On discrete properties of monotone mappings

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Outline

Section 1:

Introduction

$A \neq \emptyset$, ε partial order on the set A, $h : A \to A$
h is ε-increasing (resp. ε-decreasing) if for every $a, b \in A$:

$$\text{if } a \neq b \text{ and } (a, b) \in \varepsilon, \text{ then}$$
$$h(a) \neq h(b) \text{ and } (h(a), h(b)) \in \varepsilon \text{ (resp. } (h(b), h(a)) \in \varepsilon)$$

h is ε-monotone if it is ε-increasing or ε-decreasing

$\Delta = \{(a, a), a \in A\}$

h is Δ-increasing and Δ-decreasing

Alternative terminology:

increasing …… strictly isotone, strictly ascending or strictly order-preserving, strictly monotone

decreasing …… strictly antitone, strictly descending or strictly order-reversing

$$P \neq \emptyset,\ a, b \in P, a \neq b,\ f_{ab}(x) = \begin{cases} b & \text{if } x = a, \\ x & \text{if } x \in P, x \neq a. \end{cases}$$

Proposition (Chajda, Länger, 2023)

Let $(P, \leq)$ be a poset. TFAE

1. *f_{ab} is strictly monotone (i.e. $\leq$-increasing)*
2. *$a \parallel b, L^*(a) \subseteq L^*(b)$ and $U^*(a) \subseteq U^*(b)$*

$A \neq \emptyset,\ h : A \to A$

Is there any non-trivial

PARTIAL (LINEAR, WELL) order ε such that

h is ε-increasing (ε-decreasing)?

Theorem

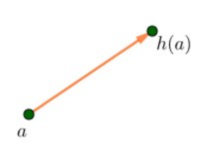
Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there is a non-trivial partial (linear, well) order ε such that h is ε-increasing (ε-decreasing),*
2. *the monounary algebra (A, h) is such that ...*

$A \neq \emptyset,\ h : A \to A$

(A, h) monounary algebra

$a \in A$

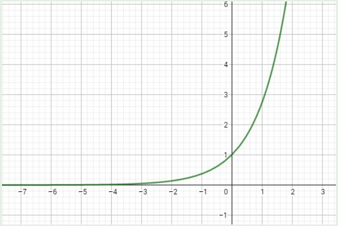


cyclic element
cycle
connected algebra
component
source

$E_h = \{[a, h(a)],\ a \in A\}$

(A, E_h) oriented graph, discrete structure of h

$(\mathbb{R}, e^x)$



every component is

sources: $r \in \mathbb{R}_0^-$

Section 2:

Partial Order

Theorem

Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there exists a partial order $\varepsilon \neq \Delta$ on A such that h is ε-increasing;*
2. *(A, h) is not connected or it contains no cycle.*

Proof.

$(1) \to (2)$

two comparable elements in a component of (A, h) with a cycle are not possible

$(2) \to (1)$

If (A, h) has cycles C_1, C_2, then $\varepsilon := (C_1 \times C_2) \cup \Delta$.

If (A, h) contains a component B without a cycle, $b \in B$, then take ε the chain which copies a ray generated by b in (A, h).

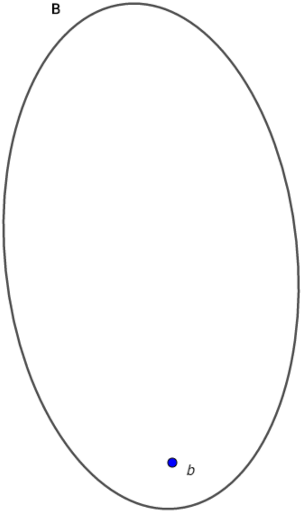
Theorem

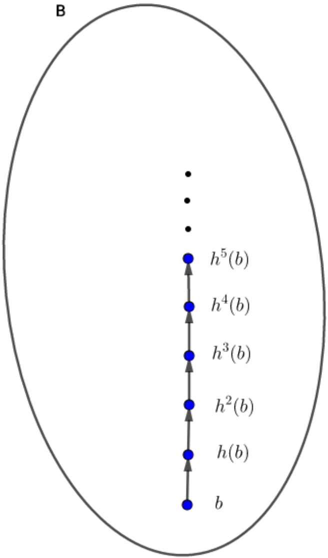
Let $A \neq \emptyset$, $h : A \to A$. TFAE

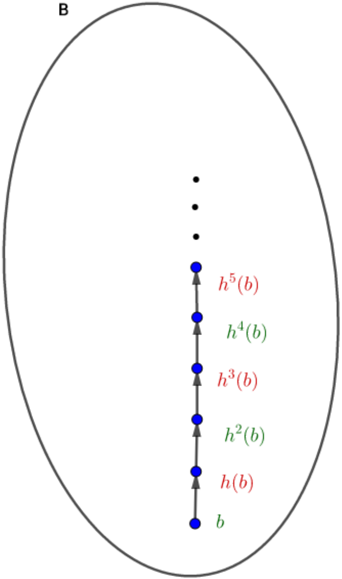
1. *there exists a partial order $\varepsilon \neq \Delta$ on A such that h is ε-decreasing;*
2. *(A, h) contains a cycle of even length or a component without a cycle.*

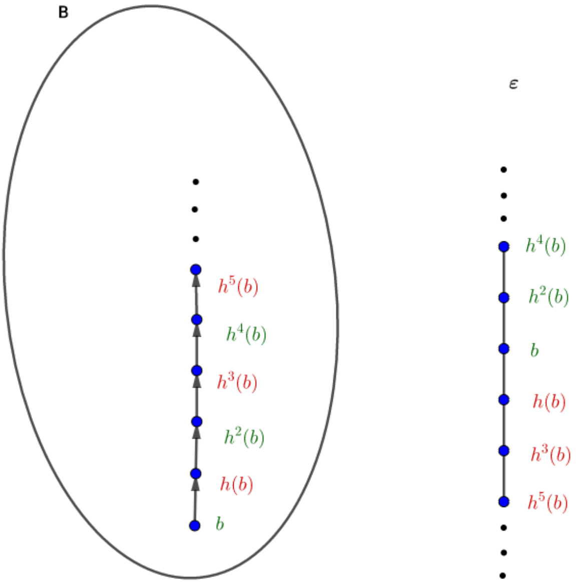
Proof. $(2) \to (1)$
If $C = \{0, 1, ..., 2n - 1\}, n \in \mathbb{N}$ is a cycle of (A, h), $h(i) = i + 1(\text{mod } 2n)$for $i \in C$, C_1 are odd and C_2 even numbers of C, then $\varepsilon := (C_1 \times C_2) \cup \Delta$.

Let B be a component of (A, h) without a cycle.




B
$h^5(b)$
$h^4(b)$
$h^3(b)$
$h^2(b)$
$h(b)$
b


B
$h^5(b)$
$h^4(b)$
$h^3(b)$
$h^2(b)$
$h(b)$
b


B
$h^5(b)$
$h^4(b)$
$h^3(b)$
$h^2(b)$
$h(b)$
b
ε
$h^4(b)$
$h^2(b)$
b
$h(b)$
$h^3(b)$
$h^5(b)$

Corollary

Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there is no non-trivial partial order ε on A such that h is ε-monotone;*
2. *(A, h) is connected with a cycle of odd length.*

Section 3:

Linear Order

Oldřich Kopeček

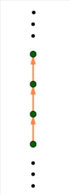
- *Equation $f(p(x)) = q(f(x))$ for given real mappings p, q*, Czech. Math. J., 62(137)(2012), 1011–1032.
- *On solvability of f(p(x)) = q(f(x)) for given real functions p,q*, Aequat. Math. 90 (2016), 471 - 494.
- *The solvability of f(p(x)) = q(f(x)) for given strictly monotonous continuous real functions p,q*, Aequat. Math. 96 (2022), 901–925.

Theorem

Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there exists a linear order ε on A such that h is ε-increasing;*
2. *algebra (A, h)*

   has at most these 3 types of components:



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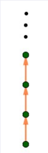


,

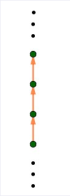
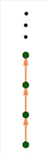
.

Theorem

Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there exists a linear order ε on A such that h is ε-decreasing;*
2. *algebra (A, h)*

   has at most these 4 types of components:

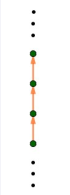
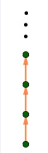
    ,  ,  ,  ,

   while  *occurs at most once.*

Corollary

Let $A \neq \emptyset$, $h : A \to A$. *TFAE*

1. *there is a linear order* ε *on* A *such that* h *is* ε*-monotone;*
2. *algebra* (A, h)

   has at most these 4 types of components:  ,  ,  , 

   and if it has  , *then*  *occurs at most once.*

Theorem

Let $A \neq \emptyset$, $h : A \to A$. TFAE

1. *there exists a well order ε on A such that h is ε-increasing;*
2. *every component of (A, h) is*  *or*  .

Proposition

Let $A \neq \emptyset$, $h : A \to A$. *Denote*

κ *the number of components of the algebra* (A, h),

μ *the number of cycles of* (A, h).

Suppose that h *is* ε*-increasing for some linear order* ε. *Then the number of linear orders* δ *such that* h *is* δ*-increasing is*

- *at least* $2^{\kappa-\mu} \cdot \kappa!$ *if* κ *is finite;*
- $2^{\|A\|}$ *else.*

Proposition

Let $A \neq \emptyset$, $h : A \to A$. *Denote*

κ *the number of components of the algebra* (A, h),

μ *the number of cycles of* (A, h).

Suppose that h *is* ε*-decreasing for some linear order* ε. *Then the number of linear orders* δ *such that* h *is* δ*-decreasing is*

- *at least* $2^{\kappa-\mu} \cdot \kappa!$ *if* κ *is finite and* h *has no fixed point;*
- *at least* $2^{2\kappa-\mu-1} \cdot (\kappa - 1)!$ *if* κ *is finite and* h *has a fixed point;*
- $2^{\|A\|}$ *else.*

Thank you very much for your attention!