On Isometries in Autometrized Algebras

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Autometrized algebras

K. L. N. Swamy, 1964

Definition

Let $(A, +, \leq, d)$ be an algebraic structure where

- $(A, +)$ is a commutative binary algebra with a distinguished element 0;
- $(A, \leq)$ is a partially ordered set;
- $d : A \times A \to A$ is a mapping satisfying the formal properties of a distance (metric) function (operation):
  1) $d(a, b) \geq 0$ with $d(a, b) = 0$ iff $a = b$;
  2) $d(a, b) = d(b, a)$;
  3) $d(a, c) \leq d(a, b) + d(b, c)$ (triangle inequality).

Then $(A, +, \leq, d)$ is called autometrized (binary) algebra.

Intrinsic metric is built from the elements of algebra involving the operations of the concerned algebra.

Classical examples of autometrized algebras

- Real numbers $d(a,b) = |a-b|$
  classical metric space;
- Boolean algebras $d(a,b) = (a \wedge b') \vee (a' \wedge b)$ (Ellis, 1951, Blumenthal, 1952)
  Boolean geometry;
- Brouwerian algebras (Nordhaus, Lapidus, 1962);
  $d(a,b) = (a-b) \vee (b-a)$ where $a-b$ is the smallest element x such that $b \vee x > a$
  Browerian geometry (no equilateral triangles, Brow. geometry is a Bool. geometry $\Leftrightarrow$ it is free of isosceles triangles iff symmetric difference is a group operation.);
- Newmanian algebras (Roy, 1960)
  Newmanian geometry (no isoscelet triangles).

Isometries on autometrized algebras

Definition

An isometry on an autometrized algebra A with distance function d is a bijection $f\colon A \to A$ such that, for all $x, y \in A$,

$$d(f(x), f(y)) = d(x, y).$$

Remark

Jakubík's definition

Isometries in Boolean algebras

D. Ellis, 1950

Autometrized Boolean algebra:

$(B, \wedge, \vee, ', 0, 1, d)$, where $d(a, b) = (a \wedge b') \vee (a' \wedge b)$.

Associated Boolean ring: $(B, \oplus, \otimes)$, where
$a \oplus b = (a \wedge b') \vee (a' \wedge b) = d(a, b), a \otimes b = a \wedge b$.

$M(B)$ ... set of all isometries in B (group under composition)
$(M(B), \circ) \simeq$(B, $\oplus$) (Every isometry is a group translation.)

J. G. Elliott, 1952

Symmetric difference is the only metric group operation in BA.

K. L. N. Swamy, P. R. Rao, K. Venkateswarlu, 2004

The only intrinsic metric on BA (a boolean polynomial $d(a, b)$ such that $d(a, b) = 0$ iff $a = b$) is the symmetric difference.

Isometries in commutative lattice orderd groups (l-spaces)

K. L. N. Swamy, 1964

Autometrized commutative l-group:

$(G, +, \leq, d)$, where

$d(a, b) = |a - b| = (a - b) \vee (b - a) = (a \vee b) - (a \wedge b)$

Classes of isometries in commutative l-groups

- translations;
- involutions;
- isometries which are also group automorphism;
- others.

K. L. N. Swamy, 1978

If f is an isometry of a commutative l-group G then there exists just one involutory isometric group automorphism T such that $f(x) = T(x) + f(0)$ for every $x \in G$.

Geometry of l-space (no equilateral triangles)

Isometries in non-commutative l-groups

J. Jakubík, 1980

$d(a, b) = |a - b| = (a - b) \vee (b - a) = (a \vee b) - (a \wedge b)$

Non-commutative l-groups cannot be autometrized as above (triangle inequality).

An isometry is a bijection $f : G \to G$ satisfying the following conditions:

1) $d(x, y) = d(f(x), f(y))$;

2) $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

If G is abelian and if $f : G \to G$ is a bijection then 1) implies 2).

J. Jakubík, 1981

The implication 1) $\Rightarrow$ 2) holds for each l-group and each bijection $f : G \to G$.

Isometries in non-commutative *l*-groups

J. Jakubík, 1980, 1981

For any isometry g in an *l*-group G there exists a uniquely determined direct decomposition $G = A \times B$ with B abelian such that $g(x) = x_a - x_b + g(0)$ for each $x \in G$. Conversely, if $G = A \times B$ is a direct decomposition of *l*-group G with B abelian and b is an element of G then the mapping g defined by $g(x) = x_a - x_b + b$ is an isometry in G and $b = g(0)$.

Isometries in non-commutative *l*-groups

Ch. Holland, 1984

Holland has formulated the definition of an intrinsic metric for a class of lattice ordered groups as a word $d(x, y)$ in the free *l*-group generated by x and y satisfying $d(a + c, b + c) = d(a, b)$, $d(a, b) = d(b, a)$ for all a, b, c in every *l*-group of that class.

The only intrinsic metrics on an *l*-group are given by the function $n|a - b|$ for some integer n.

The triangle inequality is satisfied by such a metric iff the group is abelian. There are isometries for each of these metrics, but they are rare.

Isometries in ordered groups

J. Rachůnek, 1984, M. Jasem, 1993

Definition

An autometrized ordered group is a system $(G, +, \leq, d)$ where

- (i) $(G, +, \leq)$ is an ordered group;
- (ii) $d : G \times G \to expG$ is a mapping such that for all $a, b, c \in G$
  1) $d(a, b) \subseteq U(0)$ with $d(a, b) = U(0)$ iff $a = b$;
  2) $d(a, b) = d(b, a)$;
  3) $d(a, c) \supseteq d(a, b) + d(b, c)$.

For $A \subseteq G$ we denote $U(A) = \{x \in G; a \leq x \text{ for each } a \in A\}$.

Any 2-isolated commutative Riesz group G is autometrized by $d(a, b) = |a - b|$ for each $a, b \in G$.

Isometries on MV-algebras

C. C. Chang, 1958

Definition

By an MV-algebra is meant an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the equations:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$
(MV2) $x \oplus y = y \oplus x$
(MV3) $x \oplus 0 = x$
(MV4) $\neg\neg x = x$
(MV5) $x \oplus \neg 0 = \neg 0$
(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

$x \odot y = \neg(\neg x \oplus \neg y)$ a $\vee y = (x \odot \neg y) \oplus y, x \wedge y = (x \oplus \neg y) \odot y$.

$(A, \vee, \wedge)$ ... distributive lattice with the least element 0 and the greatest element $1 = \neg 0$

Isometries on MV-algebras

D. Mundici, 1984

Any MV-algebra is an interval in an abelian *l*-group with a strong unit.

J. Jakubík, 2004

$$d(a,b) = (a \vee b) - (a \wedge b) = \neg((a \wedge b) \oplus \neg(a \vee b)),$$

where $x - y = \neg(y \oplus \neg x)$ (substraction in the abelian *l*-group with a strong unit).

Each isometry of an MV-algebra is 2-periodic, i.e. $f(f(x)) = x$.

Isometries on GMV-algebras

J. Rachůnek, 2002: GMV-algebras
Georgescu + Iorgulescu: pseudo-MV-algebras

Definition

A GMV-algebra is an algebra $(A, \oplus, \odot, ^-, ^\sim, 0, 1)$ such that

- $(A, \oplus, 0)$ is a monoid,
- $0^- = 1 = 0^\sim$,
- $x \oplus 1 = 1 = 1 \oplus x$,
- $x^{-\sim} = x = x^{\sim -}$,
- $x \odot y = (x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,
- $x \oplus (y \odot x^\sim) = (y^- \odot x) \oplus y = y \oplus (x \odot y^\sim) = (x^- \odot y) \oplus x$.

MV-algebras = commutative GMV-algebras
Boolean algebras = idempotent GMV-algebras
J. Rachůnek
GMV-algebras = special kind of bounded DRl-monoids

A. Dvurečenskij, 2002

Any GMV-algebra is an interval in an l-group with a strong unit.

J. Jakubík, 2007

$d(a, b) = (a \vee b) - (a \wedge b)$.

Isometry is a bijection $f : A \to A$ satisfying the following conditions:

1) $d(x, y) = d(f(x), f(y))$;
2) $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

If A is abelian (i.e. MV-algebra) and if $f : A \to A$ is a bijection then 1) implies 2).

M. Jasem, 2011

The implication 1) $\Rightarrow$ 2) holds for each GMV-algebra and each bijection $f : G \to G$.

Isometries on GMV-algebras

Georgescu + Iorgulescu:

$$d(x,y) = (x^- \odot y) \oplus (y^- \odot x)$$

M. Jasem, 2007
This distance function coincides with the Jakubík's one in any GMV-algebra.

An isometry on a GMV-algebra A is a bijection $f\colon A \to A$ such that, for all $x, y \in A$,

$$d(f(x), f(y)) = d(x,y),$$

where the distance function d is defined by

$$d(x,y) = (x^- \odot y) \vee (y^- \odot x),$$

or by

$$d(x,y) = (x \odot y^\sim) \vee (y \odot x^\sim).$$

Isometries on GMV-algebras

M. Jasem, 2007

For every isometry f in a GMV-algebra $A = \langle A, \oplus, \odot, ^-, ^\sim, 0, 1\rangle$ there exists an internal direct decomposition $A = B \times C$ with C commutative such that $f(0) = 1_C$ and
$f(x) = x_B \oplus (1_C \odot x_C{}^-) = x_B \oplus (1_C - x_C)$ for each $x \in A$.
Conversely, if $A = P \times Q$ is an internal direct decomposition of a GMV-algebra A with Q abelian then the mapping $g : A \to A$ defined by $g(x) = x_P \oplus (1_Q - x_Q)$ is an isometry in A and $g(0) = 1_Q$.

Isometries in other algebraic structures

R. A. Melter, 1968
Isometries in unary algebras
P. V. Ramana Murty, 1974
Isometries in semi Browerian algebras
K. L. M. Swamy, B. V. Subba Rao, 1980
Isometries in commutative DRl-monoids (DRl-semigroups)
J. Rachůnek, 1984, M. Jasem, 1986
Isometries in Riesz groups
J. Jakubík - M. Kolibiar, 1983, M. Jasem, 1985
Isometries in multilattice groups
J. Kühr, 2022
Isometries in effect algebras
J. Kühr, J. Rachůnek, D. Šarounová, 2022
Isometries in involutive pocrims

Isometries in basic algebras

Definition (Chajda, Halaš, and Kühr AU 2009)

A basic algebra is an algebra $(A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ that satisfies the equations

$$
\begin{aligned}
x \oplus 0 &= x, \\
\neg\neg x &= x, \\
\neg(\neg x \oplus y) \oplus y &= \neg(\neg y \oplus x) \oplus x, \\
\neg\big(\neg(\neg(x \oplus y) \oplus y) \oplus z\big) \oplus (x \oplus z) &= 1.
\end{aligned}
$$

MV-algebras = associative basic algebras

Orthomodular lattices = basic algebras satisfying
$x \leq y \Rightarrow y \oplus x = y$

.

Sharp, central and boolean elements in basic algebra

Definition

Let $(A, \oplus, \neg, 0, 1)$ be a basic algebra. An element $a \in A$ is said to be:

- sharp if $a \wedge \neg a = 0$; equivalently, $a \vee \neg a = 1$, (equivalently, $a \oplus a = a$). The sharp elements form neither a subalgebra nor a sublattice in general.
- central if the mapping $x \mapsto (x \wedge a, x \wedge \neg a)$ is an isomorphism of A onto $[0, a] \times [0, \neg a]$, or equivalently, $(x, y) \mapsto x \vee y$ is an isomorphism of $[0, a] \times [0, \neg a]$ onto A. The central elements form a subalgebra (Boolean algebra).
- boolean if $a \oplus x = a \vee x$ for all $x \in A$. The boolean elements form a subalgebra (Boolean algebra).

$\mathcal{C}(A) \subsetneqq \mathcal{B}(A) \subsetneqq \mathcal{S}(A)$ $\qquad$ $\mathcal{C}(A) = \mathcal{S}(A)$ for any MV-algebra A

Isometries in basic algebras

P. E., J. K., 2023

$d(x,y) = (x \vee y) \oslash (x \wedge y)$, where $x \oslash y = \neg(\neg x \oplus y)$

Theorem

Let $(A, \oplus, \neg, 0, 1, d)$ be a basic algebra with the distance function $d(x,y) = (x \vee y) \oslash (x \wedge y)$ and f be an isometry on A. Then

(i) $f(0) \wedge f(1) = 0$, $f(0) \vee f(1) = 1$;

(ii) Elements $f(0), f(1)$ are sharp;

(iii) $f(f(0)) = 0$;

(iv) $f(x) = d(x, f(0))$ for every $x \in A$;

(v) $\neg f(0) = f(1)$, f is an involution;

(vi) $x \vee f(0) = x \oplus f(0)$ for every $x \in A$;

(vii) $f(0)$ is boolean: $f(0) \vee x = f(0) \oplus x$ for every $x \in A$.

Lattice effect algebras as special class of basic algebras

An effect algebra (Foulis, Bennett, 1994) is a partial structure $(A; +, 0, 1)$ satisfying:

- $x + y = y + x$ if one side is defined;
- $(x + y) + z = x + (y + z)$ if one side is defined;
- for every x there is a unique x' such that $x' + x = 1$;
- if $x + 1$ is defined, then $x = 0$.
  The underlying order: $x \leq y$ iff $y = x + z$ for some z.
  Effect algebras are equivalent to D-posets $(A; \leq, -, 0, 1)$ (Kopka, Chovanec, 1994).
  Lattice-ordered effect algebras (lattice effect algebras) are equivalent to effect basic algebras, i.e., basic algebras satisfying

$$x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y).$$

Lattice effect algebras as special class of basic algebras

The class of lattice effect algebras includes both MV-algebras and orthomodular lattices:

Relative to the variety of lattice effect algebras:
MV- algebras ... $x \oplus y = y \oplus x$
orthomodular lattices ... $x \oplus x = x$

The smallest variety containing both the variety of MV-algebras and the variety of orthomodular lattices was recently axiomatized by Kühr et. al. (2015).

Lattice effect algebras as special class of basic algebras

$(A; +, 0, 1)$ ... Lattice effect algebra

$(A; \leq, -, 0, 1)$ ... D-lattice (Kopka, Chovanec, 1995)

$(A, \oplus, ', 0, 1)$

$x \oplus y = (x \wedge y') + y$

$x \oslash y = (x \vee y) - y = (x' \oplus y)'$

$x \ominus y = x - (x \wedge y) = (y + x')'$

$x \vee y = (x \oslash y) \oslash y$

$x \wedge y = x \ominus (x \ominus y)$

Natural intrinsic metric on lattice effect algebras

$$d(x,y) = (x \vee y) - (x \wedge y) = (x \oslash y) \vee (y \oslash x) = (x \ominus y) \vee (y \ominus x)$$

Theorem

Let A be a lattice effect algebra with the distance function $d(x,y) = (x \vee y) - (x \wedge y)$. Then all $x, y, z \in A$ satisfy:

(i) $d(x,y) = 0$ iff $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,0) = x$;

(iv) $d(x,y) = x - y$ for $y \leq x$;

(v) $d(z - x, z - y) = d(x,y)$ for $x \vee y \leq z$;

(vi) $d(x,y) \leq x \vee y$.

Natural intrinsic metric on lattice effect algebras

Theorem

Let A be a lattice effect algebra with the distance function d satisfying the conditions (i) - (vi). Then all $x, y, z \in A$ satisfy:

(1) $d(x', y') = d(x, y)$;

(2) $d(x, y) = d(x \oplus y, y \oplus x)$;

(3) $d(x, y) = d(x + z, y + z)$ for $x \vee y \leq z$;

(4) $d(x, y) = x - y$ for $y \leq x$;

(5) $d(x, y) = d(x \ominus y, y \ominus x)$;

(6) $d(x, y) \leq (x \vee y) - (x \wedge y)$;

(7) $d(x, y) \leq (x \vee y) \wedge (x \wedge y)$.

Isometries in lattice effect algebras

Examples of isometries:

- Identity ... $id : x \mapsto x$
- Negation ...$' : x \mapsto x'$
- A mapping $(x, y) \mapsto (x', y)$ on a direct product $A \times B$
- Let $a \in \mathcal{C}(A)$ (central element). Then mappings $x \mapsto (x \wedge a, x \wedge a')$ and $(x, y) \mapsto x \vee y$ are mutually inverse isomorphisms between A and $[0, a] \times [0, a']$. The mapping $(x, y) \mapsto (a - x, y)$ is isometry on $[0, a] \times [0, a']$. We obtain an isometry $x \mapsto d(a, x)$ by composition of this isometry and the isomorphisms. All isometries on lattice effect algebras are in this form.

Isometries in lattice effect algebras

Theorem

Let f be an isometry on a lattice effect algebra A with the distance function $d(x,y) = (x \vee y) - (x \wedge y)$. Then

(i) $f(0) = f(1)'$ is the central element and $A \simeq [0, f(0)] \times [0, f(1)]$;

(ii) $f(x) = d(x, f(0))$ for all $x, y \in A$;

(iii) $[0, f(1)] \simeq [f(0), 1]$, i.e. $A \simeq [0, f(1)] \times [f(0), 1]$;

(iv) f is an involution.

Isometries in lattice effect algebras

Theorem

Let A be a lattice effect algebra with the distance function $d(x, y) = (x \vee y) - (x \wedge y)$. Then every central element $a \in \mathcal{C}(A)$ determine an isometry $f : x \mapsto d(x, a)$. Conversely, each isometry f specifies a central element $a = f(0)$. The isometry is determined by this element ($f(x) = d(x, a)$).

$I(A)$ ... set of all isometries on A

Corollary

The groups $(\mathcal{C}(A), \triangle)$ and $(I(A), \circ)$ are isomorphic. ($\triangle$ is the symmetric difference.

Triangle inequality

(1) $d(a,c) \leq d(a,b) \vee d(b,c)$ ... OML

(2) $d(a,c) \leq d(a,b) \oplus d(b,c)$ ... MVA

If a lattice effect algebra A satisfies the inequality (1) then it is an ortomodular lattice.

If a lattice effect algebra A satisfies the inequality (2) then it is an MV-algebra. ??????

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THANK YOU
FOR YOUR ATTENTION!

BA = bounded lattices with antitone involutions

- The relation $\leq = \{(x,y) \in A^2 \mid \neg x \oplus y = 1\}$ is a partial order on A such that 0 and 1 are the least and the greatest element of A.
- The poset $(A, \leq)$ is a bounded lattice $(A, \vee, \wedge, 0, 1)$ where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \text{ and } x \wedge y = \neg(\neg x \vee \neg y)$$

- For each $a \in A$, the map $\gamma_a \colon x \mapsto \neg x \oplus a$ is an antitone involution on $[a, 1]$.
- For each $a \in A$, the map $\delta_a \colon x \mapsto \neg(x \oplus \neg a)$ is an antitone involution on $[0, a]$.
- $(A, \oplus, \neg, 0, 1)$ is determined by $(A, \vee, \wedge, 0, 1, (\gamma_a)_{a \in A})$ as follows:

$$\neg x = \gamma_0(x) \text{ and } x \oplus y = \gamma_y(\neg x \vee y).$$

- $(A, \oplus, \neg, 0, 1)$ is determined by $(A, \vee, \wedge, 0, 1, (\delta_a)_{a \in A})$ as follows:

$$\neg x = \delta_1(x) \text{ and } x \oplus y = \neg \delta_{\neg y}(x \wedge \neg y).$$

What is behind the axioms?

- Every basic algebra satisfies the following conditions:

  1) $0 \oplus x = x$,
  2) $\neg x \oplus x = 1$,
  3) $x \oplus 1 = 1 \oplus x = 1$,
  4) $x \leq y \;\Rightarrow\; \neg y \leq \neg x$,
  5) $x \leq y \;\Rightarrow\; x \oplus z \leq y \oplus z$,
  6) $\neg x \leq y \oplus z$ iff $\neg y \leq x \oplus z$,
  7) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$,
  8) $y \leq x \oplus y$,
  9) $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$,

- The variety of basic algebras is arithmetical and congruence regular.

What is behind the axioms?

In addition to the negation $\neg$ and the addition $\oplus$, it is useful to define multiplication $\odot$ and two substractions ($\ominus$, $\oslash$) by

$$x \odot y = \neg(\neg x \oplus \neg y),\ x \ominus y = \neg(y \oplus \neg x),\ x \oslash y = \neg(\neg x \oplus y).$$

- Every basic algebra satisfies the following conditions:

1) $0 \odot x = 0 = x \odot 0$,
2) $\neg x \odot x = 0$,
3) $x \odot 1 = x = 1 \odot x$,
4) $x \odot y \leq y$,
5) $x \leq y \ \Rightarrow\ x \odot z \leq y \odot z, \quad x \oslash z \leq y \oslash z, \quad z \ominus x \leq z \ominus y$,
6) $x \leq y$ iff $x \odot \neg y = 0$ iff $x \ominus y = 0$ iff $x \oslash y = 0$,
7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$,
$(x \vee y) \oslash z = (x \oslash z) \vee (y \oslash z)$,
8) $x \ominus (y \wedge z) = (x \ominus y) \vee (x \ominus z)$,
9) $\neg x \odot y \leq z$ iff $\neg z \odot y \leq x$.

What is behind the axioms?

The following (dual) identities 1) - 4) are equivalent to one another and they are equivalent to lattice distributivity:

1) $(x \vee y) \oplus z = (x \oplus z) \vee (y \oplus z)$,
2) $(x \wedge y) \odot z = (x \odot z) \wedge (y \odot z)$,
3) $(x \wedge y) \oslash z = (x \oslash z) \wedge (y \oslash z)$,
4) $x \ominus (y \vee z) = (x \ominus y) \wedge (x \ominus z)$.

The identities 5) - 8) are equivalent to one another and they are stronger than lattice distributivity:

5) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$, (M)
6) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$,
7) $x \oslash (y \wedge z) = (x \oslash y) \vee (x \oslash z)$,
8) $(x \vee y) \ominus z = (x \ominus z) \vee (y \ominus z)$.