Special filters in bounded lattices

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M.S. Rao recently studied various types of filters and ideals in distributive lattices with pseudocomplements. There is the question whether the distributivity assumption can be relaxed and, more specifically, if the pseudocomplementation assumption can be eliminated. This approach, in which we consider instead of the pseudocomplement a certain subset of elements behaving similar to the pseudocomplement, has already been introduced by I. Chajda and H. Länger. They show that this construction can be successfully used in order to introduce the implication and negation connectives in an intuitionistic-like logic.

We will deal with bounded lattices $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ satisfying the Ascending Chain Condition (ACC). We identify singletons with their unique element, i.e. we will write a instead of $\{a\}$. For $a \in L$ and for subset A of L we define

$$A^0 := \mathrm{Max}\{x \in L \mid x \wedge y = 0 \text{ for all } y \in A\},$$
$$a^0 := \mathrm{Max}\{x \in L \mid x \wedge a = 0\}.$$

The set a^0 is in fact a generalization of the *pseudocomplement* of a introduced by O. Frink or a modification of the annihilator since the set

$$\{x \in L \mid x \wedge y = 0 \text{ for all } y \in A\}$$

is in fact the *annihilator* of the set A as known in lattice theory. (The pseudocomplement of a is the greatest element of $\{x \in L \mid x \wedge a = 0\}$.) The advantage of our approach is that a^0 can be defined in any lattice with 0. Of course, we must pay for this advantage by the fact that a^0 need not be an element of L, but may be a subset of L, namely an antichain of $\mathbf{L}$.

The element a is called *dense* if $a^0 = 0$ and *sharp* if $a^{00} = a$.
Let D (S) denote the set of all dense (sharp) elements of L. Clearly:

$$a \in D \text{ if and only if } a^{00} = 1,$$
$$1 \in D, \qquad 0, 1 \in S, \qquad D \cap S = 1.$$

Moreover, we define the following binary relations and unary operators on 2^L:

$$A \leq B \text{ if } x \leq y \text{ for all } x \in A \text{ and all } y \in B,$$
$$A \leq_1 B \text{ if for every } x \in A \text{ there exists some } y \in B \text{ with } x \leq y,$$
$$A =_1 B \text{ if both } A \leq_1 B \text{ and } B \leq_1 A,$$
$$\overline{A} := \{x \in L \mid 1 \in x^{00} \vee y^{00} \text{ for each } y \in A\},$$
$$A^D := \{x \in L \mid x \vee y \in D \text{ for all } y \in A\}.$$

The set A is called *closed* if $\overline{\overline{A}} = A$. Observe that $\overline{0} = D$ and $\overline{1} = L$.

Lemma 1

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. Then $\overline{D} = L$ and $\overline{L} = D$.

Recall that a non-empty subset F of L is called a *filter* of $\mathbf{L}$ if $x \wedge y, x \vee z \in F$ for all $x, y \in F$ and all $z \in L$. Observe that for every $x \in L$ the set $F_x := \{y \in L \mid x \leq y\}$ is a filter of $\mathbf{L}$, the so-called *principal filter* generated by x. A filter F of $\mathbf{L}$ is called a *D-filter* if $D \subseteq F$.

It is evident that if a given bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ is pseudocomplemented then for each element $x \in L$ we have $x^0 = x^*$. We start this section with emphasizing certain differences between the properties of our concept x^0 and pseudocomplements.
It is well-known that for distributive pseudocomplemented lattices $(L, \vee, \wedge, 0)$ with bottom element 0 and $a, b \in L$ the following holds:

(i) $a \leq b$ implies $b^* \leq a^*$,

(ii) $a \leq a^{**}$ and $a^{***} = a^*$,

(iii) $(a \vee b)^* = a^* \wedge b^*$,

(iv) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

The following example shows that not all being valid for pseudocomplements, also holds for x^0.

Consider the non-distributive lattice $\mathbf{L}$ depicted in Fig. 1:

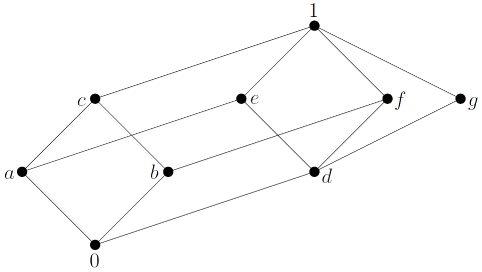

Fig. 1
Non-distributive non-pseudocomplemented lattice

We have

x	0	a	b	c	d	e	f	g	1
x^0	1	fg	eg	g	c	c	a	c	0
x^{00}	0	a	b	c	g	g	fg	g	1
$\overline{x}$	F_1	F_d	F_d	F_d	$abcf1$	$abcf1$	$abcdefg1$	$abcf1$	L

$D = F_1$ and $S := \{0, a, b, c, g, 1\}$. Hence $\mathbf{L}$ is not pseudocomplemented.
(Here and in the following we often write fg instead of $\{f, g\}$, and so on.)

Example 1 (2/2)

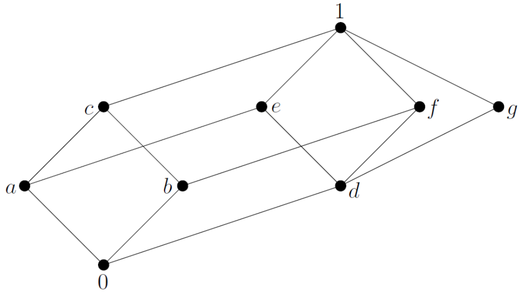

Fig. 1
Non-distributive non-pseudocomplemented lattice

In $\mathbf{L}$ we have:

(i) $a \leq c$ and $c^0 = g \not\leq \{f, g\} = a^0$, but $c^0 \leq_1 a^0$.

(ii) $f \not\leq \{f, g\} = f^{00}$, but $f \leq_1 f^{00}$.

(iii) $(a \vee b)^0 = c^0 = g \neq \{d, g\} = \{f, g\} \wedge \{e, g\} = a^0 \wedge b^0$, but $(a \vee b)^0 \leq_1 a^0 \wedge b^0$.

(iv) $(d \wedge f)^{00} = d^{00} = g \neq \{d, g\} = g \wedge \{f, g\} = d^{00} \wedge f^{00}$, but $(d \wedge f)^{00} =_1 d^{00} \wedge f^{00}$.

On the other hand, some of the properties of pseudocomplements are preserved also here.

Theorem 1

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC, $x, y \in L$ and $A, B \subseteq L$. Then:

(i) $a \wedge b = 0$ for all $a \in A$ and all $b \in A^0$, especially, $x \wedge x^0 = 0$,

(ii) If $a \wedge b = 0 \ \forall a \in A$ and $\forall b \in B$ then $A \leq_1 B^0$, esp., $x \wedge y = 0$ implies $x \leq_1 y^0$,

(iii) $A \leq_1 A^{00}$, especially, $x \leq_1 x^{00}$,

(iv) $A \leq_1 B$ implies $B^0 \leq_1 A^0$, especially, $x \leq y$ implies $y^0 \leq_1 x^0$,

(v) $A \leq_1 B^0$ if and only if $B \leq_1 A^0$, especially, $x \leq_1 y^0$ if and only if $y \leq_1 x^0$,

(vi) $a =_1 B$ implies $a = B$,

(vii) $A^0 =_1 B^0$ implies $A^0 = B^0$,

(viii) $A^{000} = A^0$, especially, $x^{000} = x^0$,

(ix) $(A \vee B)^0 \leq_1 A^0 \wedge B^0$, especially, $(x \vee y)^0 \leq_1 x^0 \wedge y^0$,

(x) $(A \wedge B)^{00} =_1 A^{00} \wedge B^{00}$, especially, $(x \wedge y)^{00} =_1 x^{00} \wedge y^{00}$,

(xi) $A^{00} \vee B^{00} \leq_1 (A^0 \wedge B^0)^0$ implies $(A^{00} \vee B^{00})^0 =_1 A^0 \wedge B^0$, especially, $x^{00} \vee y^{00} \leq_1 (x^0 \wedge y^0)^0$ implies $(x^{00} \vee y^{00})^0 =_1 x^0 \wedge y^0$.

The set of sharp elements of $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ forms a subsemilattice of $(L, \wedge)$:

Proposition 1

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and $a, b \in S$. Then $a \wedge b \in S$.

The following is an expected statement about the set of dense elements:

Proposition 2

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. Then D forms a D-filter of $\mathbf{L}$.

It is well-known that the set $\mathcal{F}$ of all filters of $\mathbf{L}$ forms a complete lattice with respect to inclusion. Using the Proposition 2 one can recognize that the set of all D-filters of $\mathbf{L}$ forms a complete sublattice of $\mathcal{F}$ with bottom element D.

D-filters were described for distributive pseudocomplemented lattices by M.S. Rao. However, we are going to show that similar result can be stated also for lattices being neither pseudocomplemented nor distributive, but satisfying a weaker condition.

Theorem 2

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and A a non-empty subset of L such that for each $x, y \in \overline{A}$ the following condition holds:

$$\text{if } 1 \in (x^{00} \vee z^{00}) \wedge (y^{00} \vee z^{00}) \text{ for all } z \in A \text{ then } 1 \in (x^{00} \wedge y^{00}) \vee z^{00} \text{ for all } z \in A.$$

Then $\overline{A}$ is a D-filter of $\mathbf{L}$.

In the next example we show a lattice being neither distributive nor pseudocomplemented, but satisfying the condition from Theorem 2.

Example 2

Consider the non-distributive lattice $\mathbf{L}$ depicted in Fig. 2:

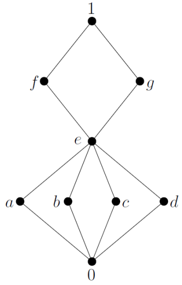

Fig. 2
Non-distributive non-pseudocomplemented lattice

We have

x	0	a	b	c	d	e	f	g	1
x^0	1	bcd	acd	abd	abc	0	0	0	0
x^{00}	0	a	b	c	d	1	1	1	1
$\overline{x}$	F_e	F_e	F_e	F_e	F_e	F_0	F_0	F_0	F_0

$D = F_e$ and $S = \{0, a, b, c, d, 1\}$. Hence $\mathbf{L}$ is not pseudocomplemented. Further, F_a is a D-filter of $\mathbf{L}$, $\overline{\{a, b\}} = F_e$ is a D-filter of $\mathbf{L}$ in accordance with Theorem 2 and $\{a, b\}^D = \{c, d, e, f, g, 1\}$.

The concept of a Stone lattice was introduced by R. Balbes and A. Horn, and also by T.P. Speed. Recall that a bounded pseudocomplemented lattice $(L, \vee, \wedge, 0, 1)$ is called *Stone* if

$$x^* \vee x^{**} = 1 \text{ and } x^* \vee y^* = (x \wedge y)^* \text{ for all } x, y \in L.$$

In analogy to this definition we define for our sake the following two concepts.

Definition 1

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. Then $\mathbf{L}$ is called *Stonean* if

$$1 \in x^{00} \vee y^{00} \text{ for every } x \in L \text{ and every } y \in x^0 \tag{1}$$

and *D-Stonean* if it is both Stonean and if

$$\text{for all } x, y \in L, x \vee y \in D \text{ is equivalent to } 1 \in x^{00} \vee y^{00}. \tag{2}$$

Observe that (1) is equivalent to $x^0 \subseteq \overline{x}$ for all $x \in L$, and (2) is equivalent to $\overline{x} = x^D$ for all $x \in L$. Hence $\mathbf{L}$ is Stonean if and only if $x^0 \subseteq \overline{x}$ for all $x \in L$, and $\mathbf{L}$ is D-Stonean if $x^0 \subseteq \overline{x} = x^D$ for all $x \in L$.

Example 3

Consider the non-distributive lattice $\mathbf{L}$ depicted in Fig. 3. We have $D = F_f$, $S = \{0, b, c, e, g, 1\}$.

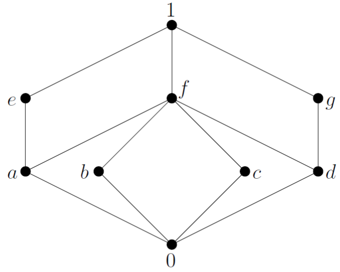

Fig. 3
Non-distributive non-pseudocomplemented non-Stonean lattice

x	0	a	b	c	d	e	f	g	1
x^0	1	bcg	ceg	beg	bce	bcg	0	bce	0
x^{00}	0	e	b	c	g	e	1	g	1
$\overline{x}$	F_f	$bcdfg1$	$adefg1$	$adefg1$	$abcef1$	$bcdfg1$	L	$abcef1$	F_0

Hence $\mathbf{L}$ is not pseudocomplemented. Moreover, $\mathbf{L}$ satisfies neither (1) nor (2) since $c \in b^0$ and $b \vee c = f \in D$, but $1 \notin f = b \vee c = b^{00} \vee c^{00}$. Therefore $\mathbf{L}$ is not Stonean.

Theorem 3

Conditions (1) and (2) of Definition 1 are independent.

How the concept of a Stonean lattice is related with its filters is shown in the following theorem.

Theorem 4

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a Stonean lattice satisfying the ACC. Then (2) is equivalent to any single of the following statements:

(i) $\overline{F} = F^D$ for all filters F of $\mathbf{L}$,

(ii) $\overline{x} = x^D$ for all $x \in L$,

(iii) for every two filters F, G of $\mathbf{L}$, $F \cap G \subseteq D$ is equivalent to $F \subseteq \overline{G}$.

The results of the previous theorem can be checked in the following example.

Example 4

Consider the non-distributive lattice $\mathbf{L}$ depicted in Fig. 5. We have $D = F_1$ and $S = \{0, b, c, d, 1\}$.

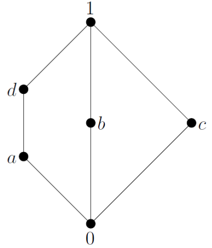

Fig. 5
Non-distributive non-pseudocomplemented D-Stonean lattice

x	0	a	b	c	d	1
x^0	1	bc	cd	bd	bc	0
x^{00}	0	d	b	c	d	1
$\overline{F_x} = F_x^D = \overline{x} = x^D$	F_1	$bc1$	$acd1$	$abd1$	$bc1$	F_0

Hence $\mathbf{L}$ is not pseudocomplemented, but it is D-Stonean. Moreover, for $x, y \in L$ both $F_x \cap F_y \subseteq D$ and $F_x \subseteq \overline{F_y}$ are (in accordance with Theorem 4) equivalent to

$$1 \in \{x, y\} \text{ or } (x, y \in \{a, b, c, d\} \text{ and } x \neq y \text{ and } \{x, y\} \neq \{a, d\}).$$

We are going to introduce the so-called coherent filters. For this, let us define the operator c on filters of $\mathbf{L}$ as follows:

$$c(F) := \{x \in L \mid \overline{x} \wedge F = L\}.$$

Definition 2

A *filter* F of a bounded lattice $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ satisfying the ACC is called *coherent* if $c(F) = F$.

One can easily show that if F and G are filters of $\mathbf{L}$ with $F \subseteq G$ then $c(F) \subseteq c(G)$. Hence $c(F \cap G) \subseteq c(F) \cap c(G)$ for all filters F, G of $\mathbf{L}$.

Example 5

Consider the lattice from Fig. 2. Then we have

x	0	a	b	c	d	e	f	g	1
$c(F_x)$	F_0	F_e	F_e	F_e	F_e	F_e	F_e	F_e	F_e

and hence F_0 and F_e are the only coherent filters. The filters F_a and F_f of the lattice depicted in Fig. 3 are coherent, but the filter F_b is not since $b \in F_b \setminus c(F_b)$.

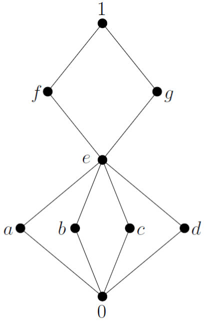

Fig. 2
Non-distributive non-pseudocomplemented lattice

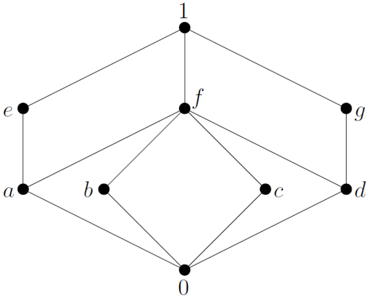

Fig. 3
Non-distributive non-pseudocomplemented non-Stonean lattice

It is easy to see that if a lattice $\mathbf{L}$ is distributive then $c(F)$ is closed under $\wedge$ for every filter F of $\mathbf{L}$. However, we can show that this holds for not necessarily distributive lattices under a weak condition.

Proposition 3

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and F a proper filter of $\mathbf{L}$ and $a, b \in L$. Then the following holds:

(i) If $c(F)$ is closed with respect to $\wedge$ then $c(F)$ is a D-filter of F,

(ii) the inclusion $\overline{a} \wedge \overline{b} \subseteq \overline{a \wedge b}$ holds if and only if $x, y \in L$ and $1 \in (x^{00} \vee a^{00}) \wedge (y^{00} \vee b^{00})$ together imply $1 \in (x \wedge y)^{00} \vee (a \wedge b)^{00}$,

(iii) if $\overline{x} \wedge \overline{y} \subseteq \overline{x \wedge y}$ for all $x, y \in c(F)$ then $c(F)$ is closed with respect to $\wedge$ and hence a D-filter of $\mathbf{L}$.

The condition in (ii) of Proposition 3 holds for all proper filters of the lattice from Fig. 2.

For D-Stonean lattices we can prove the following result.

Theorem 5

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a D-Stonean lattice and F a D-filter of $\mathbf{L}$. Then $c(F) \subseteq F$.

Let us note that the condition that $\mathbf{L}$ is D-Stonean is only sufficient but not necessary.

Example 6

The filter F_a of the non-Stonean lattice visualized in Fig. 2 is not coherent since $a \in F_a \setminus c(F_a)$, but $c(F_a) = F_e \subseteq F_a$.

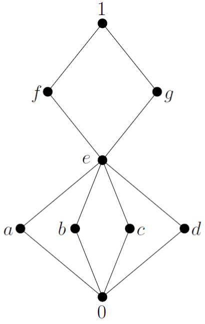

Fig. 2
Non-distributive non-pseudocomplemented lattice

Now we turn our attention to the so-called closed filters.

A *filter* F of $\mathbf{L}$ is called *closed* if it is a closed subset of L as defined in the introduction, i.e. if $\overline{\overline{F}} = F$. Of course, $F \subseteq \overline{\overline{F}}$ holds for every filter F of $\mathbf{L}$.

Example 7

The filter F_a of the D-Stonean lattice visualized in Fig. 5 is closed and coherent since $\overline{\overline{F_a}} = \overline{\{b, c, 1\}} = F_a$ and $c(F_a) = F_a$.

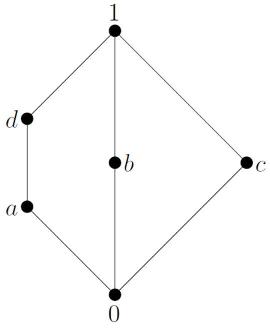

Fig. 5
Non-distributive non-pseudocomplemented D-Stonean lattice

Corollary 1

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and $A \subseteq L$. Then the following holds:

(i) $D \subseteq \overline{A}$,

(ii) $\overline{A} = L$ if and only if $A \subseteq D$,

(iii) every closed filter of $\mathbf{L}$ is a D-filter.

Example 8

The filter F_e of the lattice depicted in Fig. 2 is both coherent and closed since $c(F_e) = F_e$ and $\overline{\overline{F_e}} = \overline{F_0} = F_e$, but its subfilter F_f is neither coherent nor closed since $e \in c(F_f) \setminus F_f$ and $\overline{\overline{F_f}} = \overline{F_0} = F_e \neq F_f$.

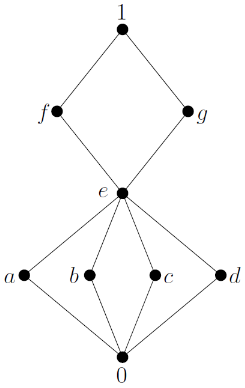

Fig. 2

Non-distributive non-pseudocomplemented lattice

Recall that a *filter* F of a lattice $(L, \vee, \wedge)$ is

- *proper* if $F \neq L$,
- *maximal* if it is a maximal proper filter,
- *prime* if $x, y \in L$ and $x \vee y \in F$ imply $x \in F$ or $y \in F$.

It is well-known that every maximal filter of a distributive lattice is prime. Unfortunately, this does not hold for non-distributive lattices. For example, the filter F_a of the lattice in Fig. 5 is maximal, but it is not prime since $b \vee c = 1 \in F_a$, but $b, c \notin F_a$.

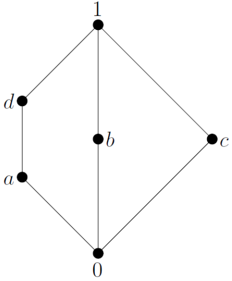

Fig. 5

Non-distributive non-pseudocomplemented D-Stonean lattice

However, we can prove the following:

Theorem 6

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and F a proper filter of $\mathbf{L}$. Then the following holds:

(i) F is maximal if and only if $x^0 \cap F \neq \emptyset$ for all $x \in L \setminus F$,

(ii) if F is maximal then F is a D-filter,

(iii) if F is maximal and $(x \vee y) \wedge z = 0$ for all $x, y \in L \setminus F$ and all $z \in F$ with $x \wedge z = y \wedge z = 0$ then F is prime.

Of course, condition (iii) of Theorem 6 holds for any distributive lattice. However, the following example shows that it may be satisfied also in a non-distributive lattice.

Example 9

The non-distributive lattice visualized in Fig. 6 satisfies the condition in (iii) of Theorem 6 for the maximal proper filter F_a. In accordance with this theorem, this filter is prime.

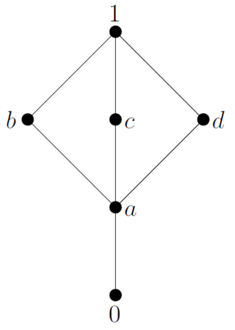

Fig. 6
Non-distributive lattice

Definition 3

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC. The filter F of $\mathbf{L}$ is called *median* if it is maximal and if for each $x \in F$ there exists some $y \in L \setminus F$ with $1 \in x^{00} \vee y^{00}$.

Example 10

The filter F_a of the non-Stonean lattice $(L, \vee, \wedge, 0, 1)$ visualized in Fig. 3 is median, closed and coherent since $b \in L \setminus F_a$, $1 \in x^{00} \vee b^{00}$ for every $x \in F_a$ and $\overline{\overline{F_a}} = \overline{\{b, c, d, f, g, 1\}} = F_a$.

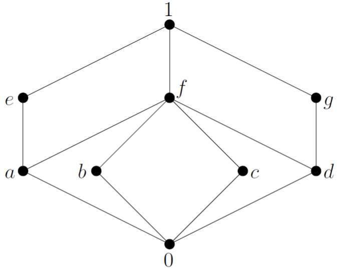

Fig. 3

Non-distributive non-pseudocomplemented non-Stonean lattice

Now we present several basic properties of proper, prime and median filters.

Theorem 7

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC, F a proper filter of $\mathbf{L}$ and $a \in L$. Then the following holds:

(i) If $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ for all $x, y \in L$, F is a prime D-filter and $a \in L \setminus F$ then $\overline{a} \subseteq F$,

(ii) if $x^{00} \vee y^{00} \leq_1 (x \vee y)^{00}$ for all $x, y \in L$, F is a median D-filter and $a \in F$ then $\overline{\overline{a}} \subseteq F$,

(iii) If $a \in F$ then $a^0 \not\subseteq F$,

(iv) if $\mathbf{L}$ is D-Stonean and F a prime D-filter of $\mathbf{L}$ then $a \in F$ if and only if $a^0 \not\subseteq F$.

The next result illuminates the relationship between coherent and median filters.

Theorem 8

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a bounded lattice satisfying the ACC and F a maximal filter of $\mathbf{L}$. Then the following holds:

(i) If F is coherent then it is median,

(ii) if $\overline{F} \not\subseteq F$ then F is median,

(iii) if $F = \overline{L \setminus F}$ then F is median.

An interesting property of filters being both median and prime shown for distributive pseudocomplemented lattices by M.S. Rao can be proved also in a more general case.

Proposition 4

Let $\mathbf{L} = (L, \vee, \wedge, 0, 1)$ be a D-Stonean lattice, F a median prime filter of $\mathbf{L}$ and $a, b \in L$. Then the following holds:

(i) If F is a D-filter of $\mathbf{L}$, $a \in F$ and $\overline{a} = \overline{b}$ then $b \in F$,

(ii) if $a \vee b \in F$ then there exists some $c \in L \setminus F$ such that $\{a \vee c, b \vee c\} \cap D \neq \emptyset$.

References

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Thank you for your attention!