$

- Set $\sim_i := \lesssim_i \cap \gtrsim_i$. Then $(A_1, A_2, A_3) \sim_i (B_1, B_2, B_3) \iff A_i = B_i$.

(1) $x \lesssim_i y$ and $x \lesssim_j y$, imply $x \gtrsim_k y$, (**antiordinality**)

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What about completeness?

Concept hierarchy

- For $A_k \subseteq K_k$, set $\mathbb{K}^{ij}_{A_k} := (K_i, K_j, \mathcal{R}^{ij}_{A_k})$, with $i < j$ and define: $x\mathcal{R}^{ij}_{A_k}y$ iff x, y and z are (after reordering) in $\mathcal{R}$, for all $z \in A_k$.

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- The **triadic concept generated** by (X_i, X_k), $X_i \subseteq K_i$ and $X_k \subseteq K_k$, is $b_{ik}(X_i, X_k) := (B_1, B_2, B_3)$, where:

$$B_j = X_i^{(i,j,X_k)_{\mathcal{R}}}, \quad B_i = X_i^{(i,j,X_k)_{\mathcal{R}}(i,j,X_k)_{\mathcal{R}}} \text{ and } B_k = (B_i \times B_j)^{(k)_{\mathcal{R}}}$$

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- Let α and β be sets of concepts of a triadic context $\mathbb{K}$. We set $X_i = \bigcup\{A_i : (A_1, A_2, A_3) \in \alpha\}$ and $X_k = \bigcup\{A_k : (A_1, A_2, A_3) \in \beta\}$. The *ik*-join of (α, β) is the triadic concept $\nabla_{ik}(\alpha, \beta) := b_{ik}(X_i, X_k)$.

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- Quite complicated structure: Biedermann

Trilattices

A trilattice is a structure $(T, \nabla_{12}, \nabla_{21}, \nabla_{13}, \nabla_{31}, \nabla_{23}, \nabla_{32})$ where T is a non-empty set and ∇_{ik} are six (2,2)-ary operators which satisfying

Idempotent laws : $x\nabla_{ik}x = x$ (T1)

1st comp. comm. laws : $x_1x_2\nabla_{ik}\bar{y} = x_2x_1\nabla_{ik}\bar{y}$ (T2)

Bound laws : $x_1(x_1x_2\nabla_{ik}\bar{y})\nabla_{ik'}\bar{z} = (x_1x_2\nabla_{ik}\bar{y})\nabla_{ik'}\bar{z}$ (T3)

Limit laws : $(\bar{x}\nabla_{ik}\bar{y})(\bar{y}\nabla_{ki}\bar{x})\nabla_{jk}(\bar{x}\nabla_{ik}\bar{y}) = \bar{x}\nabla_{ik}\bar{y}$ (T4)

Antiordinal laws : $(x_1\nabla_{ik}\bar{y})(x_1x_2\nabla_{ik}\bar{y})\nabla_{jk}(x_1\nabla_{ik}\bar{y}) = x_1\nabla_{ik}\bar{y}$ (T5)

Commutative laws : $\bar{x}\nabla_{ik}\bar{y} = \bar{y}\nabla_{kj}(\bar{y}\nabla_{ki}\bar{x})$ (T6)

Separation laws : $x_1x_2\nabla_{ik}\bar{y} = (x_1\nabla_{ik}\bar{y})(x_2\nabla_{ik}\bar{y})\nabla_{ik}\bar{y}$ (T7)

Absobtion laws : $\bar{x}\nabla_{ik}\bar{y} = (\bar{x}\nabla_{ik}\bar{y})\nabla_{ik}\bar{y}$ (T8)

Assoc laws : $(x_1x_2\nabla_{ik}\bar{y})(x_3\nabla_{ik}\bar{y})\nabla_{ik}\bar{y} = (x_1\nabla_{ik}\bar{y})(x_2x_3\nabla_{ik}\bar{y})\nabla_{ik}\bar{y}$ (T9)

$\{i, j, k\} = \{1, 2, 3\}$ and $k' \neq i$.
$x_1, x_2, x_3, y_1, y_2, z_1, z_2 \in T$,
$\bar{x} = x_1x_2$, $\bar{y} = y_1y_2$ and $\bar{z} = z_1z_2$.

Line Diagrams

$\mathbb{O}_3$ with $K_1 = K_2 = K_3 = \{1, 2, 3\}$ and $(i, j, k) \in \mathcal{R}$ iff $i \leq j \leq k$.

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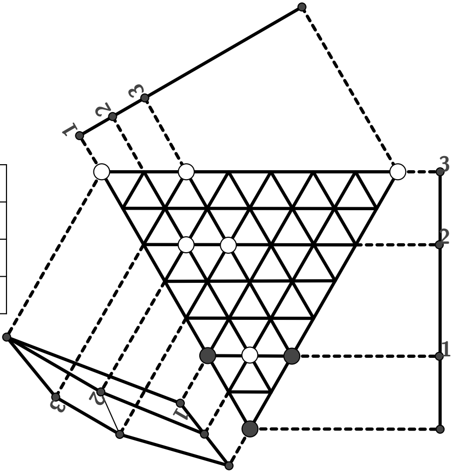
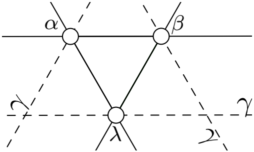
$\mathbb{O}_3$	1	2	3
1	1	12	123
2		2	23
3			3



9 concepts: $(123, 3, 3)$; $(12, 23, 3)$; $(1, 12, 23)$; $(12, 2, 23)$; $(1, 1, 123)$; $(1, 123, 3)$; $(123, 123; \emptyset)$; $(123, \emptyset, 123)$; $(\emptyset, 123, 123)$.

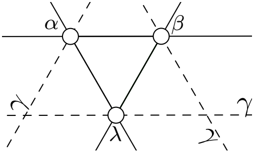
Line diagrams (cont')

	P		F	
	a	b	a	b
1		x	x	
2	x			x



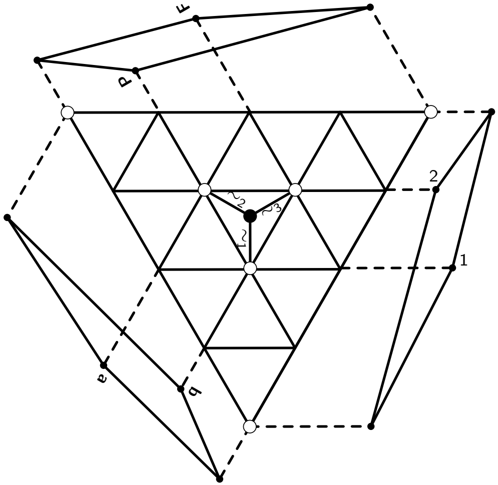
Line diagrams (cont')

	P		F	
	a	b	a	b
1		x	x	
2	x			x

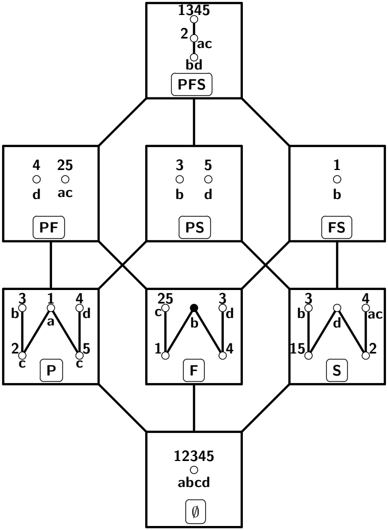


- concepts: $O_1 := (\emptyset, ab, PF)$, $O_2 := (12, \emptyset, PF)$, $O_3 := (12, ab, \emptyset)$, $\alpha := (2, a, P)$, $\beta := (2, b, F)$, $\lambda := (1, b, P)$ and $\gamma := (1, a, F)$.
- $\alpha \sim_1 \beta$, $\beta \sim_2 \lambda$ and $\lambda \sim_3 \alpha$, a form a **cycle**, (α, β, λ).
- In addition, γ verifies $\gamma \sim_1 \lambda$, $\gamma \sim_3 \beta$ and $\gamma \sim_2 \alpha$.
- Once α, β, and λ are represented, it is possible to represent γ, since there is no common point to the three dotted lines.

Line diagram: an alternative



Line diagram (cont')



Implications in Formal Contexts

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$$\begin{aligned}
\mathbb{K} \models B_1 \to B_2 &\iff \forall g \in G, g \in B_1' \implies g \in B_2' \\
&\iff B_2 \subseteq B_1'' \\
&\iff \bigwedge \{\mu a \mid a \in B_1\} \leq \mu m \text{ for all } m \in B_2
\end{aligned}$$

Thus, implications can be read from line diagrams

An example of data analysis (cont')

	Shirt	Jacket	Hiking Boots	Ski Pants	Shoes	Outerwear	Clothes	Footwear
100	×						×	
200		×	×			×	×	×
300			×	×		×	×	×
400					×			×
500					×			×
600		×				×	×	

An example of data analysis (cont')

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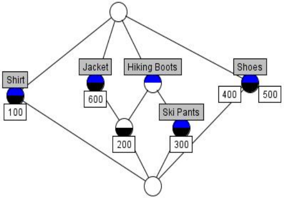
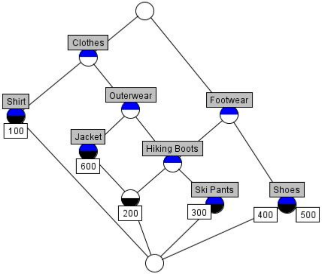
300			×	×		×	×	×
400					×			×
500					×			×
600		×				×	×	





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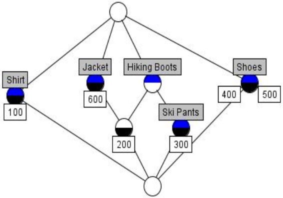

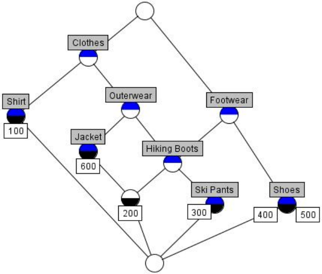

Any transaction with clothes and footwear has hiking boots.

$\{$Clothes, Footwear $\} \implies \{$ Outerwear, Hiking Boots$\}$.

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- An implication basis of $\mathbb{K}$ is a set $\mathcal{L}$ that is sound, complete and non redundant.

Triadic implications

Biedermann

An implication is a relation of the form $(E \to F)_G$ valid in the context with $E, F \subseteq K_j$ and $G \subseteq K_k$ with $\{j, k\} = \{2, 3\}$.

(1) conditional attribute implications: $(A_1 \to A_2)_C$

(2) attributional condition implications: $(C_1 \to C_2)_A$

$A_1, A_2, A \subseteq K_2$ and $C_1, C_2, C \subseteq K_3$.