The combinatorics of weak congruences of lattices

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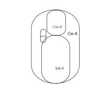

Weak congruence

symmetric, transitive, compatible binary relation on an algebra

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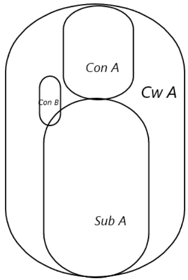
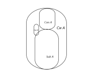
The collection $\mathrm{Cw}(A)$ of weak congruences on an algebra A is an algebraic lattice under inclusion. The sublattices of the weak congruence lattice of A are $\mathrm{Con}(A)$, $\mathrm{Sub}(A)$ and $\mathrm{Con}(B)$, for every subalgebra B. See [3].



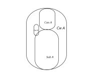


Some preliminaries

How to calculate the number of weak congruences?

$$|\mathrm{Cw}(B,\vee)| = 1 + \sum_{\substack{B^* \in \mathrm{Sub}\, B \\ B^* \neq \emptyset}} |\mathrm{Con}\, B^*|$$

First see [4].



Theorem (H., D. Ahmed and Z. Németh)

If $(B, \vee)$ is a semilattice defined by a binary tree B, then

$$|\mathrm{Cw}(B, \vee)| = 4(|\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)|) - (|\mathrm{Cw}(B_1, \vee)| + |\mathrm{Cw}(B_2, \vee)|),$$

where B_1, B_2 are the left and right maximal subtrees of the tree, respectively.

The calculation

Proof. Both $\mathrm{Cw}(B_1 +_{ord} 1', \vee)$ and $\mathrm{Cw}(B_2 +_{ord} 1', \vee)$ contain tl
only congruence on the singleton $\{1\}$, is

$$|\mathrm{Cw}(B, \vee)| = 1 + \sum_{\substack{B^* \in \mathrm{Sub}\, B \\ B^* \neq \emptyset}} |\mathrm{Con}(B^*, \vee)|$$

$$= 1 + \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)\, \mathrm{Con}(B_2^*, \vee)| + |\mathrm{Cw}(B_1 +_{ord} \{$$

$$+ |\mathrm{Cw}(B_2 +_{ord} \{1'\}, \vee)$$

using Lemma 4.4.1,

$$= \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)| \cdot |\mathrm{Con}(B_2^*, \vee)| + (3\, |\mathrm{Cw}(B_1, \vee)| - 1)$$

$$+ (3\, |\mathrm{Cw}(B_2, \vee$$

$$= \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i, \\ B_i^* \neq \emptyset}} 4\, |\mathrm{Con}(B_1^*, \vee)| \cdot |\mathrm{Con}(B_2^*, \vee)| + 4\, |\mathrm{Cw}(B_1, \vee)| + 4\, |\mathrm{Cw}(B_2, \vee)|$$

$$- |\mathrm{Cw}(B_1, \vee)| - |\mathrm{Cw}(B_2, \vee)| - 4.$$

Now for B_i,

$$|\mathrm{Cw}(B_i, \vee)| = 1 + \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i \\ B_i^* \neq \emptyset}} |\mathrm{Con}(B_i^*, \vee)|\,,$$

and let us use

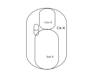
$$|\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)| = (1 + \sum_{\substack{B_1^* \in \mathrm{Sub}\, B_1 \\ B_1^* \neq \emptyset}} |\mathrm{Con}(B_1^*, \vee)|)$$

$$(1 + \sum_{\substack{B_2^* \in \mathrm{Sub}\, B_2 \\ B_i^* \neq \emptyset}} |\mathrm{Con}(B_2^*, \vee)|)$$

$$= \sum_{\substack{B_i^* \in \mathrm{Sub}\, B_i \\ B_i^* \neq \emptyset}} |\mathrm{Con}(B_1^*, \vee)| \cdot |\mathrm{Con}(B_2^*, \vee)| + |\mathrm{Cw}(B_1, \vee)| + |\mathrm{Cw}(B_2, \vee)| - 1.$$

Then we arrive at

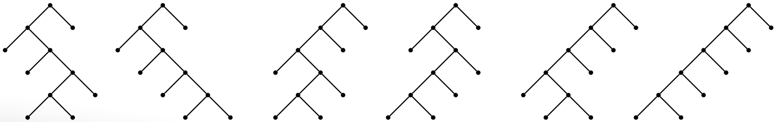
$$|\mathrm{Cw}(B, \vee)| = 4\, |\mathrm{Cw}(B_1, \vee)| \cdot |\mathrm{Cw}(B_2, \vee)| - |\mathrm{Cw}(B_1, \vee)| - |\mathrm{Cw}(B_2, \vee)|\,.$$



Theorem (H., D. Ahmed and Z. Németh)
If $(B, \vee)$ is a semilattice defined by a prickly-snake B of height h, then

$$|\mathrm{Cw}(B, \vee)| = 7 \cdot |\mathrm{Cw}(B_1, \vee)| - 2 = \frac{5 \cdot 7^h + 1}{3},$$

where B_1 is the left maximal subtree of the tree.



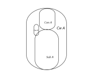
Theorem (H., D. Ahmed and Z. Németh)
If $(B, \vee)$ is a semilattice defined by a perfect binary tree B of height h, then

$$|\mathrm{Cw}(B, \vee)| = 4 \cdot |\mathrm{Cw}(B_1, \vee)|^2 - 2 \cdot |\mathrm{Cw}(B_1, \vee)|\,,$$

where B_1 is the left maximal subtree of the tree.
Moreover,

$$|\mathrm{Cw}(B, \vee)| = \left\lceil \tfrac{1}{4}\, C^{2^{h+1}} \right\rceil, \quad C = 2.61803398874989\ldots$$

where $\lceil x \rceil$ denotes the least integer greater than or equal to x.



Conjecture

The constant C in this Theorem seems to be equal to $\frac{3+\sqrt{5}}{2}$, i.e. the Golden Ratio plus 1. Our numerical experiments with Mathlab, Mathematica and Maple affirm this idea; there are differences in the 16th decimal places but they could be numerical errors. On the other hand, we have been unable to prove or disprove the conjecture.

The greatest number of weak congruences of finite lattices

Theorem (H., A. Tepavčević)
If L is a finite lattice of size $n = |L|$, then L has at most $\frac{3^n+1}{2}$ weak congruences. Furthermore, $|\operatorname{Cw} L| = \frac{3^n+1}{2}$ if and only if L is a chain.

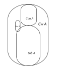
Proof of the greatest case: chain, $\frac{3^n+1}{2}$

First, we prove that if L is a chain, then $|\operatorname{Cw} L| = \frac{3^n+1}{2}$. An n-element lattice can have at most 2^n subuniverses. Furthermore, by [2], $|\operatorname{Sub} L| = 2^n$ if and only if it is a chain. By [1], an n-element lattice can have at most 2^{n-1} congruences; furthermore, $|\operatorname{Con} L| = 2^{n-1}$ if and only if it is a chain. Now

$$|\operatorname{Cw} L| = 1 + \sum_{\substack{L^* \in \operatorname{Sub} L \\ L^* \neq \emptyset}} |\operatorname{Con} L^*| = 1 + \sum_{i=1}^{n} \binom{n}{i} 2^{i-1} =$$

$$1 + \frac{\sum_{i=1}^{n} \binom{n}{i} 2^i}{2} = 1 + \frac{-1 + \sum_{i=0}^{n} \binom{n}{i} 2^i}{2} =$$

$$= 1 + \frac{-1 + (1+2)^n}{2} = \frac{3^n + 1}{2}.$$

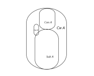


Proof of the greatest case: chain, $\frac{3^n+1}{2}$

We have to show that all the n-element lattices have fewer weak congruences than $\frac{3^n+1}{2}$. We denote the elements of L by $a_1 \prec \cdots \prec a_n$. If L' is not a chain, then it has at least two incomparable elements, say $p||q$. Of course $p \vee q \in L'$ and $p \wedge q \in L'$. We denote the remaining elements of L' by $b_1, \ldots, b_{n-4}$ arbitrarily. Now

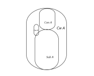
$$\left|\mathrm{Cw}\, L'\right| = 1 + \sum_{\substack{L^* \in \mathrm{Sub}\, L' \\ L^* \neq \emptyset}} \left|\mathrm{Con}\, L^*\right|$$

By [2], the sum $|\mathrm{Cw}\, L'|$ has less summands than the sum $|\mathrm{Cw}\, L|$. We make an **injection** from the summands of $|\mathrm{Cw}\, L'|$ to the summands of $|\mathrm{Cw}\, L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi : L' \to L$, $(p \wedge q) \mapsto a_1$, $p \mapsto a_2$, $q \mapsto a_3$, $(p \vee q) \mapsto a_4$, and if $x \notin \{p, q, p \wedge q, p \vee q\}$, then $\varphi(x) \in L \setminus \{a_1, a_2, a_3, a_4\}$ arbitrarily...



The second greatest number of weak congruences of finite lattices

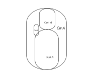
Theorem (H., A. Tepavčević)
If $|L| = n \geq 4$ and L has less than $\frac{3^n+1}{2}$ weak congruences, then the second greatest value in weak congruences is $\frac{53 \cdot 3^{n-4}+1}{2}$. Furthermore, L has $\frac{53 \cdot 3^{n-4}+1}{2}$ weak congruences if and only if $L \simeq C_1 +_{glu} B_4 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set and B_4 is the four element Boolean lattice.



We prove that all the other n-element lattices have less weak congruences. To show this, first we calculate the above number in a different way. By [2], L has $13 \cdot n^{n-4}$ subuniverses. By [1], this form of an n-element lattice L has 2^{n-2} congruences. We denote the non-comparable elements of B_4 by a and b. Now

$$|\mathrm{Cw}\, L| = 1 + \sum_{\substack{L^* \in \mathrm{Sub}\, L \\ L^* \neq \emptyset}} |\mathrm{Con}\, L^*| =$$

$$= 1 + \sum_{\substack{L^* \in \mathrm{Sub}\, L \\ B_4 \subseteq L^*}} |\mathrm{Con}\, L^*| + \sum_{\substack{L^* \in \mathrm{Sub}\, L \\ b \notin L^* \\ a \in L^*}} |\mathrm{Con}\, L^*| + \sum_{\substack{L^* \in \mathrm{Sub}\, L \\ \{a\} \cap L^* = \emptyset}} |\mathrm{Con}\, L^*| = (**)$$



Now

$$1+\sum_{\substack{L^*\in\operatorname{Sub}L\\ \{a\}\cap L^*=\emptyset}}|\operatorname{Con}L^*| = |\operatorname{Cw}C_{n-1}|$$

so

$$(**) = \sum_{i=0}^{n-4}\binom{n-4}{i}2^{i+4-2}+\sum_{i=0}^{n-2}\binom{n-2}{i}2^{i+1-1}+\frac{3^{n-1}+1}{2} =$$

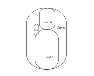
$$= 4(1+2)^{n-4}+(1+2)^{n-2}+\frac{3^{n-1}+1}{2} = \frac{53\cdot 3^{n-4}+1}{2}.$$

Consider an arbitrary n-element lattice L' that is neither a chain, nor of form $C_1 +_{glu} B_4 +_{glu} C_2$,. Clearly

$$\left|\operatorname{Cw} L'\right| = 1 + \sum_{\substack{L^* \in \operatorname{Sub} L' \\ L^* \neq \emptyset}} \left|\operatorname{Con} L^*\right|.$$

This sum contains not more summands than that of L by [2].

We show that $|\mathrm{Cw}\, L'| \leq |\mathrm{Cw}\, L|$. If L' is neither a chain, nor of the form $C_1 +_{glu} B_4 +_{glu} C_2$, then it has antichains, let $a||b$ one of them. We make an injection from the summands of $|\mathrm{Cw}\, L'|$ to the summands of $|\mathrm{Cw}\, L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi : L' \to L$. Denote the elements of B_4 in L by $\{p, q, p \wedge q, p \vee q\}$. Let $a\varphi = p$, $b\varphi = q$, $(a \wedge b)\varphi = p \wedge q$, $(a \vee b)\varphi = p \vee q$; otherwise we define φ arbitrarily but bijectively. The image of any sublattice of L' is a sublattice of L because if the considered sublattice contains both a and b, then the image of it is a sublattice of form $C_1 +_{glu} B_4 +_{glu} C_2$. If the considered sublattice contains at most one of a and b, then its image is a chain. Now clearly by [1], the image of each summand is not greater than the summand itself because the image of a sublattice is a chain or of form $C_1 +_{glu} B_4 +_{glu} C_2$, but the latter case happens only when the sublattice is not a chain.



The "“third" greatest number of weak congruences of finite lattices

Theorem (H., A. Tepavčević)

If $|L| = n \geq 4$ and $L \simeq C_1 +_{glu} N_5 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set, then the number of weak congruences of finite lattices is $\frac{125 \cdot 3^{n-5}+1}{2}$.

The "“third" greatest number of weak congruences of finite lattices

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If $|L| = n \geq 4$ and $L \simeq C_1 +_{glu} N_5 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set, then the number of weak congruences of finite lattices is $\frac{125 \cdot 3^{n-5}+1}{2}$.

Conjecture (H., A. Tepavčević)

We conjecture that $\frac{125 \cdot 3^{n-5}+1}{2}$ is the third greatest number of weak congruences of finite lattices, and the corresponding lattice is $L \simeq C_1 +_{glu} N_5 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set.

Lantern: the n-element lattice M_{n-2}

We use the notation M_1 for the 3-element chain and M_2 for the 4-element Boolean lattice. For $n \geq 3$, M_{n-2} consists of $n-2$ atoms, which are also coatoms, and of 0 and 1. So, the lattice M_{n-2} has $n-2$ atoms and n elements. We call the lattice M_{n-2} a **lantern**.

Theorem (H., A. Tepavčević)
For $n \geq 3$, the lantern M_{n-2} has $2^{n-1} + n^2 + 2n - 5$ weak congruences.

The number of weak congruences of ordinal sum of lattices

Lemma (H., A. Tepavčević)
Given finite lattices L_1 and L_2, let $L = L_1 +_{ord} L_2$. Then

$$|\operatorname{Sub} L| = |\operatorname{Sub} L_1||\operatorname{Sub} L_2|.$$

Lemma (H., A. Tepavčević)
Given finite lattices L_1 and L_2, let $L = L_1 +_{glu} L_2$. Then

$$|\operatorname{Con} L| = |\operatorname{Con} L_1||\operatorname{Con} L_2|.$$

Lemma (H., A. Tepavčević)
Given finite lattices L_1 and L_2, Let $L = L_1 +_{ord} L_2$. Then

$$|\operatorname{Cw} L| = 2 \cdot (|\operatorname{Cw} L_1| - 1)(|\operatorname{Cw} L_2| - 1) + |\operatorname{Cw} L_1| + |\operatorname{Cw} L_2| - 1.$$

Lantern on a chain

Lemma (H., A. Tepavčević)

If $L \simeq C_1 +_{ord} M_{k-2} +_{ord} C_2$, where C_1 and C_2 are chains or the empty set, and $|C_1| + |C_2| = l$, then

$$|\mathrm{Cw}\, L| = \frac{(2^k + 2k^2 + 4k - 11) \cdot 3^l + 1}{2}$$

By using this, we also obtain the result $\frac{53 \cdot 3^{n-4} + 1}{2}$.

Chandelier

Let $N_{m_1,m_2,\ldots,m_n}$ be a lattice of width n, containing n chains with m_1, m_2,...,m_n elements. They have intersection $\{0, 1\}$, any other element of it belongs exactly to one chain.
The index i in m_i denote the i-th chain. We call the lattice $N_{m_1,m_2,\ldots,m_n}$ a **chandelier**.

Lemma (H., A. Tepavčević)
The chandelier $N_{m,k}$ has

$$\frac{3^m - 1}{2}\frac{3^k - 1}{2} + 3 \cdot (2^m - 1)(2^k - 1) + \frac{3^{m+2} + 3^{k+2}}{2} - 4$$

weak congruences.

Chandelier

Let $N_{m_1,m_2,\ldots,m_n}$ be a chandelier of width n, containing n chains with m_1, $m_2,\ldots,m_n$ elements. Let $w^{(k)}(m_{l_1},\ldots,m_{l_k})$ be the number of special weak congruences on $N_{m_1,m_2,\ldots,m_n}$, which are weak congruences of sublattices of $N_{m_1,m_2,\ldots,m_n}$ of width k where $\{m_{l_1},\ldots m_{l_k}\}$ is a fixed subset of the set $\{m_1, m_2, \ldots, m_n\}$ containing k different elements.

Chandelier

Let $N_{m_1,m_2,\ldots,m_n}$ be a chandelier of width n, containing n chains with m_1, $m_2,\ldots,m_n$ elements. Let $w^{(k)}(m_{l_1},\ldots,m_{l_k})$ be the number of special weak congruences on $N_{m_1,m_2,\ldots,m_n}$, which are weak congruences of sublattices of $N_{m_1,m_2,\ldots,m_n}$ of width k where $\{m_{l_1},\ldots m_{l_k}\}$ is a fixed subset of the set $\{m_1, m_2, \ldots, m_n\}$ containing k different elements.

Lemma (H., A. Tepavčević)

Let $k \geq 3$. Then,

$$w^{(k)}(m_1,\ldots m_n) = \prod_{i=1}^{k} \frac{3^{m_i}-1}{2} + (2^{m_1}-1)\cdot(2^{m_2}-1)\cdot\ldots(2^{m_k}-1)$$

It is easy to see that

$$|\operatorname{Cw} N_{m,k}| = w^{(1)}(m) + w^{(1)}(k) + w^{(2)}(m,k) - 3.$$

Further, $|\operatorname{Cw} N_{m,k,l}| =$
$w^{(1)}(m)+w^{(1)}(k)+w^{(1)}(l)+w^{(2)}(m,k)+w^{(2)}(m,l)+w^{(2)}(k,l)+w^{(3)}(m,k,l)-7.$

It is easy to see that

$$|\operatorname{Cw} N_{m,k}| = w^{(1)}(m) + w^{(1)}(k) + w^{(2)}(m,k) - 3.$$

Further, $|\operatorname{Cw} N_{m,k,l}| = w^{(1)}(m)+w^{(1)}(k)+w^{(1)}(l)+w^{(2)}(m,k)+w^{(2)}(m,l)+w^{(2)}(k,l)+w^{(3)}(m,k,l)-7.$

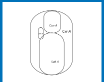
Theorem (H., A. Tepavčević)

The number of weak congruences of a chandelier of width n is

$$|\operatorname{Cw} N_{m_1,m_2,\dots,m_n}| = \sum_{i=1}^{n} \sum_{A \in \mathcal{P}^i(\{m_1,m_2,\dots,m_n\})} w^{(i)} A - 4n + 5,$$

where $\mathcal{P}^i(\{m_1, m_2, \dots, m_n\})$ is the set of all subsets of $\{m_1, m_2, \dots, m_n\}$ with i elements.

5. References I



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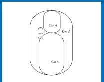

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Thank you for your attention!