On varieties of commutative BCK-algebras

Covers of some varieties

Jan Kühr

Joint work with Václav Cenker and Petr Ševčík

Palacký University in Olomouc
Czechia

Varieties of ŁBCK-algebras

- ŁBCK-algebras are
  - the algebraic model of the implicational fragment of the Łukasiewicz propositional logic [Komori 1978];

Varieties of ŁBCK-algebras

- ŁBCK-algebras are
  - the algebraic model of the implicational fragment of the Łukasiewicz propositional logic [Komori 1978];
  - algebras $(A, \ominus, 0)$ where

$$x \ominus y = (x - y) \vee 0,$$

for some lattice-ordered abelian group $(G, \leq, +, -, 0)$ and a convex subset $A \subseteq G^+$.

Varieties of ŁBCK-algebras

- ŁBCK-algebras are
  - the algebraic model of the implicational fragment of the Łukasiewicz propositional logic [Komori 1978];
  - algebras $(A, \ominus, 0)$ where

$$x \ominus y = (x - y) \vee 0,$$

    for some lattice-ordered abelian group $(G, \leq, +, -, 0)$ and a convex subset $A \subseteq G^+$.
- The varieties of ŁBCK-algebras are

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \cdots \subset \mathcal{L},$$

where $\mathcal{S}_n = \mathrm{V}(S_n)$ and $\mathcal{L} = \mathrm{V}(\mathbb{Z}^+)$ [Komori 1978].

Varieties of ŁBCK-algebras

- ŁBCK-algebras are
  - the algebraic model of the implicational fragment of the Łukasiewicz propositional logic [Komori 1978];
  - algebras $(A, \ominus, 0)$ where

$$x \ominus y = (x - y) \vee 0,$$

    for some lattice-ordered abelian group $(G, \leq, +, -, 0)$ and a convex subset $A \subseteq G^+$.

- The varieties of ŁBCK-algebras are

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{S}_{n+1} \subset \cdots \subset \mathcal{L},$$

  where $\mathcal{S}_n = \mathrm{V}(S_n)$ and $\mathcal{L} = \mathrm{V}(\mathbb{Z}^+)$ [Komori 1978].

- $\mathcal{L}$ is a subvariety of the variety of commutative BCK-algebras, $\mathcal{C}$. So, what are the covers of the $\mathcal{S}_n$'s (and of $\mathcal{L}$) in the subvariety lattice of $\mathcal{C}$?

Commutative BCK-algebras are algebras $(A, \ominus, 0)$ of type $(2, 0)$ satisfying the equations

$$x \ominus 0 = x,$$
$$x \ominus x = 0,$$
$$(x \ominus y) \ominus z = (x \ominus z) \ominus y,$$
$$x \ominus (x \ominus y) = y \ominus (y \ominus x).$$

Commutative BCK-algebras

Commutative BCK-algebras are algebras $(A, \ominus, 0)$ of type $(2, 0)$ satisfying the equations

$$x \ominus 0 = x,$$
$$x \ominus x = 0,$$
$$(x \ominus y) \ominus z = (x \ominus z) \ominus y,$$
$$x \ominus (x \ominus y) = y \ominus (y \ominus x).$$

The underlying poset defined by $x \leq y$ iff $x \ominus y = 0$ is a meet-semilattice with

$$x \wedge y = x \ominus (x \ominus y).$$

Commutative BCK-algebras

Commutative BCK-algebras are algebras $(A, \ominus, 0)$ of type $(2, 0)$ satisfying the equations

$$x \ominus 0 = x,$$
$$x \ominus x = 0,$$
$$(x \ominus y) \ominus z = (x \ominus z) \ominus y,$$
$$x \ominus (x \ominus y) = y \ominus (y \ominus x).$$

The underlying poset defined by $x \leq y$ iff $x \ominus y = 0$ is a meet-semilattice with

$$x \wedge y = x \ominus (x \ominus y).$$

Bounded commutative BCK-algebras are term-equivalent to MV-algebras.

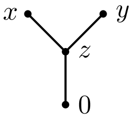
ŁBCK-algebras are commutative BCK-algebras satisfying

1. the equation
$$(x \ominus y) \wedge (y \ominus x) = 0;$$
2. the quasi-equation
$$x \wedge y \geq z \quad \& \quad x \ominus z = y \ominus z \quad \Rightarrow \quad x = y.$$

ŁBCK-algebras are commutative BCK-algebras satisfying

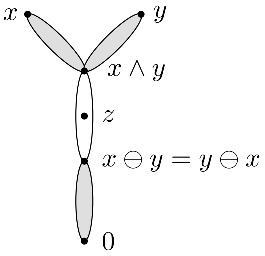
1. the equation
$$(x \ominus y) \wedge (y \ominus x) = 0;$$
2. the quasi-equation
$$x \wedge y \geq z \quad \& \quad x \ominus z = y \ominus z \quad \Rightarrow \quad x = y.$$

The smallest commutative BCK-algebra that is not an ŁBCK-algebra:



We say that

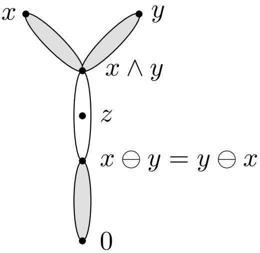
- (x, y, z) is a forbidden triple if $x \wedge y \geq z$, $x \ominus z = y \ominus z$ and $x \neq y$;



Forbidden triples/pairs

We say that

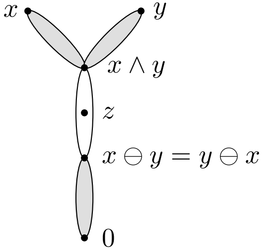
- (x, y, z) is a forbidden triple if $x \wedge y \geq z$, $x \ominus z = y \ominus z$ and $x \neq y$;
- (x, y) is a forbidden pair if $x \ominus y = y \ominus x \neq 0$.



Forbidden triples/pairs

We say that

- (x, y, z) is a forbidden triple if $x \wedge y \geq z$, $x \ominus z = y \ominus z$ and $x \neq y$;
- (x, y) is a forbidden pair if $x \ominus y = y \ominus x \neq 0$.

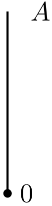


Lemma

A commutative BCK-algebra is not an ŁBCK-algebra iff it has a forbidden triple/pair.

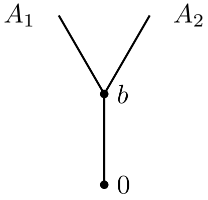
Forbidden subalgebras

Simple construction – splitting a totally ordered ŁBCK-algebra:



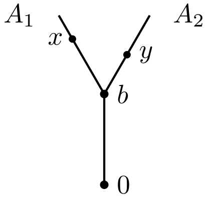
Forbidden subalgebras

Simple construction – splitting a totally ordered ŁBCK-algebra:



Forbidden subalgebras

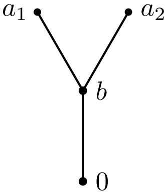
Simple construction – splitting a totally ordered ŁBCK-algebra:



We define $x \ominus y = x \ominus b$ in A_1 and $y \ominus x = y \ominus b$ in A_2.

Forbidden subalgebras

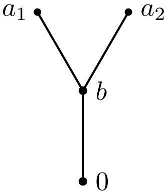
Simple construction – splitting a totally ordered ŁBCK-algebra:



We may assume that the "branches" are bounded, with (a_1, a_2) being a forbidden pair, in which case $[0, a_1] \cong [0, a_2]$.

Forbidden subalgebras

Simple construction – splitting a totally ordered ŁBCK-algebra:

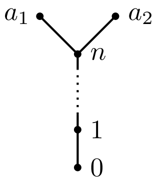


We may assume that the "branches" are bounded, with (a_1, a_2) being a forbidden pair, in which case $[0, a_1] \cong [0, a_2]$.

This is a simple algebra.

Forbidden subalgebras

Splitting $S_{n+1} = \{0, 1, \ldots, n+1\}$:



$a_1 \ominus a_2 = a_1 \ominus n = 1$ and $a_2 \ominus a_1 = a_2 \ominus n = 1$

Notation: $S_{n,2}$ or $M_2(S_n)$

Forbidden subalgebras

Theorem

The are uncountably many varieties of commutative BCK-algebras.

The map $K \mapsto \mathrm{V}(\{S_{n,2} \mid n \in K\})$ from nonempty subsets of positive integers to subvarieties of $\mathscr{C}$ is one-to-one.

Forbidden subalgebras

Theorem

The are uncountably many varieties of commutative BCK-algebras.

The map $K \mapsto \mathrm{V}(\{S_{n,2} \mid n \in K\})$ from nonempty subsets of positive integers to subvarieties of $\mathscr{C}$ is one-to-one.

We say that A is sectionally of finite length if, for every $a \in A$, $[0, a]$ is of finite length.

Theorem

Suppose that A is sectionally of finite length. Then A is not an ŁBCK-algebra iff it has a subalgebra isomorphic to some $S_{n,2}$.

Theorem

The (strict) covers of $\mathcal{S}_n$ in $\Lambda(\mathscr{C})$ are:

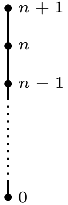
- $\mathcal{S}_{n+1}$,
- $\mathcal{S}_{n-1,2}$ if $n \geq 2$, and
- $\mathcal{S}_n \vee \mathcal{S}_{m,2}$ (for every $m < n-1$) if $n \geq 3$.

Covers of $\mathcal{S}_n$

Theorem

The (strict) covers of $\mathcal{S}_n$ in $\Lambda(\mathscr{C})$ are:

- $\mathcal{S}_{n+1}$,
- $\mathcal{S}_{n-1,2}$ if $n \geq 2$, and
- $\mathcal{S}_n \vee \mathcal{S}_{m,2}$ (for every $m < n-1$) if $n \geq 3$.

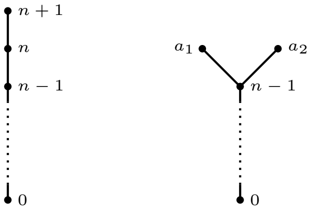


$\mathcal{S}_{n+1}$

Theorem

The (strict) covers of $\mathcal{S}_n$ in $\Lambda(\mathscr{C})$ are:

- $\mathcal{S}_{n+1}$,
- $\mathcal{S}_{n-1,2}$ if $n \geq 2$, and
- $\mathcal{S}_n \vee \mathcal{S}_{m,2}$ (for every $m < n-1$) if $n \geq 3$.



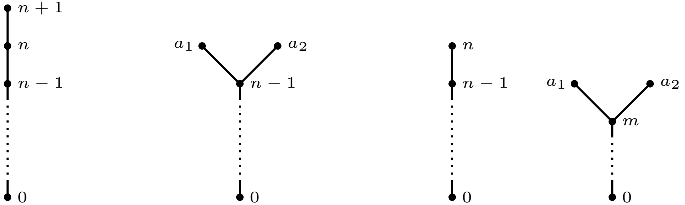
$\mathcal{S}_{n+1}$ $\mathcal{S}_{n-1,2}$ $(n \geq 2)$

Covers of $\mathcal{S}_n$

Theorem

The (strict) covers of $\mathcal{S}_n$ in $\Lambda(\mathscr{C})$ are:

- $\mathcal{S}_{n+1}$,
- $\mathcal{S}_{n-1,2}$ if $n \geq 2$, and
- $\mathcal{S}_n \vee \mathcal{S}_{m,2}$ (for every $m < n-1$) if $n \geq 3$.



$\mathcal{S}_{n+1}$ $\qquad$ $\mathcal{S}_{n-1,2}$ $(n \geq 2)$ $\qquad$ $\mathcal{S}_n \vee \mathcal{S}_{m,2}$ $(n \geq 3)$

Covers of $\mathcal{S}_{n,p}$

Theorem

For any integers $n \geq 1$ and $p \geq 2$, the (strict) covers of $\mathcal{S}_{n,p}$ in $\Lambda(\mathscr{L})$ are:

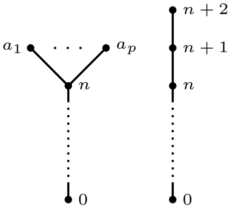
- $\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$,
- $\mathcal{S}_{n,p+1}$, and
- $\mathcal{S}_{n,p} \vee \mathcal{S}_{m,2}$ (for every $m < n$) if $n \geq 2$.

Covers of $\mathcal{S}_{n,p}$

Theorem

For any integers $n \geq 1$ and $p \geq 2$, the (strict) covers of $\mathcal{S}_{n,p}$ in $\Lambda(\mathscr{L})$ are:

- $\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$,
- $\mathcal{S}_{n,p+1}$, and
- $\mathcal{S}_{n,p} \vee \mathcal{S}_{m,2}$ (for every $m < n$) if $n \geq 2$.



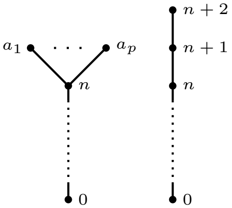
$\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$

Covers of $\mathcal{S}_{n,p}$

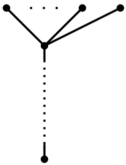
Theorem

For any integers $n \geq 1$ and $p \geq 2$, the (strict) covers of $\mathcal{S}_{n,p}$ in $\Lambda(\mathscr{L})$ are:

- $\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$,
- $\mathcal{S}_{n,p+1}$, and
- $\mathcal{S}_{n,p} \vee \mathcal{S}_{m,2}$ (for every $m < n$) if $n \geq 2$.



$\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$



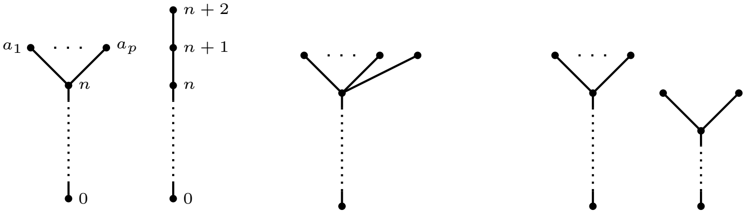
$\mathcal{S}_{n,p+1}$

Covers of $\mathcal{S}_{n,p}$

Theorem

For any integers $n \geq 1$ and $p \geq 2$, the (strict) covers of $\mathcal{S}_{n,p}$ in $\Lambda(\mathscr{L})$ are:

- $\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$,
- $\mathcal{S}_{n,p+1}$, and
- $\mathcal{S}_{n,p} \vee \mathcal{S}_{m,2}$ (for every $m < n$) if $n \geq 2$.



$\mathcal{S}_{n,p} \vee \mathcal{S}_{n+2}$ $\mathcal{S}_{n,p+1}$ $\mathcal{S}_{n,p} \vee \mathcal{S}_{m,2}$ $(n \geq 2)$

For any integer $k \geq 1$, we define

$$x \ominus ky = \big(\dots((x \ominus y) \ominus y) \ominus \dots\big) \ominus y.$$

For any integer $k \geq 1$, we define

$$x \ominus ky = \Big(\ldots((x \ominus y) \ominus y) \ominus \ldots\Big) \ominus y.$$

Lemma (Cornish 1980, Komori 1978)

1. A totally ordered commutative BCK-algebra A satisfies

$$(x \ominus ky) \wedge y = 0. \qquad (\mathsf{E}_k)$$

iff $A \cong S_n$ for some $n \leq k$.

For any integer $k \geq 1$, we define

$$x \ominus ky = \Big(\dots((x \ominus y) \ominus y) \ominus \dots\Big) \ominus y.$$

Lemma (Cornish 1980, Komori 1978)

1. A totally ordered commutative BCK-algebra A satisfies

$$(x \ominus ky) \wedge y = 0. \tag{E$_k$}$$

   iff $A \cong S_n$ for some $n \leq k$.

2. A subdirectly irreducible commutative BCK-algebra is a tree with meet-irreducible 0. It satisfies (E$_k$) iff its length is $\leq k$.

Digression – $\mathscr{E}_k$

Let $\mathscr{E}_k$ be the subvariety of $\mathscr{C}$ defined by (E$_k$).

The subdirectly irreducible members of $\mathscr{E}_k$ are trees (with meet-irreducible 0) of length $\leq k$.

Note that $\mathscr{E}_k \cap \mathscr{L} = \mathscr{S}_k$.

Let $\mathscr{E}_k$ be the subvariety of $\mathscr{C}$ defined by (E_k).

The subdirectly irreducible members of $\mathscr{E}_k$ are trees (with meet-irreducible 0) of length $\leq k$.

Note that $\mathscr{E}_k \cap \mathscr{L} = \mathscr{S}_k$.

Theorem

The only (strict) cover of $\mathscr{E}_k$ in $\Lambda(\mathscr{C})$ is $\mathscr{E}_k \vee \mathscr{S}_{k+1}$; it is axiomatized by the equations

$$(x \ominus (k+1)y) \wedge y = 0 \qquad (\mathsf{E}_{k+1})$$

and

$$(x \ominus ky) \wedge y \wedge (u \ominus v) \wedge (v \ominus u) = 0.$$

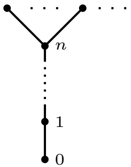
Axiomatization of $\mathcal{S}_{n,p}$

Lemma

For any integer $n \geq 1$ and any cardinal $\kappa \geq 2$, the algebra $S_{n,\kappa}$ satisfies the equation

$$(x \ominus ky) \wedge y = 0 \qquad (\mathsf{E}_k)$$

iff $n < k$.

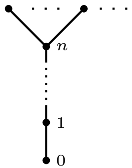


Lemma

For any integer $n \geq 1$ and any cardinal $\kappa \geq 2$, the algebra $S_{n,\kappa}$ satisfies the equation

$$(x \ominus ky) \wedge y = 0 \qquad (\mathsf{E}_k)$$

iff $n < k$.



Thus, also $S_{m,\kappa}$ with $m < n$ satisfy (E_k), but they are not in $\mathcal{S}_{n,\kappa}$ because $S_{n,\kappa}$ is a simple algebra and its subalgebras are

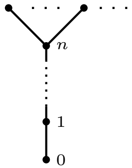
- S_m for $m \leq n+1$ and
- $S_{n,\lambda}$ for $\lambda \leq \kappa$.

Proposition

For any integer $n \geq 1$ and any cardinal $\kappa \geq 2$, the algebra $S_{n,\kappa}$ satisfies the equation

$$(x \ominus y) \wedge (y \ominus x) \leq \\ \leq x \ominus k\big((x \ominus y) \wedge (y \ominus x)\big) \quad (\mathsf{F}_k)$$

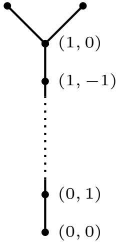
iff $n \geq k$.



Every cBCK-algebra satisfies (F_1).

Digression – $\mathcal{S}_{1,2}^{\infty}$

$$S_1^{\infty} = [(0,0),(1,0)] \text{ in } (\mathbb{Z} \vec{\times} \mathbb{Z})^{+} = (\{0\} \times \mathbb{Z}^{+}) \cup (\{1\} \times \mathbb{Z}^{-})$$



$S_{1,2}^{\infty}$

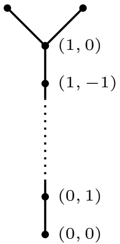
$$S_1^{\infty} = [(0,0),(1,0)] \text{ in } (\mathbb{Z} \vec{\times} \mathbb{Z})^{+} = (\{0\} \times \mathbb{Z}^{+}) \cup (\{1\} \times \mathbb{Z}^{-})$$

Proposition

The algebra $S_{1,2}^{\infty}$ satisfies the equation

$$(x \ominus y) \wedge (y \ominus x) \leq$$
$$\leq x \ominus k\big((x \ominus y) \wedge (y \ominus x)\big) \quad (\mathsf{F}_k)$$

for each k. Consequently, the variety $\mathcal{S}_{1,2}^{\infty} = \mathrm{V}(S_{1,2}^{\infty})$ is not generated by its finite members.



$S_{1,2}^{\infty}$

Axiomatization of $\mathcal{S}_{n,p}$

Theorem

The variety $\mathcal{S}_{n,\kappa}$ with κ infinite is axiomatized, relative to $\mathscr{C}$, by the equations

$$(x \ominus (n+1)y) \wedge y = 0 \qquad (\mathsf{E}_{n+1})$$

and

$$(x \ominus y) \wedge (y \ominus x) \leq x \ominus n\big((x \ominus y) \wedge (y \ominus x)\big). \qquad (\mathsf{F}_n)$$

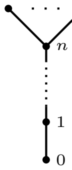
Axiomatization of $\mathcal{S}_{n,p}$

Lemma

For any integers $n, p \geq 1$, the algebra $S_{n,p}$ satisfies the equation

$$\bigwedge_{0 \leq i \neq j \leq k} (x_i \ominus x_j) \wedge (x_j \ominus x_i) = 0 \qquad (\mathsf{G}_k)$$

iff $p \leq k$.



Note that (G_1) defines $\mathscr{L}$ relative to $\mathscr{C}$.

Axiomatization of $\mathcal{S}_{n,p}$

Theorem

For any integers $n, p \geq 1$, the variety $\mathcal{S}_{n,p}$ is axiomatized, relative to $\mathscr{C}$, by the equations

$$(x \ominus (n+1)y) \wedge y = 0, \qquad (\mathsf{E}_{n+1})$$

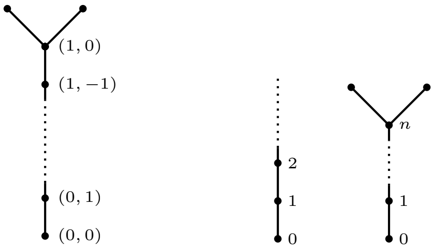
$$(x \ominus y) \wedge (y \ominus x) \leq x \ominus n\big((x \ominus y) \wedge (y \ominus x)\big), \qquad (\mathsf{F}_n)$$

$$\bigwedge_{0 \leq i \neq j \leq p} (x_i \ominus x_j) \wedge (x_j \ominus x_i) = 0. \qquad (\mathsf{G}_p)$$

Note that $\mathcal{S}_{n,1} = \mathcal{S}_{n+1} = \mathscr{E}_{n+1} \cap \mathscr{L}$.

Theorem

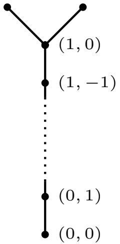
$\mathcal{S}_{1,2}^{\infty}$ and $\mathscr{L} \vee \mathcal{S}_{n,2}$ (for each $n \geq 1$) are covers of $\mathscr{L}$ in $\Lambda(\mathscr{C})$.



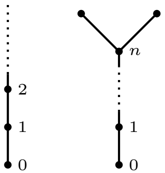
$\mathcal{S}_{1,2}^{\infty}$ $\mathscr{L} \vee \mathcal{S}_{n,2}$

Theorem

$\mathcal{S}_{1,2}^{\infty}$ and $\mathscr{L} \vee \mathcal{S}_{n,2}$ (for each $n \geq 1$) are covers of $\mathscr{L}$ in $\Lambda(\mathscr{C})$.



$\mathcal{S}_{1,2}^{\infty}$



$\mathscr{L} \vee \mathcal{S}_{n,2}$

?

Thank you!