Minimal closed monoids for the Galois connection $\mathrm{End} - \mathrm{Con}$

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Introduction

This talk is a small contribution to the topic concerning the lattice of congruence lattices of algebras on a given set.

Let A be a fixed finite set.

- algebra (A, F)
- system of all congruences of an algebra (A, F) ordered by inclusion forms a lattice $\mathrm{Con}(A, F)$
- system of all $\mathrm{Con}(A, F)$ of all algebras (A, F) ordered by inclusion forms a lattice $\mathcal{E}_A$

Each congruence lattice is a complete sublattice of $\mathrm{Eq}(A)$. Due to the Galois connection $\mathrm{End} - \mathrm{Con}$, the endomorphism monoids $M = \mathrm{End}\,\mathrm{Con}(A, F)$ of such congruence lattices also form a lattice

$$\mathcal{M}_A := \{\mathrm{End}\,\mathrm{Con}(A, F) \mid F \subseteq A^A\},$$

which is dual to $\mathcal{E}_A$.

Basic notions

To fix the notions and notation, recall that a binary relation $\theta \subseteq A \times A$ is *compatible* with (or *invariant* for) a function $f \in A^A$, we also say f *preserves* ρ, denoted by $f \triangleright \rho$, if

$$\forall x, y \in A : (x, y) \in \theta \implies (fx, fy) \in \theta.$$

Equivalently this expresses the fact that f is an *endomorphism* of θ ($f \in \operatorname{End}\theta$) and – provided that θ is an equivalence relation – that θ is a *congruence* of the algebra (A, f) ($\theta \in \operatorname{Con}(A, f)$).
The relation $\triangleright$ induces a Galois connection, namely $\operatorname{End} - \operatorname{Con}$, between unary mappings and equivalence relations, defined by

$$\operatorname{End} Q := \{f \in A^A \mid \forall \rho \in Q : f \triangleright \rho\} \quad \text{for } Q \subseteq \operatorname{Eq}(A)$$

$$\operatorname{Con}(A, F) := \operatorname{Con} F := \{\theta \in \operatorname{Eq}(A) \mid \forall f \in F : f \triangleright \rho\} \quad \text{for } F \subseteq A^A.$$

Basic notions

The least monoid $T \in \mathcal{M}_A$ consists of all unary functions that preserve *all* equivalence relations on A, that is, we have $T = \mathrm{End}\,\mathrm{Eq}(A)$. Therefore, the monoid T and the functions in it are called *trivial*.
If $3 \leq |A|$, then $T := \{\mathsf{id}_A\} \cup C_A$, where id_A is the identity mapping and C_A denotes the set of all unary constant functions on A.

The central role for describing the minimal endormorphism monoids of congruence lattices: functions of type I, II and III.

Coatoms

Theorem

The coatoms of $\mathcal{E}_A$ are exactly the congruence lattices of the form $\mathrm{Con}(A, f)$ where $f \in A^A$ satisfies:

(I) *f is nontrivial and $f^2 = f$, or*

(II) *f is nontrivial, f^2 is a constant, say u, and $|\{x \in A \mid fx = u\}| \geq 3$, or*

(III) *$f^p = id_A$ for some prime p, such that the permutation f has at least two cycles of length p.*

Theorem

The same functions also determine the coatoms $\mathrm{Quord}(A, f)$ in the lattice $\mathcal{L}_A$ of quasiorder lattices of algebras on the base set A.

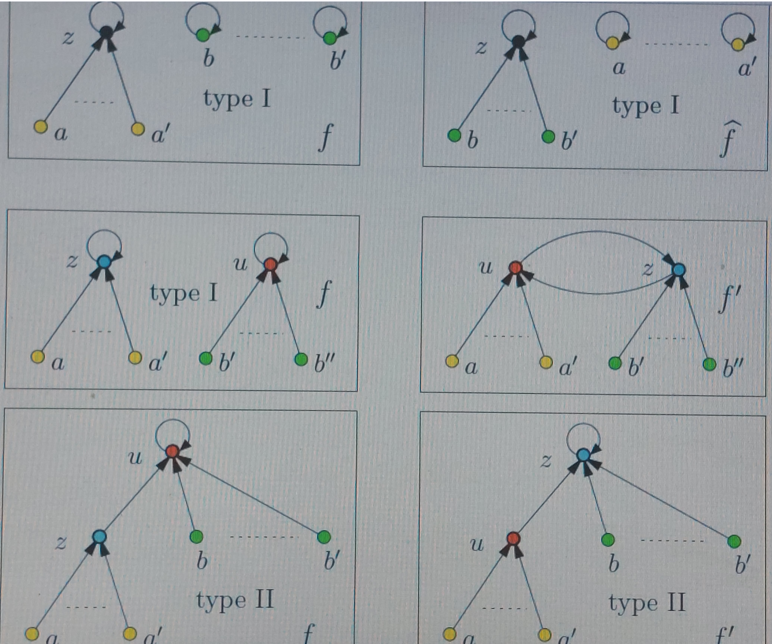
Notation

For a function f of type I with exactly one nontrivial component (whose fixed point is denoted by z) let $\widehat{f}$ be defined as follows:

$$\widehat{f}x := \begin{cases} z & \text{if } fx = x, \\ x & \text{otherwise.} \end{cases} \tag{1}$$

For a function f with a 2-element image $\mathrm{Im}(f) = \{z, u\}$, let f' be defined as follows:

$$f'x := \begin{cases} u & \text{if } fx = z, \\ z & \text{if } fx = u. \end{cases} \tag{2}$$



z
b
b′
type I
a
a′
f
z
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Result

The other side of the Galois connection $\mathrm{End} - \mathrm{Con}$, i.e., determine $\mathrm{End}\,\mathrm{Con}(A, f)$ for all coatoms $\mathrm{Con}(A, f)$, i.e., the minimal nontrivial endomorphism monoids in the lattice $\mathcal{M}_A$.

Theorem

Let $3 \leq |A| < \infty$.

(A) *The following table describes the Galois closure $\mathrm{End}\,\mathrm{Con}(A, f)$ for all functions f of type I, II or III. The number s indicates the number of nontrivial functions in the closure.*

(B) *The Galois closures $\mathrm{End}\,\mathrm{Quord}(A, f)$ for the functions of type I and II are always $\{f\} \cup T$ and for functions of type III we have*
$\mathrm{End}\,\mathrm{Quord}(A, f) = \mathrm{End}\,\mathrm{Con}(A, f) = \{f, f^2, \dots, f^{p-1}\} \cup T$.

	type of f	$\lvert \mathrm{Im}(f) \rvert$	number of nontrivial components K of f	other conditions	Galois closure $\mathrm{End}\,\mathrm{Con}(A, f)$	s
(1)	I	≥ 3	≥ 2		$\{f\} \cup T$	1
(2)	I	≥ 3	1	$\lvert K \rvert \geq 3$	$\{f, \widehat{f}\} \cup T$	2
(3)	I	≥ 3	1	$\lvert K \rvert = 2$	$\{f, \widehat{f}, (\widehat{f})'\} \cup T$	3
(4)	I	2	2		$\{f, f'\} \cup T$	2
(5)	I	2	1	$\lvert A \rvert > 3$	$\{f, f', \widehat{f}\} \cup T$	3
(6)	I	2	1	$\lvert A \rvert = 3$	$\{f, f', \widehat{f}, (\widehat{f})'\} \cup T$	4
(7)	II	≥ 3			$\{f\} \cup T$	1
(8)	II	2			$\{f, f'\} \cup T$	2
(9)	III			cycle length p	$\{f, f^2, \ldots, f^{p-1}\} \cup T$	$p-1$

Thank you for your attention.