Tensor product of effect algebras

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Let E be an effect algebra...

Definition

An effect algebra is a partial algebra $(E; \oplus, ', 0, 1)$, where $\oplus$ is a partial binary operation, $'$ is a unary operation and $0, 1$ are constants, such that for each $a, b, c \in E$:

(i) $a \oplus b = b \oplus a$;

(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(iii) $a \oplus b = 1$ if and only if $b = a'$;

(iv) $a \oplus 1$ is defined if and only if $a = 0$.

Where (i–ii) are Kleene identities.

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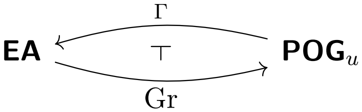
Where (i–ii) are Kleene identities.

Ordering: $a \leq b$ iff $(\exists c), a \oplus c = b$. **Partial subtraction:** For $a \leq b$, there is a unique $c = b \ominus a$, i.e., $\oplus$ **is cancellative**. **Examples:** $\mathcal{P}(H)$, $\mathcal{E}(H)$, Boolean algebras ($\oplus =$ disjoint union), orthomodular posets, orthoalgebras, MV-algebras,...

Related categories

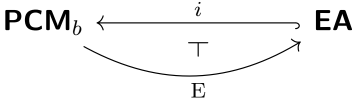
POG$_u$ = the category of partially ordered Abelian groups with order unit

$$[0,u]_A \quad \leftarrow\!\!\shortmid \quad (A,u)$$



PCM$_b$ = the category of partial commutative monoids with a top element

forgets cancellativity



forces cancellativity

Key concept: Riesz Decomposition Property

Definition

A (partial) commutative monoid M satisfies (RDP), if for all

$$a_1 \oplus a_2 = b_1 \oplus b_2$$

there exist $(c_{i,j})_{i,j=1,2}$, so that $a_i = c_{1,i} \oplus c_{2,i}$, $b_i = c_{i,1} \oplus c_{i,2}$.

$$\begin{matrix} a_1 & a_2 & & \\ \| & \| & & \\ \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} & & \begin{matrix} = \\ = \end{matrix} & \begin{matrix} b_1 \\ b_2 \end{matrix} \end{matrix}$$

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commutative monoids	effect algebras	po-groups
refinement property	Riesz Decomposition Property	interpolation

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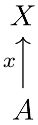
commutative monoids	effect algebras	po-groups
refinement property	Riesz Decomposition Property	interpolation

Theorem: Effect algebras with (RDP) are categorically equivalent to interpolation Abelian po-groups with order unit.

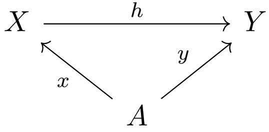
Key concept: coslice category

Let A be an object of a category $\mathcal{C}$. A coslice category $A \downarrow \mathcal{C}$ is given by the following data:

object $= \{x \colon A \to X\}$



arrows $= \{h \colon x \to y\}$



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object $= \{x \colon A \to X\}$

$$\begin{array}{c} X \\ {\scriptstyle x}\uparrow \\ A \end{array}$$

arrows $= \{h \colon x \to y\}$

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & Y \\ & {}_{x}\nwarrow \quad \nearrow^{y} & \\ & A & \end{array}$$

Given an adjunction $(L \dashv R) \colon \mathcal{C} \to \mathcal{D}$ and an object $A \in \mathcal{C}$:

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D} \;(\perp) \quad \rightsquigarrow \quad A \downarrow \mathcal{C} \underset{R'}{\overset{L'}{\rightleftarrows}} L(A) \downarrow \mathcal{D} \;(\perp)$$

Where $L' \colon x \mapsto L(x)$ and $R' \colon y \mapsto R(y) \circ \eta_A$.

(Finally) Tensor product in **EA**

Definition

Let E, F, G be effect algebras. A mapping $\beta\colon E \times F \to G$ is a *bihomomorphism* if

(i) $(\forall a \in E), \beta(a, -)\colon F \to G$ preserves orthosums;

(ii) $(\forall b \in F), \beta(-, b)\colon E \to G$ preserves orthosums;

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Definition

A tensor product of effect algebras E, F is a bihomomorphism

$$- \otimes -\colon E \times F \to E \otimes F$$

such that each bihomomorphism $\beta\colon E \times F \to G$ uniquely splits through $E \otimes F$ as $\beta = f \circ \otimes$.

Tensor product in $\mathbf{PCM}_b$, $\mathbf{EA}$ and $\mathbf{POG}_u$

$\mathbf{PCM}_b$	$\mathbf{EA}$	$\mathbf{POG}_u$

Tensor product in **PCM**$_b$, **EA** and **POG**$_u$

PCM$_b$	**EA**	**POG**$_u$

$$(A, A^+, u) \otimes (B, B^+, v) \cong (A \otimes B, A^+ \otimes B^+, u \otimes v)$$

Recall:

$$A \otimes B = \mathcal{F}_{Ab}(A \times B)/I$$

I is generated by

$$(a, c) + (b, c) - (a + b, c)$$
$$(a, c) + (a, d) - (a, c + d)$$

$a, b \in A, c, d \in B.$

Tensor product in **PCM**$_b$, **EA** and **POG**$_u$

PCM$_b$	**EA**	**POG**$_u$

When are two elements of a tensor product $M \otimes N$

$$a_1 \otimes b_1 \oplus \cdots \oplus a_n \otimes b_n$$
$$c_1 \otimes d_1 \oplus \cdots \oplus c_m \otimes d_m$$

equal?

The case M and N satisfy (RDP) is (essentially) treated by Wehrung in [4] .

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Tensor products always exist [1], but they are hard to compute.

Elements of $E \otimes F$ are of the form

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Questions

- **Which colimits (in EA) are preserved by tensor product?**
  In **POG**, dimension groups are characterized as directed colimits of the simplicial ones. Given to such examples

$$A = \varinjlim_{d \in \mathcal{D}} S_d \quad \text{and} \quad B = \varinjlim_{d' \in \mathcal{D}} S'_{d'},$$

  we have (**because filtered colimits commute with tensor product**)

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$\mathbf{PCM}_b$	**EA**	$\mathbf{POG}_u$
YES	?	NO

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- **Does tensor product in EA preserve (RDP)?**
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- **How does the tensor product $[0,1]_{\mathbb{R}} \otimes [0,1]_{\mathbb{R}}$ look like?**
  It was for a while an open problem whenever it is a lattice-effect algebra.

Question I: Colimits

Consider tensoring with an effect algebra E as a functor

$$E \otimes - \colon \mathbf{EA} \to \mathbf{EA}$$

which sends $F \mapsto E \otimes F$. Is there a right adjoint?

$$\frac{E \otimes F \to G}{F \to [E, G]} \tag{1}$$

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Consider

$$E \otimes - \colon \mathbf{EA} \to E \downarrow \mathbf{EA},$$

which sends F to $f\colon E \to E \otimes F$ ($f\colon a \mapsto a \otimes 1$).

$$\begin{array}{ccc} E \otimes F & \longrightarrow & G \\ \nwarrow & & \nearrow_{g} \\ & E & \end{array} \qquad F \longrightarrow [E, G]_g \tag{2}$$

Question I: Colimits

Theorem

Let $\mathcal{D}$ be a small category and $E \in \mathbf{EA}$. The functor

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preserves colimits over $\mathcal{D}$ whenever $\mathcal{D}$ is connected ($=$ the underlying graph is connected).

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Proof: The functor $E \otimes - \colon \textbf{EA} \to \textbf{EA}$ splits as:

$$\textbf{EA} \longrightarrow E \downarrow \textbf{EA} \longrightarrow \textbf{EA} \tag{3}$$

The first part preserves all colimits. The second part preserves all connected colimits.

EA and $\mathbf{POG}_u$

For $E \in \mathbf{EA}$ and $A = \mathrm{Gr}(E)$, consider a square:

$$\begin{array}{ccc} E \downarrow \mathbf{EA} & \xleftarrow[E\otimes -]{} & \mathbf{EA} \\ \Big\downarrow \mathrm{Gr} & & \mathrm{Gr} \Big\downarrow \\ A \downarrow \mathbf{POG}_u & \xleftarrow{A\otimes -} & \mathbf{POG}_u \end{array}$$

Commutativity (up to iso) would give us an isomorphism

$$\mathrm{Gr}(E \otimes F) \cong \mathrm{Gr}(E) \otimes \mathrm{Gr}(F). \tag{4}$$

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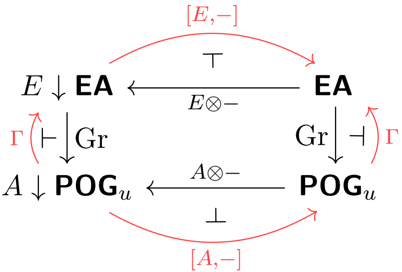

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We switch to the right adjoints: For $h\colon A \to B$ an object of $A \downarrow \mathbf{POG}_u$

$$[E, \Gamma(B, v)]_{\bar{h}} \cong \Gamma([A, B], h). \tag{5}$$

Question II: Riesz Decomposition Property

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For a pair of effect algebras E, F, we have

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Corollary

In **EA**, tensor product does not preserve (RDP).

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Proof:

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- Let $A, B \in \mathbf{POG}_u$ satisfy (RDP), but $A \otimes B$ not (example in [4]).
- There is $A \cong \mathrm{Gr}(E)$ and $B \cong \mathrm{Gr}(F)$ for some $E, F \in$ **EA** with (RDP).
- If $E \otimes F$ had (RDP), then $\mathrm{Gr}(E \otimes F) \cong \mathrm{Gr}(E) \otimes \mathrm{Gr}(F) \cong A \otimes B$ would satisfy (RDP) as well.

Question II: Riesz Decomposition Property.

Theorem

Recall the functor $\mathrm{E}\colon \mathbf{POG}_b \to \mathbf{EA}$. Let $M, N \in \mathbf{POG}_b$. Then

$$\mathrm{E}(M \otimes N) \cong \mathrm{E}(M) \otimes \mathrm{E}(N).$$

Corallary

The functor $\mathrm{E}\colon \mathbf{POG}_b \to \mathbf{EA}$ does not preserve (RDP).

Question III: $[0,1]_{\mathbb{R}} \otimes [0,1]_{\mathbb{R}}$

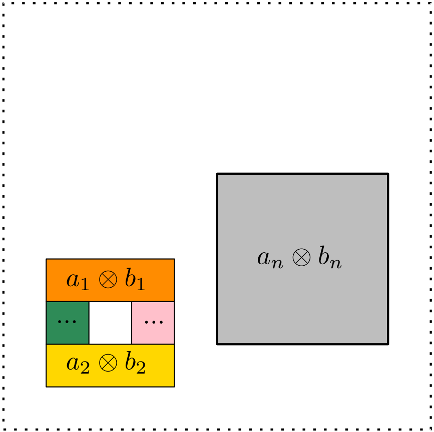
Theorem (see [3]): $\Gamma(\mathbb{R},1) \otimes \Gamma(\mathbb{R},1) \cong \Gamma(\mathbb{R} \otimes \mathbb{R}, 1 \otimes 1)$ satisfies (RDP) but it is not a lattice.
What about $\mathbf{PCM}_b$?

Question III: $[0,1]_{\mathbb{R}} \otimes [0,1]_{\mathbb{R}}$

In $\mathbf{PCM}_b$, one can represent and element

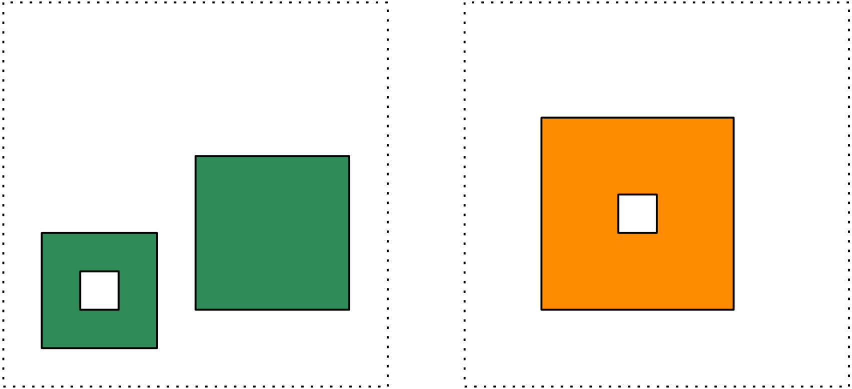
$$a_1 \otimes b_1 \oplus a_2 \otimes b_2 \oplus \cdots \oplus a_n \otimes b_n$$

of $[0,1]_{\mathbb{R}} \otimes [0,1]_{\mathbb{R}}$ as an orthogonal polygon living inside a unit square:



Question III: $[0,1]_\mathbb{R} \otimes [0,1]_\mathbb{R}$

Two elements of $[0,1] \otimes_{\mathbf{PCM}_b} [0,1]$ are equal iff the corresponding orthogonal polygons are related by an **orthogonal dissection**.



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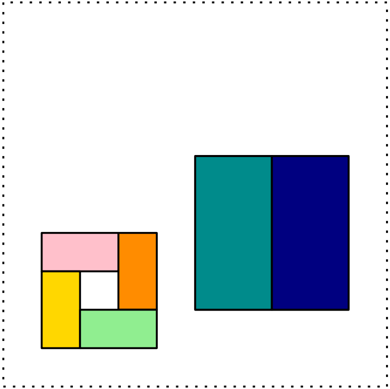

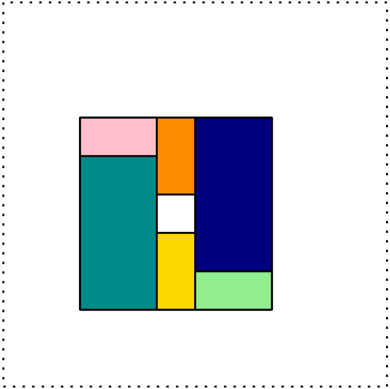

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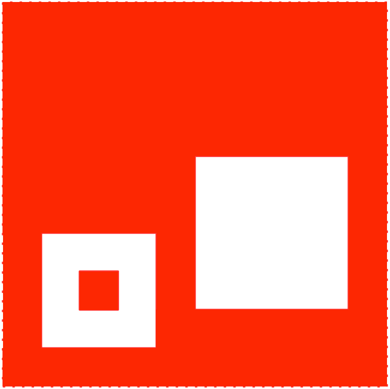
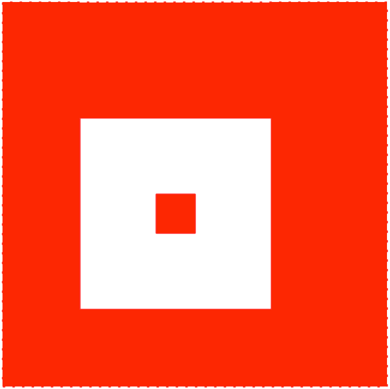
Does the equivalence induced by orthogonal dissection preserve complements?





Dehn invariant

This combinatorial problem was recently solved in [2] by establishing a full (Dehn) invariant:

$$D\colon \{\text{orthogonal polygons}\} \to \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}\ (\cong \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}).$$

Theorem

If we identify **EA** with a subcategory of $\mathbf{PCM}_b$, then we have

$$[0,1] \otimes_{\mathbf{EA}} [0,1] \cong [0,1] \otimes_{\mathbf{PCM}_b} [0,1].$$

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THANK YOU FOR YOUR ATTENTION