Multiplication of matrices over lattices

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University of Szeged

Stará Lesná, September 2023

Matrices over the two-element lattice **2**

square matrices over **2**

Matrices over the two-element lattice **2**

square matrices over **2** $\longleftrightarrow$ binary relations

Matrices over the two-element lattice $\mathbf{2}$

square matrices over $\mathbf{2}$ $\longleftrightarrow$ binary relations $\longleftrightarrow$ directed graphs

Matrices over the two-element lattice **2**

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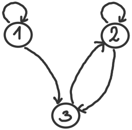
A	B	$A \cdot B$
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$		
		

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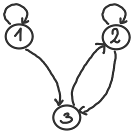
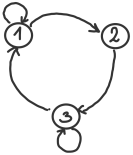
<table>
<tr><th>A</th><th>B</th><th>$A \cdot B$</th></tr>
<tr><td>$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$</td><td></td></tr>
<tr><td>
</td><td>
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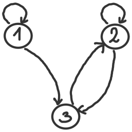
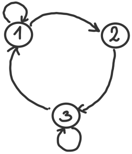
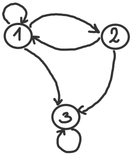
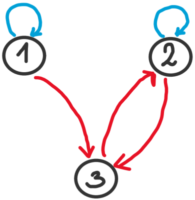
		

Transportation network



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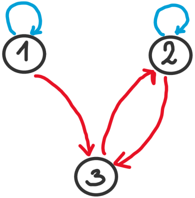

- vertices: sites

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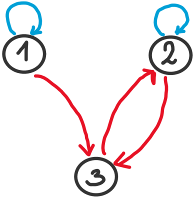

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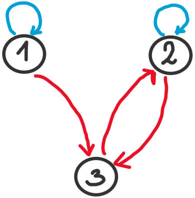
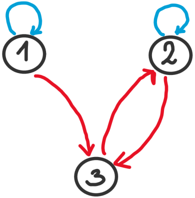


- vertices: sites
- edges: (possible one-way) roads
- loops: parking lots



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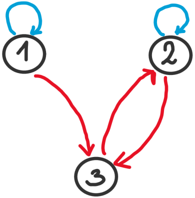

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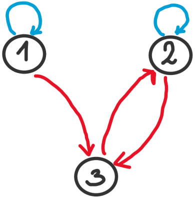

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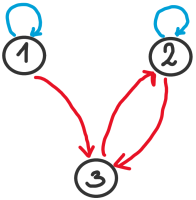

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$A^\ell = (w_{ij})_{n \times n}$, $w_{ij} = 1$: there is a directed route of length ℓ from i to j.

Idempotent matrices over the two-element lattice

Schein (1970)

idempotent matrices over $\mathbf{2} \longleftrightarrow$ pseudo-orders

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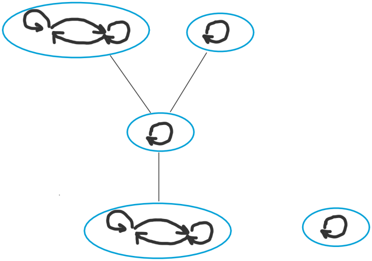
Schein (1970)

idempotent matrices over **2** $\longleftrightarrow$ pseudo-orders

- $\lesssim$ quasi-order: reflexive transitive relation
- $\sim := \lesssim \cap \lesssim^{-1}$ equivalence relation
- $(X/\sim; \leq)$ partially ordered set (poset)
- pseudo-order: remove some of the loops from a quasi-order in a certain way ☞

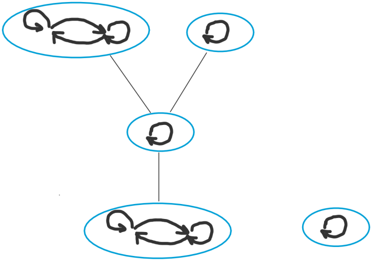
Pseudo-order

quasi-order



Pseudo-order

quasi-order



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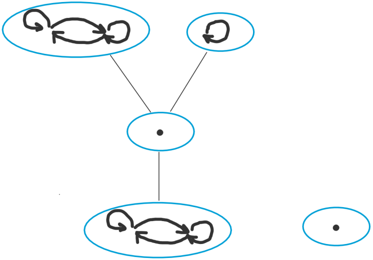
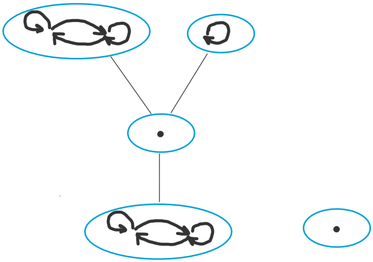


Pseudo-order as a transportation network



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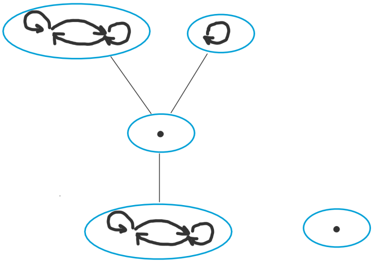

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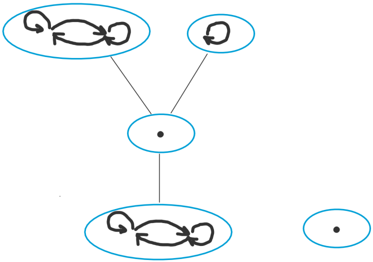

- you can choose a direct route (transitivity)
- you can plan your route to have a chance to take a rest in a parking lot

Matrices over lattices

Multiplication of matrices over a lattice L is associative $\iff$ L is a distributive lattice.

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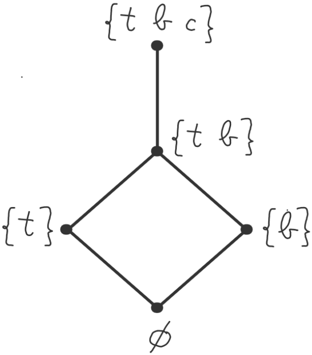

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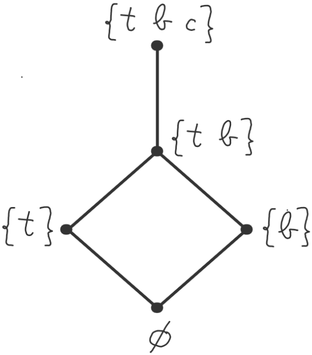

- t: trucks

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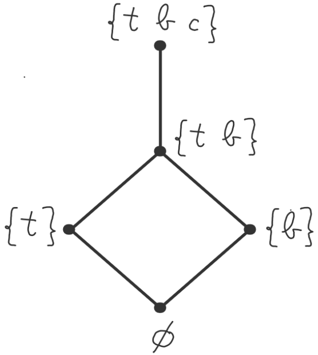

- t: trucks
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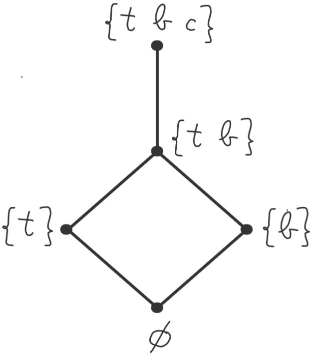
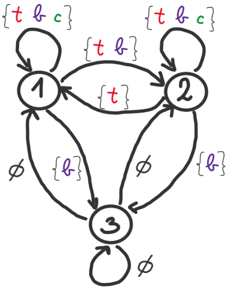


- t: trucks
- b: buses
- c: cars

Cuts

$$A = \begin{pmatrix} \{t\,b\,c\} & \{t\,b\} & \{b\} \\ \{t\} & \{t\,b\,c\} & \{b\} \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$$



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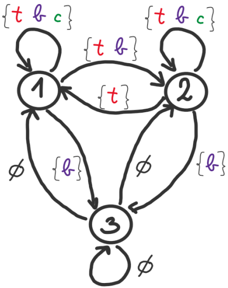
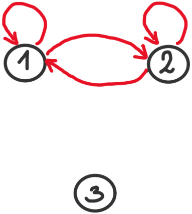


$$\Gamma_t(A) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



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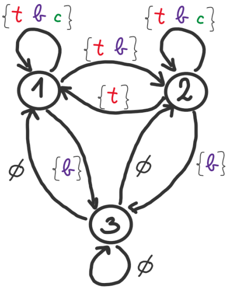

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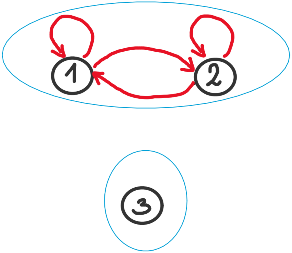

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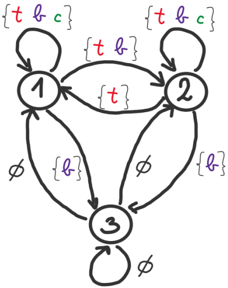

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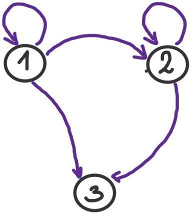

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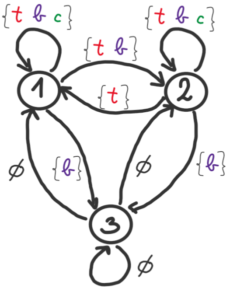

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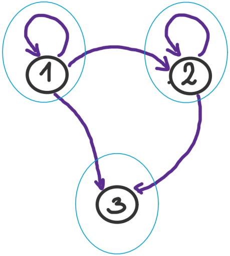

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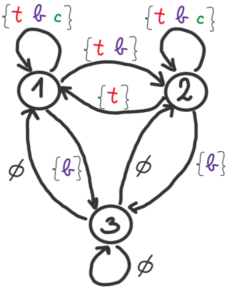

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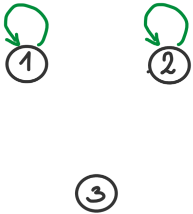

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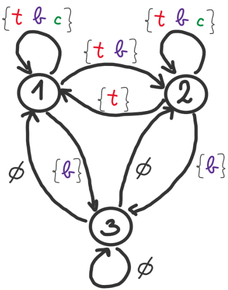

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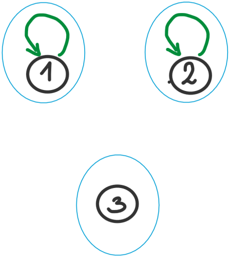

Idempotent matrices over lattices

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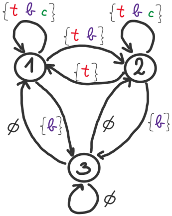

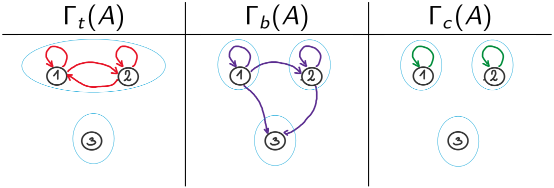

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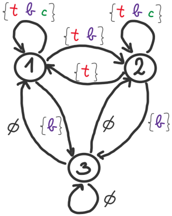
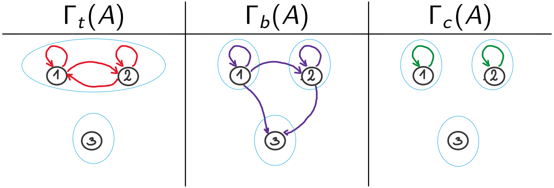
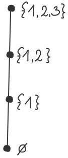




A matrix A over a distributive lattice $L \leq \mathcal{P}(\Omega)$ is idempotent $\iff$ the binary relation corresponding to the cut matrix $\Gamma_\omega(A)$ is a pseudo-order for each $\omega \in \Omega$.

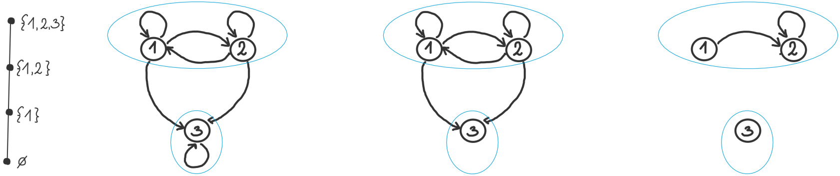
Idempotent matrices over chains

L



Idempotent matrices over chains

L α_1 α_2 α_3



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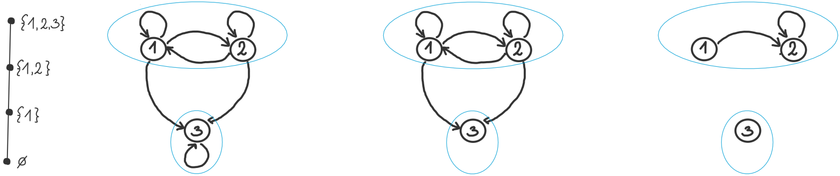
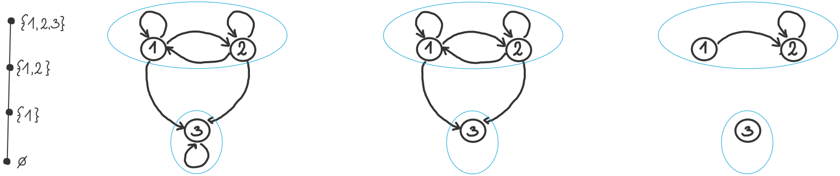


L	α_1	$\supseteq$	α_2	$\supseteq$	α_3



Theorem

L is the m-element chain. A matrix A over L is idempotent $\iff$ the binary relations corresponding to the cut matrices $\Gamma_k(A)\,(k = 1, \ldots, m-1)$ form a system of nested pseudo-orders $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$.

Nilpotent matrices over lattices

Theorem (Give'on (1964), Zhang (2001), Tan (2005))

A matrix A over a bounded distributive lattice L is nilpotent $\iff$ every cycle in the directed graph corresponding to A has capacity 0.

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Strictly upper triangular matrix: A has zeros below its main diagonal as well as on the main diagonal, i.e., $a_{ij} \neq 0 \implies i < j$.

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Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then a matrix A over L is nilpotent $\iff$ it is conjugate to a strictly upper triangular matrix, i.e., there exists a strictly upper triangular matrix U and an invertible matrix C such that $A = C^{-1}UC$.

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Theorem

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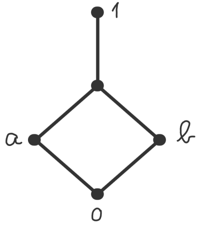

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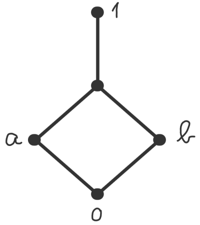

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Nilpotent matrices

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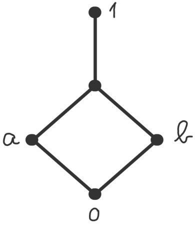

$A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$: $A^2 = 0 \Longrightarrow A$ is nilpotent, but A is not conjugate to a strictly upper triangular matrix.

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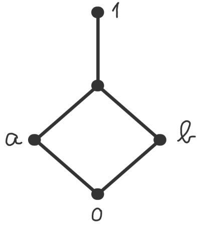

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Theorem does not necessarily remain true without the assumption on the irreducibility of 0.

Fixed point iteration

The greatest solution of $xA = x$ – Transportation network

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Fixed point iteration

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If $\lim_{k \to \infty}(A^k \mathbf{1}) = (z_1, \ldots, z_n)$, then z_i is the set of vehicles that can start arbitrarily long trips at i, i.e., z_i is the set of vehicles that can reach a directed cycle from i.

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Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then the following are equivalent for any matrix A over L:

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Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then the following are equivalent for any matrix A over L:

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Corollary

Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then the following are equivalent for any matrix A over L:

- the only solution of the fixed-point equation $xA = x$ is $x = \mathbf{0}$;
- $\lim_{k\to\infty}(\mathbf{1}A^k) = \mathbf{0}$;

Fixed point iteration

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Corollary

Let L be a bounded distributive lattice in which 0 is meet-irreducible. Then the following are equivalent for any matrix A over L:

- the only solution of the fixed-point equation $xA = x$ is $x = \mathbf{0}$;
- $\lim_{k\to\infty}(\mathbf{1}A^k) = \mathbf{0}$;
- A is nilpotent;
- there exists a permutation $\pi \in S_n$ such that $P_\pi^{-1}AP_\pi$ is strictly upper triangular.

Associative spectrum

The number of possibilities of inserting brackets into a product $x_1 \cdot \ldots \cdot x_n$ is given by the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

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If the associative spectrum is the sequence of Catalan numbers, then the multiplication is said to be antiassociative.

Associative spectrum

Proposition

If a binary operation has an identity element, then it is either associative (i.e., the associative spectrum is constant 1) or it is antiassociative (i.e., the associative spectrum consists of the Catalan numbers).

Associative spectrum

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If a binary operation has an identity element, then it is either associative (i.e., the associative spectrum is constant 1) or it is antiassociative (i.e., the associative spectrum consists of the Catalan numbers).

Corollary

If the lattice L is not distributive, then the multiplication of matrices over L is antiassociative.

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