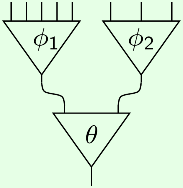
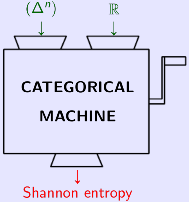
Operads

Tom Leinster
University of Edinburgh

Three talks

Sunday: Operads





Monday: An algebraic view of entropy

Thursday: Entropy modulo a prime

Trajectory of these three talks

TODAY

TOMORROW

I will explain how operads—a cousin of algebraic theories—lead to the notion of entropy (which might seem to belong to other branches of science).

Then I will show how this story leads to a mysterious construction in number theory.

THURSDAY

Today: Operads

1. What do category theorists care about?
2. A perspective on clones
3. Operads
4. Algebras for operads
5. Clones versus operads
6. Beyond sets

1. What do category theorists care about?

Two things category theorists care about

- Spotting patterns across mathematics, and connections between different parts of mathematics—especially connections that are unexpected.

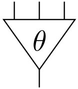
- Moving beyond the category of sets.

  E.g. given an algebraic theory, we want to consider not only its algebras in **Set**, but also its algebras in other categories where the concept of "algebra" makes sense.

2. A perspective on clones

Clones

An (abstract) clone is a sequence $(C_n)_{n\geq 0}$ of sets together with:

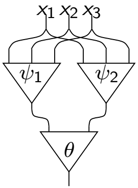


$\in C_3$

- for each $n, k \geq 0$, an operation

$$\begin{array}{ccc} C_n \times C_k^n & \to & C_k \\ (\theta, \psi_1, \ldots, \psi_n) & \mapsto & \theta(\psi_1, \ldots, \psi_n) \end{array}$$

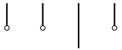
$$\theta(\psi_1, \ldots, \psi_n)(x_1, \ldots, x_k) = \theta\big(\psi_1(x_1, \ldots, x_k), \ldots, \psi_n(x_1, \ldots, x_k)\big)$$



- for each $1 \leq i \leq n$, a chosen element $\pi_i^n \in C_n$,

$$\pi_i^n(x_1, \ldots, x_n) = x_i$$

π_3^4



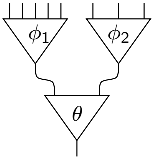
satisfying axioms.

An equivalent definition of clone

A clone is a sequence $(C_n)_{n\geq 0}$ of sets, together with:

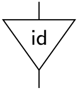
- for each $n, k_1, \ldots, k_n \geq 0$, an operation

$$\begin{array}{ccc} C_n \times C_{k_1} \times \cdots \times C_{k_n} & \to & C_{k_1+\cdots+k_n} \\ (\theta, \phi_1, \ldots, \phi_n) & \mapsto & \theta \circ (\phi_1, \ldots, \phi_n) \end{array}$$



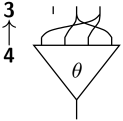
$$\theta \circ (\phi_1, \ldots, \phi_n)(x_1^1, \ldots, x_1^{k_1}, \ \ldots, \ x_n^1, \ldots, x_n^{k_n}) = \theta\big(\phi_1(x_1^1, \ldots, x_1^{k_1}), \ldots, \phi_n(x_n^1, \ldots, x_n^{k_n})\big)$$

- a chosen element $\mathrm{id} \in C_1$

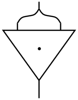


- for each function $\mathbf{n} \xrightarrow{f} \mathbf{m}$, a function $C_n \xrightarrow{f_*} C_m$ (where $\mathbf{n} = \{1, \ldots, n\}$),

$$f_*(\theta)(x_1, \ldots, x_m) = \theta(x_{f(1)}, \ldots, x_{f(n)})$$



satisfying axioms.



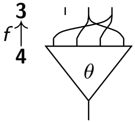
Example of f_*: If $f : \mathbf{2} \to \mathbf{1}$ and $\theta \in C_2$ is multiplication, then $f_*(\theta) \in C_1$ is squaring.

From standard clones to alternative clones

Start with a clone C in the standard sense.
Build a clone in the alternative sense like this:

- For $f : \mathbf{n} \to \mathbf{m}$ and $\theta \in C_n$, define $f_*(\theta) \in C_m$ by

$$f_*(\theta) = \theta(\pi^m_{f(1)}, \ldots, \pi^m_{f(n)}).$$



- For $\theta \in C_n, \phi_1 \in C_{k_1}, \ldots, \phi_n \in C_{k_n}$, take the functions

$$\begin{array}{rccc} f_i : & \mathbf{k_i} & \to & \mathbf{k_1} + \cdots + \mathbf{k_n} \\ & q & \mapsto & k_1 + \cdots + k_{i-1} + q \end{array}$$

  and then define

$$\theta \circ (\phi_1, \ldots, \phi_n) = \theta\big(f_{1*}(\phi_1), \ldots, f_{n*}(\phi_n)\big).$$

$f_{1*}(\phi_1)(x_1^1, \ldots, x_n^{k_n})$
$=\phi_1(x_1^1, \ldots, x_1^{k_1})$

- Put $\mathrm{id} = \pi^1_1 \in C_1$.

From alternative clones to standard clones

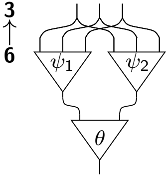
Start with a clone C in the alternative sense.
Build a clone in the standard sense like this.

- For $n, k \geq 0$, take the function $f\colon \mathbf{nk} \to \mathbf{k}$ given by

$$f(pk+q) = q \qquad (0 \leq p \leq n-1,\ 1 \leq q \leq k)$$

  and then for $(\theta, \psi_1, \ldots, \psi_n) \in C_n \times C_k^n$, define

$$\theta(\psi_1, \ldots, \psi_n) = f_*(\theta \circ (\psi_1, \ldots, \psi_n)).$$



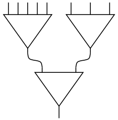
- For $1 \leq i \leq n$, take the function $f\colon \mathbf{1} \to \mathbf{n}$ given by $f(1) = i$, and then define $\pi_i^n = f_*(\mathrm{id})$.

Conclusion

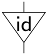
A clone can equivalently by defined as a sequence (C_n) of sets with:

- for each $n, k_1, \ldots, k_n \geq 0$, a *composition* operation

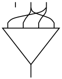
$$\begin{array}{ccc} C_n \times C_{k_1} \times \cdots \times C_{k_n} & \to & C_{k_1 + \cdots + k_n} \\ (\theta, \phi_1, \ldots, \phi_n) & \mapsto & \theta \circ (\phi_1, \ldots, \phi_n) \end{array}$$



- an *identity* element $\mathrm{id} \in C_1$



- for each $f : \mathbf{n} \to \mathbf{m}$, a *reindexing* function $f_* \colon C_n \to C_m$,



satisfying axioms.

Question *What if we remove the reindexing from the definition?*

3. Operads

The definition of operad

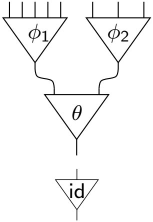
To obtain the definition of operad, simply take the alternative definition of clone and remove the reindexing. So:

An operad P is a sequence $(P_n)_{n \geq 0}$ of sets, together with:

- for each $n, k_1, \ldots, k_n \geq 0$, a composition operation

$$\begin{array}{ccc} P_n \times P_{k_1} \times \cdots \times P_{k_n} & \to & P_{k_1 + \cdots + k_n} \\ (\theta, \phi_1, \ldots, \phi_n) & \mapsto & \theta \circ (\phi_1, \ldots, \phi_n) \end{array}$$

- an *identity* element $\mathrm{id} \in P_1$,



satisfying axioms.

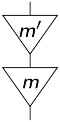
The axioms are associativity and identity laws.

They guarantee that every tree of operations has a unique composite.

Examples of operads

- The terminal operad: $P_n = \{*_n\}$ for all $n \geq 0$.
- For any monoid M, there is an operad P^M defined by

$$P_n^M = \begin{cases} M & \text{if } n = 1 \\ \varnothing & \text{otherwise.} \end{cases}$$



- Every set X has an endomorphism operad $\mathrm{End}(X)$, where

$$\mathrm{End}(X)_n = \{\text{functions } X^n \to X\}$$

  and composition in the operad is by composition of operations.
- Similarly, every vector space X has an endomorphism operad $\mathrm{End}(X)$, where

$$\mathrm{End}(X)_n = \{\text{linear maps } X^{\otimes n} \to X\}.$$

  (More generally, the same works in any monoidal category.)

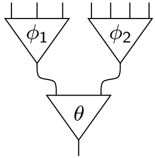
More examples of operads

The operad P of polynomials over a commutative ring R:

$$P_n = R[x_1, \ldots, x_n].$$

Composition is given by substitution and relabelling of variables: e.g. if

$$\theta = x_1^2 + x_2^3 \in P_2,$$
$$\phi_1 = 2x_1x_3 - x_2 \in P_3, \qquad \phi_2 = x_1 + x_2x_3x_4 \in P_4,$$

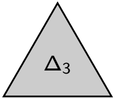


then

$$\theta \circ (\phi_1, \phi_2) = (2x_1x_3 - x_2)^2 + (x_4 + x_5x_6x_7)^3 \in P_7.$$

More examples of operads

The operad Δ of simplices:



$$\Delta_n = \{(p_1, \dots, p_n) \in \mathbb{R}^n \;:\; p_i \geq 0, \sum p_i = 1\}.$$

Composition is defined by thinking of $\mathbf{p} = (p_1, \dots, p_n)$ as a probability distribution on $\{1, \dots, n\}$. E.g. if

$\mathbf{p} =$  $= (\frac{1}{2}, \frac{1}{2}),\ \mathbf{q}_1 =$  $= (\frac{1}{6}, \dots, \frac{1}{6}),\ \mathbf{q}_2 =$  $= (\frac{1}{52}, \dots, \frac{1}{52})$

then

$$\mathbf{p} \circ (\mathbf{q}_1, \mathbf{q}_2) = (\underbrace{\tfrac{1}{12}, \dots, \tfrac{1}{12}}_{6}, \underbrace{\tfrac{1}{104}, \dots, \tfrac{1}{104}}_{52}) \in \Delta_{58}.$$

Generally, given

$$\mathbf{p} = (p_1, \dots, p_n),$$
$$\mathbf{q}_1 = (q_1^1, \dots, q_1^{k_1}),\ \dots,\ \mathbf{q}_n = (q_n^1, \dots, q_n^{k_n}),$$

define

$$\mathbf{p} \circ (\mathbf{q}_1, \dots, \mathbf{q}_n) = (p_1 q_1^1, \dots, p_1 q_1^{k_1},\ \dots,\ p_n q_n^1, \dots, p_n q_n^{k_n}) \in \Delta_{k_1 + \cdots + k_n}.$$

4. Algebras for operads

The definition of algebra for an operad

Let P be an operad.

A P-algebra is a set A together with, for each $n \geq 0$ and $\theta \in P_n$, a map

$$\overline{\theta}\colon A^n \to A,$$

satisfying action-like axioms:

- *composition*:

$$\begin{aligned} &\overline{\theta \circ (\phi_1, \ldots, \phi_n)}(a_1^1, \ldots, a_1^{k_1}, \ \ldots, \ a_n^1, \ldots, a_n^{k_n}) \\ = \ &\overline{\theta}\big(\overline{\phi_1}(a_1^1, \ldots, a_1^{k_1}), \ldots, \overline{\phi_n}(a_n^1, \ldots, a_n^{k_n})\big) \end{aligned}$$

  whenever $\theta \in P_n$, $\phi_i \in P_{k_i}$, and $a_i^j \in A$
- *identities*: $\overline{\mathrm{id}}(a) = a$ whenever $a \in A$.

Equivalently, it's a set A together with a map of operads $P \to \mathrm{End}(A)$.

Examples of algebras for operads

- Let P be the terminal operad: $P_n = \{*_n\}$ for all $n \geq 0$.

  Then a P-algebra is a set A together with one map $A^n \to A$ for each $n \geq 0$, satisfying axioms.

  It's exactly a monoid. The map $A^n \to A$ is n-ary multiplication.

- Fix a monoid M, and take the operad P^M:

$$P^M_n = \begin{cases} M & \text{if } n = 1 \\ \varnothing & \text{otherwise.} \end{cases}$$

  A P^M-algebra is a set A together with a map $\overline{m}\colon A \to A$ for each $m \in M$, satisfying axioms.

  It's exactly a set with a left M-action. The map $\overline{m}$ is $a \mapsto m \cdot a$.

More examples of algebras for operads

Let Δ be the operad of simplices:

$$\Delta_n = \{\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n \;:\; p_i \geq 0, \sum p_i = 1\}$$
$$= \big\{\text{probability distributions on } \{1, \dots, n\}\big\}.$$

Two examples of Δ-algebras:

- Any convex subset $A \subseteq \mathbb{R}^d$ is a Δ-algebra in a natural way: for $\mathbf{p} \in \Delta_n$, define

$$\begin{array}{rccc} \overline{\mathbf{p}}\colon & A^n & \to & A \\ & (\mathbf{a}_1, \dots, \mathbf{a}_n) & \mapsto & p_1\mathbf{a}_1 + \cdots + p_n\mathbf{a}_n. \end{array}$$

- The power set $\mathcal{P}(\mathbb{R}^d)$ is also a Δ-algebra: define

$$\begin{array}{rccc} \overline{\mathbf{p}}\colon & (\mathcal{P}(\mathbb{R}^d))^n & \to & \mathcal{P}(\mathbb{R}^d) \\ & (S_1, \dots, S_n) & \mapsto & p_1S_1 + \cdots + p_nS_n, \end{array}$$

  where

$$p_1S_1 + \cdots + p_nS_n = \{p_1s_1 + \cdots + p_ns_n \;:\; s_1 \in S_1, \dots, s_n \in S_n\}.$$

Which algebraic theories can operads express?

That is, given an algebraic theory T, when can we find an operad P such that the P-algebras are exactly the T-algebras?

Theorem *An algebraic theory can be expressed by an operad if and only if the theory can be presented using equations that are strongly regular:*

- *the same variables appear on each side of the equation,*
- *without repetition,*
- *and in the same order.*

Examples

- Theory of monoids: $(xy)z = x(yz)$, $1x = x$, $x1 = x$ ✓
- Theory of M-sets: $(mm')x = m(m'x)$, $1x = x$ ✓
- Theory of commutative monoids: includes $xy = yx$ **x**
- Theory of rings: includes $x(y + z) = xy + xz$ **x**
- Theory of groups: includes $x^{-1}x = 1$ **x**

There is no operad whose algebras are commutative monoids, or rings, or groups.

5. Clones versus operads

From operads to clones

Start with an operad P. We can build a clone $C(P)$, which can be described in two ways:

- Viewpoint 1: The theory of P-algebras is an algebraic theory presentable by strongly regular equations.

  In particular, it's an algebraic theory, or equivalently, a clone, $C(P)$.

- Viewpoint 2: Define

$$C(P)_n = \coprod_{k \geq 0} P(k) \times \{\text{functions } \mathbf{k} \to \mathbf{n}\}.$$

  E.g. if $P_n = \{*\}$ for all n (the operad for monoids) then an n-ary operation in $C(P)$ is a number $k \geq 0$ with a function $f : \mathbf{k} \to \mathbf{n}$, which we view as

$$(x_1, x_2, \ldots, x_n) \mapsto x_{f(1)} x_{f(2)} \cdots x_{f(k)}.$$

Fact: *The algebras for the operad P are the same as the algebras for the clone $C(P)$.*

From clones to operads

Start with a clone C. We can build an operad $P(C)$, which can be described in two ways:

- Viewpoint 1: Keep all the operations of the algebraic theory C, but forget all the equations that are *not* strongly regular. This gives an operad $P(C)$.
- Viewpoint 2: Recall that a clone is an operad together with a reindexing function $f_*: C_n \to C_m$ for each $f: \mathbf{n} \to \mathbf{m}$.

  To get $P(C)$, just forget the reindexing!

The algebras for the clone C are *not* the same as the algebras for the operad $P(C)$, since we've thrown away some equations.

Example of clone algebras versus operad algebras

Let C be the clone for the theory of convex combinations: informally, the algebraic structure possessed by any convex set in $\mathbb{R}^d$.

This theory contains operations like $(x, y) \mapsto cx + (1 - c)y$ for all $c \in [0, 1]$, and equations including

$$\tfrac{1}{2}x + \tfrac{1}{2}x = x.$$

Then $P(C)$ is the operad Δ of simplices.

But in an algebra for the operad $P(C) = \Delta$, the equation

$$\overline{(\tfrac{1}{2}, \tfrac{1}{2})}(x, x) = x$$

(which is not strongly regular) need not hold.

For example, in the Δ-algebra $\mathcal{P}(\mathbb{R})$, take $S = \{0, 1\} \in \mathcal{P}(\mathbb{R})$. Then

$$\tfrac{1}{2}S + \tfrac{1}{2}S = \{0, \tfrac{1}{2}, 1\} \neq \{0, 1\} = S.$$

Conclusion: *You can pass back and forth between clones and operads, but they're not equivalent notions.*

6. Beyond sets

Or: *Algebras in categories other than* **Set**

Monoidal categories

Roughly, a monoidal category is a category equipped with a product $\otimes$ and a unit object I satisfying

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C), \qquad A \otimes I \cong A \cong I \otimes A$$

for all objects A, B, C.

Examples

- **Set** (the category of sets), with $\otimes = \times$ and $I = \{*\}$.
- **Top** (topological spaces), with $\otimes = \times$ and $I = \{*\}$.
- **Cat** (categories), with $\otimes = \times$ and $I =$ the category with one object and only the identity morphism.

These examples all have the special feature that there are canonical maps

$$\begin{array}{ccc} X & \to & X \otimes X \\ x & \mapsto & (x, x) \end{array} \qquad \begin{array}{ccc} X & \to & I \\ x & \mapsto & *, \end{array}$$

which means we can *duplicate* and *delete* variables.

More examples of monoidal categories

Not all monoidal categories have this special feature.

Examples

- $\mathbf{Vect}_k$ (the category of vector spaces over a field k), with $\otimes = \otimes_k$ and $I = k$.

  There is no canonical map $X \to X \otimes X$ or $X \to k$.
- $\mathbf{Vect}_k^{\mathrm{op}}$ (the opposite/dual of $\mathbf{Vect}_k$), with $\otimes = \otimes_k$ and $I = k$.

  There is no canonical map $X \otimes X \to X$ or $k \to X$.
- $\mathbf{Vect}_k^{\mathbb{Z}}$ ($\mathbb{Z}$-graded vector spaces), with graded $\otimes$.
- $\mathbf{ChCx}_k$ (chain complexes), with usual $\otimes$.

Algebras in a monoidal category

Let $\mathcal{M}$ be a monoidal category.

Typically, you can't talk about algebras in $\mathcal{M}$ for an arbitrary algebraic theory. For example, what would a "group in $\mathcal{M}$" be?

But you *can* talk about algebras in $\mathcal{M}$ for an *operad*.

Let P be an operad.

A P-algebra in $\mathcal{M}$ is an object A of $\mathcal{M}$ together with, for each $n \geq 0$ and $\theta \in P_n$, a map

$$\overline{\theta} \colon A^{\otimes n} \to A,$$

satisfying action-like axioms.

Examples of operad algebras in a monoidal category

Let P be the terminal operad: $P_n = \{*_n\}$ for all $n \geq 0$.

A P-algebra...

- in **Set** is a monoid (as we've already seen)
- in **Top** is a topological monoid
- in **Cat** is a strict monoidal category ($\otimes$ is *strictly* associative and unital)
- in $\mathbf{Vect}_k$ is an algebra over the field k
- in $\mathbf{Vect}_k^{\mathrm{op}}$ is a coalgebra over k
- in $\mathbf{Vect}_k^{\mathbb{Z}}$ is a $\mathbb{Z}$-graded algebra over k
- in $\mathbf{ChCx}_k$ is a differential graded algebra over k.

Preview of tomorrow

Tomorrow, I'll explain how if we keep pursuing these ideas, we arrive at the concept of entropy.