Endomorphism kernel property for finite groups

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Outline

We will talk about endomorphism kernel property for finite groups. Blyth, Fang, and Silva (2004) defined an endomorhism kernel property (EKP) for an universal algebra as follows:

Definition

An algebra A has the *endomorphism kernel property* (=EKP) if every congruence relation on A different from the universal congruence ι_A is the kernel of an endomorphism on A.

This equivalent to the fact that every non-trivial epimorphic image of A is isomorphic to a subalgebra of A.
They proved that finite Boolean algebras, finite chains as bounded distributive lattices posses EKP, finite bounded distributive lattice has EKP if and only if it is a product of chains. They also proved a full characterisation of finite de Morgan algebras having EKP, also other classes of algebras were studied from this point of view later.

The notion of strong endomorphism property was defined by Blyth and Silva in 2008.

Let A be a universal algebra, $f : A \to A$ be an endomorphism, $\Theta \in Con(A)$ be a congruence on A. We say that f is *compatible* with Θ if $a \equiv b(\Theta) \Rightarrow f(a) \equiv f(b)(\Theta)$.

Endomorphism f is *strong* (*congruence preserving*) (on A), if it is compatible with every congruence $\Theta \in Con(A)$.

Definition

An algebra A has the *strong endomorphism kernel property* (=SEKP) if and only if every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A.

Concerning SEKP, Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras. For instance, a Boolean algebra has SEKP if and only if it has exactly two elements. A full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP was proved by Blyth, J. Fang and Wang in 2013. SEKP for distributive p-algebras and Stone algebras has been studied and fully characterized by G. Fang and J. Fang in 2013. Semilattices with SEKP were fully described by J. Fang and Z.-J. Sun in 2013. Guričan and Ploščica described unbounded distributive lattices with SEKP in 2016. Halušková described monounary algebras with SEKP in 2018. Double MS-algebras with SEKP were described by J. Fang in 2017.

Most of classes of algebras considered so far were classes which have a lattice reduct. First result concerning classical structures is in the paper by J. Fang and Z.-J. Sun from 2020 - they proved that finite abelian group has SEKP if and only if it is a cyclic group.

We are going to talk about EKP for finite groups. Our main result is that every finite abelian group has EKP. We shall also discuss some results about non-abelian finite groups.

Outline

Usual method to prove some results for EKP is by using direct product decomposition. But as congruences in (abelian) groups are not factorable, we must be quite carefull. Nilpotency is of great help here. There is an important structural characterization of finite nilpotent groups:
Let G be a finite group, $|G| = p_1^{a_1} \cdots p_k^{a_k}$, where $p_1, \ldots, p_k$ are pairwise different prime numbers. Then G is nilpotent if and only if

$$G \cong G_1 \times G_2 \times \cdots \times G_k \tag{1}$$

where G_i is (isomorphic to) a Sylow p_i-subgroup of G for every $i \in \{1, \ldots, k\}$, it means that $|G_1| = p_1^{a_1}, \ldots, |G_k| = p_k^{a_k}$.

We shall use the following well known theorem

Theorem

Let G be a finite nilpotent group written in this way as a product of its Sylow p_i-groups G_i,

$$G = G_1 \times G_2 \times \cdots \times G_k$$

Let H be a subgroup of G. Then there exist subgroups H_i of G_i, $i = 1, \ldots, k$, such that

$$H = H_1 \times H_2 \times \cdots \times H_k$$

Moreover, if $H \triangleleft G$, then $H_i \triangleleft G_i$ for $i = 1, \ldots, k$.

Using this decomposition, the factor group G/H (in case when $H \triangleleft G$) can be written as a product of factor groups in the form

$$G/H \cong G_1/H_1 \times \cdots \times G_k/H_k$$

Combining previous theorems we get

Theorem

Let each of Sylow subgroups $G_1, \ldots, G_k$ of a finite nilpotent group G (written in the form (1)) have EKP. Then also G has EKP.

As (finite) abelian groups are nilpotent, we can use this theorem and what is left is to prove that every finite abelian $p-$group has EKP.
Let us start with a special case of homomorphic images of a finite abelian $p-$group.
Cyclic group with n elements will be denoted by Z_n. Let p be a prime number.
By a structure theorem for finite abelian groups a finite abelian $p-$group G can be uniquely written as $G \cong Z_{p^{a_1}} \times \cdots \times Z_{p^{a_n}}$, $a_1 \leq \cdots \leq a_n$.
Numbers $p^{a_1}, \ldots, p^{a_n}$ are called *abelian invariants* of a $p-$group G.
Unfortunately, congruences of such factorization need not be factorable, e.g. $Z_2 \times Z_2$ has a (normal) subgroup $\{(0,0),(1,1)\}$.
But we can prove the following:

Lemma

Let G be a finite abelian $p-$group, $G = Z_{p^{a_1}} \times \cdots \times Z_{p^{a_n}}$, K be a subgroup of G, $|K| = p$. Then there exist $1 \leq i \leq n$ such that

$$G/K \cong Z_{p^{a_1}} \times \cdots \times Z_{p^{a_{i-1}}} \times Z_{p^{a_i - 1}} \times Z_{p^{a_{i+1}}} \times \cdots \times Z_{p^{a_n}}$$

which means that the group G/K is isomorphic to a subgroup of G.

Using this it is possible to use Cauchy's theorem and a kind of induction to prove that

Theorem

Let G be a finite abelian $p-$group, $|G| = p^n$. Then for any subgroup H of G, the factor group G/H is isomorphic to a subgroup of G.

which means that all finite abelian $p-$groups (and all finite abelian groups) have EKP.

Outline

We shall show infinitely many finite non-abelian groups with EKP.
Again, let p be a prime number.
Let G be a group, $Z(G)$ be the center of G. Let us start with some well known facts/theorems.

Theorem

1. *Let G be a finite p-group. Then $Z(G)$ is nontrivial.*
2. *Let G be a group. If $G/Z(G)$ is cyclic, then G is abelian.*
3. *Let G be a finite p-group, $H \lhd G$, $|H| = p$. Then $H \subseteq Z(G)$.*
4. *Let G be a group, $|G| = p^2$. Then G is abelian, it means G is either cyclic or $G \cong Z_p \times Z_p$.*

We claim that all but one (now non-abelian) groups with p^3 elements have EKP.
The consequence of previous facts is

Corollary

Let G be a non-abelian group, $|G| = p^3$. Then there is exactly one normal subgroup of G which has p elements. Moreover, this normal subgroup is the center $Z(G)$ and

$$G/Z(G) \cong Z_p \times Z_p$$

This is the only non trivial case for non-abelian groups with p^3 elements.

As for each prime number p there are exactly 2 non-abelian groups with p^3 elements, there can be more straightforward way to prove what we want, but we would like to use the following beautiful statement

Theorem

Suppose that G is a $p-$group all of whose abelian subgroups are cyclic. Then G is cyclic or it is the quaternion group.

Hence, as a direct consequence we have

Theorem

Let G be a non-abelian group, $|G| = p^3$, where $p > 2$, or $G \cong D_4$ (dihedral 8 element group). Then G has a non-cyclic abelian subgroup H, it means a subgroup H such that

$$H \cong Z_p \times Z_p$$

Therefore we know the following

Lemma

Let G be a non-abelian group, $|G| = p^3$.
1. If $p > 2$, then G has EKP.
2. If $p = 2$ and $G \cong D_4$, then G has EKP (quaternion group does not have EKP).

The previous results can be extended by multiplication using factor Z_p^k and using this and ideas about nilpotent groups, the most general result we can say is

Theorem

Let G be a finite nilpotent group written in the form (1). Let each Sylow group G_i be (isomorphic to) one of the following groups:

1. *an abelian group,*
2. $Z_{p_i}^{k_i} \times P_i$, *where* $k_i \geq 0$, $p_i > 2$ *and* P_i *is a non-abelian group of order* p_i^3,
3. $Z_2^{k_i} \times D_4$, *where* $k_i \geq 0$ *and* D_4 *is a dihedral* $8-$*element group.*

Then G has EKP.

Remark: our results does not provide all non-abelian $p-$groups which have EKP. Direct computation in GAP shows that for example there are 6 non-abelian groups of order $3^4 = 81$ which have EKP (GAP identifications returned by **IdSmallGroup()** of these groups are [81,6], [81,7], [81,8], [81,9], [81,12], [81,13]), but only 2 of them are of the form $Z_3 \times P$, where P is a non-abelian group of order 3^3. There are 4 non-abelian groups of order 81 which do not have EKP (GAP id's of these groups are [81,3], [81,4], [81,10], [81,14]).

Thank you for the attention.

Outline

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