Digraph representations of certain Mal'tsev conditions

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Congruence conditions

By a **congruence condition** we mean any property for varieties that is described by the congruence relations of algebras of the variety. For example:

Definition

A variety is **congruence distributive/modular** if any algebra of the variety has distributive/modular congruence lattice.

Definition

For a fixed natural n, a variety $\mathcal{V}$ is called **congruence n-permutable** if for any algebra $\mathbf{A} \in \mathcal{V}$ and any congruences $\alpha, \beta \in \operatorname{Con} \mathbf{A}$ the following identity holds:

$$\underbrace{\alpha \circ \beta \circ \alpha \circ \dots}_{n} = \underbrace{\beta \circ \alpha \circ \beta \circ \dots}_{n}$$

Graph conditions

By a **graph condition** we mean any property for varieties that is described by the set of (directed) graphs compatible with an algebra of the variety. For example:

- all compatible graphs admitted by the variety are symmetric
- all reflexive compatible graphs admitted by the variety are symmetric
- all reflexive compatible graphs admitted by the variety are antisymmetric
- the variety admits the graph $\mathbb{C}$:

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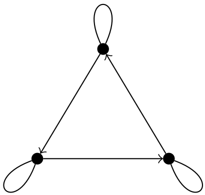

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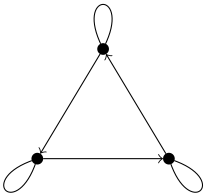
- all compatible graphs admitted by the variety are symmetric
- all reflexive compatible graphs admitted by the variety are symmetric
- all reflexive compatible graphs admitted by the variety are equivalences
- the variety admits a graph containing $\mathbb{C}$ as an induced subgraph:



Conditions for n-permutability

Theorem (Hagemann, Mitschke)

For a natural n and a variety $\mathcal{V}$ the following are equivalent:

1. *$\mathcal{V}$ is congruence n-permutable*
2. *there are ternary terms $t_1, \ldots, t_{n-1}$ of $\mathcal{V}$ satisfying*

$$\begin{array}{rcll} t_1(x,x,y) & \approx & y & \\ t_{i+1}(x,x,y) & \approx & t_i(x,y,y) & \text{for all } i \\ x & \approx & t_{n-1}(x,y,y) & \end{array}$$

3. *for any compatible reflexive digraph of any algebra of $\mathcal{V}$, any edge of the graph is part of an n-circle*

The first is a congruence, the second a Mal'tsev, the third a graph condition.

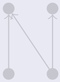
Graph conditions for congruence permutability

A Mal'tsev condition may be characterized by many graph conditions.

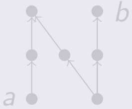
Proposition

A variety admits a Mal'tsev term ($p(x, x, y) \approx p(y, x, x) \approx y$) if and only if it satisfies any/all of the following graph conditions.

- *Any reflexive graph is symmetric.*
- *Any reflexive graph is an equivalence.*
- *No graph contains the following as an induced subgraph:*



- *In any graph containing the following as a subgraph, there is a path from a to b:*



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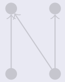

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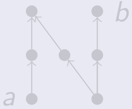

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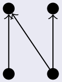

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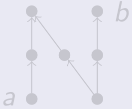

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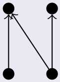

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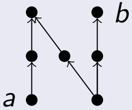

Connectivity

Definition

A directed graph $(V, \to)$ is

- **strongly connected** if from any two vertices a and b there is a path from a to b
- **weakly connected** if $(V, \to \cup \leftarrow)$ is strongly connected
- **extremely connected** if $(V, \to \cap \leftarrow)$ is strongly connected
- **simply connected** if it is weakly connected and for any circlic path $c^{(0)} \to d_1^{(0)} \to \cdots \to d_{k-1}^{(0)} \to c^{(0)}$ there are circlic paths $c^{(i)} \to d_1^{(i)} \to \cdots \to d_{k-1}^{(i)} \to c^{(i)}$ so that for each i and j, $c^{(i)} \to c^{(i+1)}$ and $d_j^{(i)} \to d_j^{(i+1)}$ hold, and there is an m so that $d_1^{(m)} = \cdots = d_{k-1}^{(m)} = c^{(m)}$

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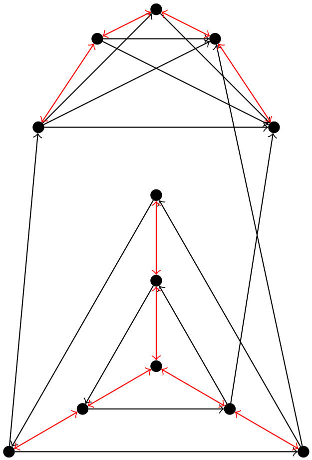
An example



Strong versus extreme connectivity

Theorem

If a variety is congruence modular, then every reflexive strongly connected graph admitted by it is extremely connected.

Theorem

For a locally finite variety $\mathcal{V}$, the following are equivalent:

1. *every reflexive strongly connected graph admitted by $\mathcal{V}$ is extremely connected,*
2. *$\mathcal{V}$ satisfies an idempotent Mal'tsev condition that fails in the variety of semilattices.*

We conjecture the latter to be true without the assumption of local finiteness.

Strong versus less extreme connectivity

Theorem

For a variety $\mathcal{V}$, the following are equivalent:

1. *if $\mathbb{G}$ is a reflexive strongly connected graph admitted by $\mathcal{V}$, then factoring $\mathbb{G}$ by its extremely connected components results in an extremely connected graph*
2. *if $\mathbb{G}$ is a reflexive strongly connected graph admitted by $\mathcal{V}$, then factoring $\mathbb{G}$ by its extremely connected components results in a graph $\mathbb{G}'$ such that factoring $\mathbb{G}'$ by its extremely connected components results in an extremely connected graph*
3. $\dots$
4. *any reflexive and antisymmetric graph admitted by $\mathcal{V}$ contains no (loopless) circle*
5. *$\mathcal{V}$ admits a Taylor operation.*

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Another graph condition for Taylor

Consider the graph $\mathbb{C}$ (the three-element reflexive directed cycle).

Definition

For a cardinality κ denote by $\mathbb{C}_\kappa$ the graph

$$\bigcup_{\nu \leq \kappa} \mathbb{C}^\nu \times \mathbb{E}_\kappa,$$

where $\mathbb{E}_\kappa$ denotes the reflexive, symmetric and antisymmetric graph of cardinality κ.

Theorem

A variety admits a Taylor operation if and only if it does not admit $\mathbb{C}_\kappa$ for any cardinality κ.

Majority and near unanimity

Theorem (Kazda, 2011)

Any finite graph that admits a Mal'tsev operation also admits a majority operation.

Corollary

The varieties admitting a majority operation cannot be described by a graph condition referencing only finite graphs.

Theorem (Maróti, Zádori, 2012)

Any finite reflexive graph that admits Gumm operations also admits a near unanimity operation.

Corollary

The varieties admitting a near unanimity operation cannot be described by a graph condition referencing only finite graphs.

Congruence modularity

Polin algebras form a locally finite variety that is as close to congruence modular as possible without being congruence modular.

Theorem

Suppose $\mathbb{G}$ is a finite graph admitted by a Polin algebra. Then $\mathbb{G}$ admits a $|G|+1$-ary near unanimity operation.

This suggests (admittedly quite weakly):

Conjecture

Congruence modularity can not be characterized by a graph condition.

Three open problems

Problem

Is the "Corollary" true?

Problem

Is it true that if any reflexive connected graph admitted by a variety is simply connected, then the variety is congruence permutable?

Problem

Is there a fixed k so that any (finite) graph admitted by a Polin algebra has a k-ary near unanimity term, or a Gumm/Day chain of length k?