Completely hereditarily atomic OMLs
Part I – tutorial

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Tutorial

Today is a tutorial that will give some basics of OMLs along with two methods to construct OMLs.

OMLs

Definition An ortholattice (OL) is a bounded lattice L with a unary operation $'$: L $\to$ L that is order-inverting, period 2, and with x, x' complements. It is an orthomodular lattice (OML) if it satisfies

$$x \leq y \Rightarrow x \vee (x' \wedge y) = y$$

Example 1 Any Boolean algebra (BA) is an OML.

Example 2 Gluing together BA's at 0,1 gives an OML.

Example 3 The closed subspaces $\mathcal{C}(\mathrm{H})$ of a Hilbert space is an OML.

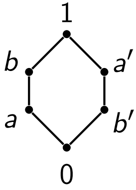
Definition A block of an OL is a maximal Boolean subalgebra.

Note Each OL is the union of its blocks. OMLs are those OLs where the order is determined by its blocks.

Proposition Let B be a block of an OML L.

1. B is closed under existing joins and meets in L.
2. An atom of B is an atom of L.
3. If L is atomic, it is atomistic

Atomic OMLs are atomistic. The same is not true for OLs.

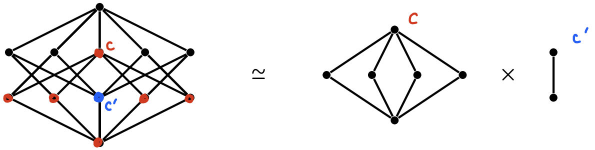


Proposition An interval of an OML is an OML.

Definition The center C(L) of an OML L is the set of elements that are in all blocks.

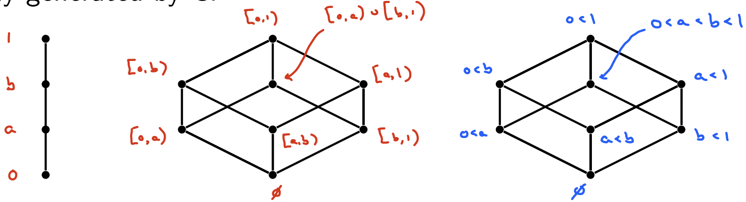
Proposition If L is a complete OML and C(L) is atomic with A its set of atoms, then $\mathrm{L} \simeq \prod_A [0, a]$ and each $[0, a]$ is irreducible.

Example



Kalmbach's construction

Definition For a bounded chain C let B(C) be the Boolean algebra freely generated by C.



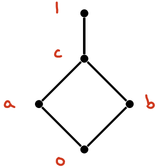
Proposition B(C) is realized as the subalgebra of Pow(C−{1}) of finite unions of half-open intervals [a,b).

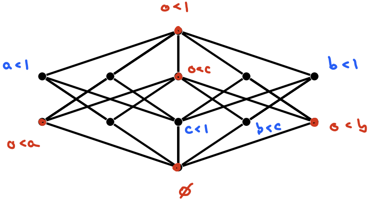
Elements of B(C) correspond to even-length strictly increasing sequences of C with appropriate order and orthocomplementation.

Definition For L a bounded lattice, K(L) is all even-length sequences $x : x_1 < \cdots < x_{2n}$ and order

$$x \leq y \quad \text{iff} \quad \forall_i \; \exists_j \quad y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}$$

and orthocomplementation $x' = x \oplus \{0, 1\}$





Theorem (Kalmbach) K(L) is an OML with L as a sublattice.

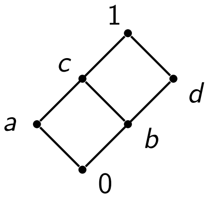
Theorem

1. Blocks of K(L) are the B(C) where C is a max chain of L.
2. Atoms of K(L) are the $a < b$ where b covers a in L.
3. If L is complete, the MacNeille completion of K(L) is an OML.

Further results we won't need ...

4. K(L) is concrete, i.e. it has enough 2-valued states. (JH, MN)
5. Poset $\underset{U}{\overset{K(\cdot)}{\rightleftarrows}}$ OMP is an adjunction (JH) and ...
6. Its Eilenberg-Moore algebras are the effect algebras. (GJ)

Quiz



1. How many blocks does K(L) have?
2. How many atoms does each block have?
3. How many atoms does K(L) have?
4. Is K(L) irreducible?
5. For $x : 0 < a$ and $y : b < d$ what is $x \vee y$?
6. What is the height of $[y, x \vee y]$?

Geometric considerations

Our next topic is ties between OMLs and geometry. We first recall some very old results from projective geometry.

Theorem For a vector space V over a division ring there is a projective geometry $\mathbb{G}(V)$ whose points are the 1-dimensional subspaces of V and whose lines are the 2-dimensional subspaces.

Theorem For a projective geometry G of dimension ≥ 3, analogs of straight-edge compass constructions yield a division ring on the points of a line of G. Then one has a vector space $\mathbb{V}(G)$.

Theorem $V \simeq \mathbb{V}\mathbb{G}(V)$ and $G \simeq \mathbb{G}\mathbb{V}(G)$ when dimensions ≥ 3.

Lattices and geometry

Definition A geomodular lattice is one that is irreducible, algebraic, complemented and modular.

Theorem (Birkhoff, Menger) The subspace lattice of a vector space over a division ring is a geomodular lattices and every geomodular lattice of height ≥ 4 arises this way.

projective geometries $\equiv$ vector spaces $\equiv$ geomodular lattices

Note We wish to consider OLs and geometry. This will use tools familiar from Hilbert spaces.

Orthogonality spaces

Definition Let $(X, \perp)$ be a set with a symmetric binary relation $\perp$ that satisfies $x \perp x \Rightarrow x \perp y$ for all $y \in X$. For $A \subseteq X$ set

$$A^{\perp} = \{x : x \perp a \text{ for all } a \in A\}$$

Then put $L_{\perp\perp} = \{A\colon A = A^{\perp\perp}\}$.

Theorem $L_{\perp\perp}$ is a complete OL. Each complete OL arises this way.

Note this construction is familiar from inner product spaces where the $\perp$ is the relation of orthogonality of vectors. Here, 0 is the unique vector with $0 \perp 0$. We extend this.

Hermitian spaces

Definition A Hermitian space is a vector space E over a $\ast$-field k with a Hermitian form $\langle \cdot | \cdot \rangle : E \times E \to k$, i.e.

$$\langle xf + yg | h \rangle = x \langle f | h \rangle + y \langle g | h \rangle, \; \langle f | g \rangle = \langle g | f \rangle^*, \; \langle f | f \rangle = 0 \Rightarrow f = 0$$

Definition For a Hermitian space E let $L_{\perp\perp}$ be its complete OL of biorthogonally closed subsets, which are necessarily subspaces.

Characterizing OLs arising from Hermitian spaces

Definition A bounded lattice has the covering property if a atom and $a \nleq x$ implies that $x \vee a$ covers x.

Theorem (Birkhoff, von Neumann, Baer) If E is Hermitian, $L_{\perp\perp}$ is an irreducible, complete, atomistic OL with the CP. Conversely, any irreducible, complete, atomistic OL with the CP and height ≥ 4 is isomorphic to $L_{\perp\perp}$ for some Hermitian space E.

Note! We have moved from the setting of OMLs.

Orthomodular spaces

Definition A Hermitian space is orthomodular if $L_{\perp\perp}$ an OML. A complete, irreducible atomic OML with the CP is called a propositional system.

orthomodular spaces ≡ propositional systems (for height ≥ 4)

Example A Hilbert space over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ is an orthomodular space.

Example Any finite-dimensional Hermitian space is orthomodular.

For a long time, there were no other known examples, and building evidence that there are none. Keller was the first to construct an infinite-dimensional orthomodular space that is not a Hilbert space. The construction is necessarily involved.

What makes a space orthomodular

Definition A subspace A of E is splitting if $E = A + A^{\perp}$. Let L_s be the set of all splitting subspaces.

Theorem We always have $L_s \subseteq L_{\perp\perp}$. There is equality iff E is an orthomodular space.

Proof We show part, that L_s satisfies $A \subseteq B \Rightarrow A + (A^{\perp} \cap B) \supseteq B$.

$$x \in B \Rightarrow x = x_A + x_{A^{\perp}}$$

So $x_{A^{\perp}} = x - x_A$ is in $A^{\perp} \cap B$ since $x, x_A \in B$.

Keller's example

Definition A Laurent series $\sum c_n t^n$ has coefficients $c_n \in \mathbb{R}$ and exponents $n \in \mathbb{Z}$ with $\{n : c_n \neq 0\}$ bounded below.

A Laurent series is a map $x : \mathbb{Z} \to \mathbb{R}$ with support bounded below. The Laurent series form a field with obvious analogs of addition and polynomial multiplication. Let

$$\varphi(x) = \begin{cases} \min\{n : c_n \neq 0\} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

Definition For Γ a totally ordered abelian group, a generalized power series over Γ is $x : \Gamma \to \mathbb{R}$ whose support is well-ordered.

Theorem The generalized power series k over Γ is a field with valuation $\varphi : k \to \Gamma \cup \{\infty\}$.

Setting $d(x,y) = \varphi(x-y)$ yields a sort of ultrametric on k, so a means to talk about convergence of series in k.

Definition Let $\Gamma = \bigoplus_{\mathbb{N}} \mathbb{Z}$ with reverse lexicographic order and then let $\gamma_n \in \Gamma$ be the generator $(0,\ldots,0,1,0,\ldots)$.

Now a construction analogous to the construction of ℓ^2.

Definition E is all sequences $f : \mathbb{N} \to k$ that are "square-summable" in the sense that $\sum f(n)^2 t^{\gamma_n}$ converges. For $f, g \in$ E set

$$\langle f \mid g \rangle = \sum f(n) g(n) t^{\gamma_n}$$

Theorem E is an infinite-dimensional orthomodular space and $\mathrm{L}_{\perp\perp}$ is not isomorphic to the closed subspaces of a Hilbert space.