Meet-irreducibility of congruence lattices of prime-cycled algebras

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SSAOS 2023, 2.9.–8.9.2023
Stará Lesná, Slovensko

Let A be given, finite set.

- **a congruence** of an algebra (A, F) is an equivalence on A, which is compatible with all operations in F
- the system of all congruences of an algebra (A, F), ordered by inclusion, forms a lattice $\mathrm{Con}(A, F)$
- the system of all lattices $\mathrm{Con}(A, F)$ with a given base set A forms a lattice $\mathcal{E}_A$

- L - lattice, $x, a, b \in L$
  - x is $\wedge$-irreducible if $x = a \wedge b$ implies $x \in \{a, b\}$
  - x is $\vee$-irreducible if $x = a \vee b$ implies $x \in \{a, b\}$
- $F \subseteq G$ implies $\mathrm{Con}(A, G) \subseteq \mathrm{Con}(A, F)$, hence $\mathrm{Con}(A, \{f_1, f_2, \dots\}) = \bigwedge \mathrm{Con}(A, f_i)$, $i \in 1, 2, \dots$
- all $\wedge$-irreducible elements in $\mathcal{E}_A$ are of the form $\mathrm{Con}(A, f)$ for a single mapping f

$\Longrightarrow$ **it is sufficient to consider monounary algebras**

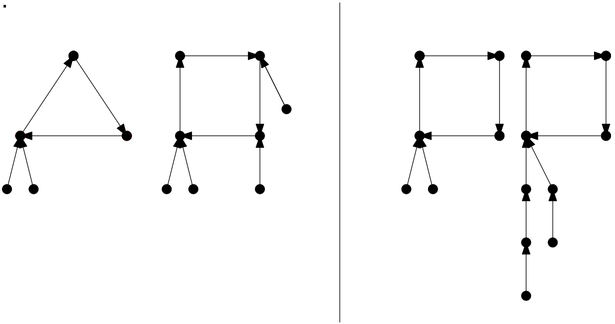
Let (A, f) be a monounary algebra.

- operation $f \in A^A$ is called **trivial**, if it is an identity of a constant
- element $x \in A$ is called **cyclic** if $\exists n \in \mathbb{N} : f^n(x) = x$, otherwise it is called **noncyclic**
- let $x \in A$ be noncyclic and let $y \in A$ be a cyclic element such that $f^k(y) = f^k(x), k = \min\{n \in \mathbb{N} : f^n(y) = f^n(x)\}$. We call y **a colleague of** x and we denote it x'.

Preliminary

- if $f(x)$ belongs to a cycle for $\forall x \in A$, then (A, f) is called **an algebra with short tails**
- if there exists $x \in A$ such that $f(x)$ is noncyclic, we say that (A, f) **contains long tails**

Example:



- (A, f) is called **connected algebra** if for every $x, y \in A$ there exist $m, n \in \mathbb{N}_0$ such that $f^m(x) = f^n(y)$, otherwise it is called **non-connected**
- the maximal connected subalgebras of (A, f) are called **components** of (A, f)

Preliminary

Lemma 1

$\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$ $\iff$ there exists a set G of nontrivial operations such that

$$\mathrm{Con}(A, f) = \bigcap_{g \in G} \mathrm{Con}(A, g), \ (\forall g \in G)\, \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g).$$

Lemma 2

Let (A, f) be a monounary algebra containing two distinct noncyclic elements a, b such that $f(b) = a$. If $\mathrm{Con}(A, f)$ is $\wedge$-reducible, then there exists a nontrivial openeration g on A such that $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)$ and $g(b) = a$.

Meet-irreducibility in $\mathcal{E}_A$

- **characterization of all meet-irreducible elements** in $\mathcal{E}_A$ is an open problem
- partial answers are known in some cases

Known cases

- 2018 (Studenovská, Pöschel, Radeleczki):
  - (A, f) is **an acyclic algebra**, i.e. each cycle contains 1 element
  - (A, f) is **a permutation algebra**, i.e. contains only cyclic elements
- 2018 (Studenovská, Janičková)
  - each cycle of (A, f) contains at most 2 elements
  - (A, f) is **an algebra with short tails**, i.e. each element of A maps into cycle
- 2020 (Studenovská, Janičková)
  - (A, f) is **a connected algebra**
- 2022 (Janičková)
  - some sufficient conditions if (A, f) contains a connected subalgebra with $\wedge$-irreducible congruence lattice

Open cases

$\Longrightarrow$ **it remains to consider monounary algebras such that:**

- (A, f) is non-connected algebra
- (A, f) contains at least one cycle with at least 3 elements
- (A, f) contains at least one element which maps into non-cyclic element (i.e. contains long tails)

Today´s case

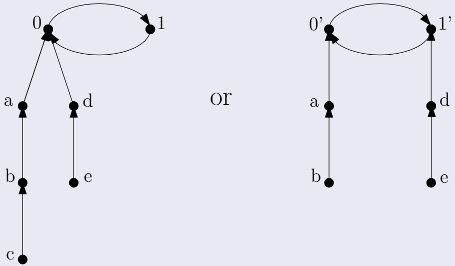
- (A, f) such that each cycle contains prime number of elements - **prime-cycled algebras**

Prime-cycled algebras

Proposition 3 (SJ, 2018)

Let (A, f) be a prime-cycled algebra such that $|A| > 2$ and each cycle contains 2 elements. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible $\iff$ one of the following holds:

1. *(A, f) contains a subalgebra of the type:*



2. *(A, f) is an algebra with short tails and at least 2 cycles.*

Prime-cycled algebras

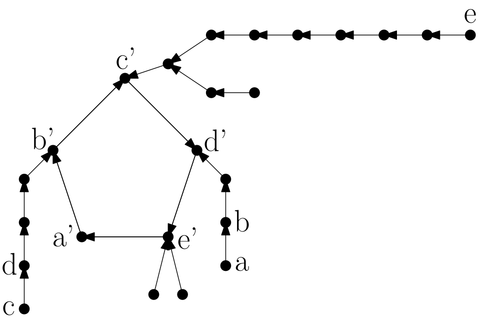
Proposition 4 (SPR, 2018)

Let (A, f) *be a prime-cycled algebra with short tails. Then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible* $\iff$ *each cycle contains same number of elements, and there are at least 2 cycles in* (A, f).

Definition 5

Let (A, f) be a connected monounary algebra with p cyclic elements such that $p \geq 3$ is a prime number. We say that **the set of cyclic elements of** (A, f) **is covered** if for every cyclic element $c \in A$ there exist a noncyclic element $x \in A$ such that $c = x'$ and $f(x)$ is noncyclic.

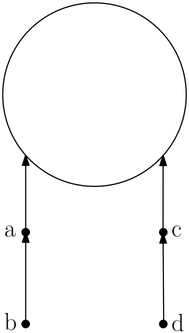
Example:



Prime-cycled algebras

Proposition 6 (SJ, 2020)

Let (A, f) be a connected prime-cycled algebra with cycle containing at least 3 elements. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible $\iff$ the set of cyclic elements is covered and there exist distinct noncyclic elements $a, b, c, d \in A$ such that $f(a), f(c)$ are cyclic and $f(b) = a, f(d) = c$.



Prime-cycled algebras

Further, we assume that there is

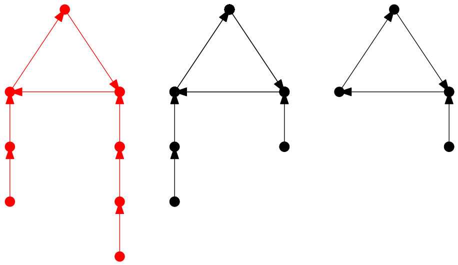
- at least one cycle with ≥ 3 elements
- at least one element which maps into noncyclic element

Prime-cycled algebras

Proposition 7 (J, 2022)

Let (A, f) be a prime-cycled algebra such that each cycle contains $p \geq 3$ elements, and there is a connected subalgebra B of (A, f) such that $\mathrm{Con}(B, f \restriction B)$ is $\wedge$-irreducible in $\mathcal{E}_B$. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$.

Example:



Idea of proof:

1) By way of contradiction, if $\mathrm{Con}(A, f)$ is $\wedge$-reducible, then by Lemma 2, there must exist a nontrivial openeration g on A such that $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)$ and $g(b) = a$.

2) We show that for such g, it holds $g(x) = f(x)$ for every $x \in A$ which implies $\mathrm{Con}(A, f) = \mathrm{Con}(A, g)$, a contradiction.

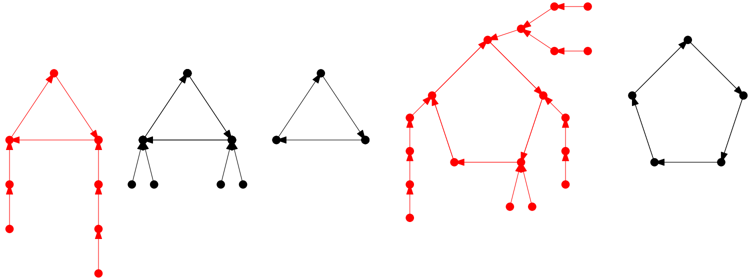
$\implies \mathrm{Con}(A, f)$ is $\wedge$-irreducible. □

Prime-cycled algebras

Proposition 8 (J, 2022)

Let (A, f) be a prime-cycled algebra with cycles containing $p_1, \ldots, p_k$ elements where $p_1, \ldots, p_k \geq 3$. Let for each $i \in \{1, \ldots, k\}$ exist a connected subalgebra B of (A, f) with p_i cyclic elements such that $\mathrm{Con}(B, f \restriction B)$ is $\wedge$-irreducible in $\mathcal{E}_B$. Then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$.

Example:

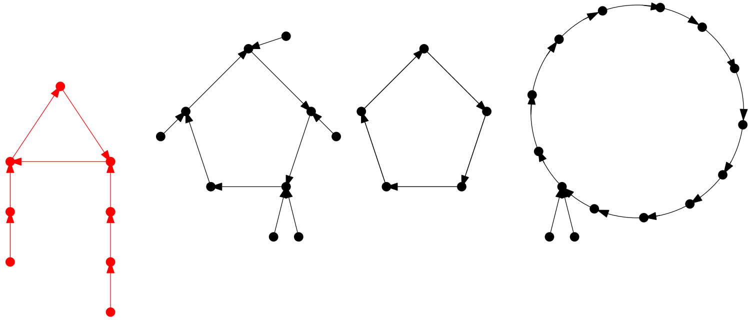


Prime-cycled algebras

Proposition 9

Let (A, f) be a non-connected prime-cycled algebra containing exactly one component T with long tail. Let T contain p cyclic elements, and let there be no other component of (A, f) which would contain p cyclic elements. Then $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Example:



Idea of proof:

$$g_1(x) = \begin{cases} f(x), & \text{if } x \in T \\ x', & \text{if } x \notin T \end{cases}$$

$$g_2(x) = \begin{cases} x', & \text{if } x \in T \\ f(x), & \text{if } x \notin T \end{cases}$$

1) g_1, g_2 are nontrivial, $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g_1), \mathrm{Con}(A, g_2)$.

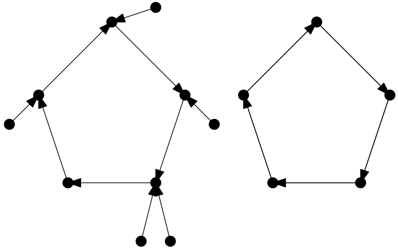
2) $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_1) \cap \mathrm{Con}(A, g_2)$.

$\implies$ $\mathrm{Con}(A, f)$ is $\wedge$-reducible. □

Corollary 10

Let (A, f) be a non-connected prime-cycled algebra containing exactly one component T with long tail. Then $\mathrm{Con}(A, f)$ is $\wedge$-reducible.

Example:



Idea of proof:

Let S be the set of elements from components which have the same number of cyclic elements as T. Then:

$$g_3(x) = \begin{cases} f(x), & \text{if } x \in S \\ x', & \text{if } x \notin S \end{cases}$$

$$g_4(x) = \begin{cases} x', & \text{if } x \in S \\ f(x), & \text{if } x \notin S \end{cases}$$

1) g_3, g_4 are nontrivial, $\mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g_3), \mathrm{Con}(A, g_4)$.

2) $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_3) \cap \mathrm{Con}(A, g_4)$.

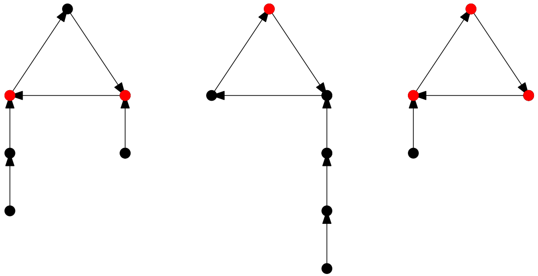
$\implies$ $\mathrm{Con}(A, f)$ is $\wedge$-reducible. □

Prime-cycled algebras

Proposition 11

Let (A, f) be a non-connected prime-cycled algebra such that each cycle contains $p \geq 3$ elements, there is at least one components with long tails, and there is no connected subalgebra B of (A, f) such that $\mathrm{Con}(B, f \restriction B)$ is $\wedge$-irreducible in $\mathcal{E}_B$. Then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$.

Example:



Idea of proof:

Each cycle contains at least 1 not covered element, let us denote it $n_1, n_2, ..., n_k$.

$$g_5(x) = \begin{cases} x, & \text{if } x \in \{n_1, n_2, ..., n_k\} \\ x', & \text{if } x \text{ is noncyclic and } f(x) \text{ is cyclic} \\ f(x), & \text{otherwise} \end{cases}$$

$$g_6(x) = \begin{cases} f(x), & \text{if } x \text{ is cyclic} \\ f(x'), & \text{if } x \text{ is noncyclic} \end{cases}$$

Then $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_3) \cap \mathrm{Con}(A, g_4)$.

$\implies \mathrm{Con}(A, f)$ is $\wedge$-reducible. □

Last open cases for prime-cycled algebras

- (A, f) contains at least one cycle with 2 elements, and at least one cycle with ≥ 3 elements
- each cycle has ≥ 3 elements, and there is a cycle with p elements such that there is no connected subalgebra B of (A, f) with p cyclic elements such that $\mathrm{Con}(B, f \restriction B)$ is $\wedge$-irreducible in $\mathcal{E}_B$

What next?

- **Necessary and sufficient conditions** under which congruence lattice of a **prime-cycled algebra** is $\wedge$-irreducible.
- Characterization of all $\wedge$-irreducible elements in $\mathcal{E}_A$.

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Thank you for your attention.

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