Implications in effect algebras

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As explained by D. J. Foulis and M. K. Bennett [14] and by R. Giuntini and H. Greuling, effect algebras were introduced in order to formalize effects of quantum mechanics. This process of formalization is described in these papers and in detail also in the monograph by A. Dvurečenskij and S. Pulmannová and hence it is not necessary to repeat it here. However, there are still two aspects which were not investigated in these sources.

Namely, if one considers effect algebras as a formalization of the logic of quantum mechanics then the natural question arises what are the logical connectives derived from them. The aims of our paper is to analyze three possibilities how to define the logical connectives implication and conjunction such that these form adjoint pairs. The importance of this requirement is that when the adjoint pair is established than the corresponding logic is equipped with the derivation rule Modus Ponens.

Preliminaries

Consider a poset $\mathbf{P} = (P, \leq)$ and let $A, B \subseteq P$. If it has a bottom element, this element will be denoted by 0. If $\mathbf{P}$ has a top element, this element will be denoted by 1. The *poset* $\mathbf{P}$ is called *bounded* if it has both 0 and 1, and in this case it will be denoted by $\mathbf{P} = (P, \leq, 0, 1)$.

By a *binary operator* on P we understand a mapping from P^2 to 2^P, i.e. it assigns to every pair (x, y) of elements of P a subset of P. In what follows, for the sake of brevity, we will not distinguish between a singleton $\{a\}$ and its unique element a.

The poset $\mathbf{P}$ is said to satisfy the

- Ascending Chain Condition (shortly ACC) if there are no infinite ascending chains in $\mathbf{P}$,
- Descending Chain Condition (shortly DCC) if there are no infinite descending chains in $\mathbf{P}$.

Notice that every finite poset satisfies both the ACC and the DCC. Further, let $\operatorname{Max} A$ and $\operatorname{Min} A$ denote the set of all maximal and minimal elements of A, respectively. If P satisfies the ACC then for every $a \in A$ there exists some $b \in \operatorname{Max} A$ with $a \leq b$. This implies that if A is not empty the same is true for $\operatorname{Max} A$. The corresponding assertion holds for the DCC and $\operatorname{Min} A$.
We define

$$
\begin{aligned}
A \leq_1 B & \text{ if for every } a \in A \text{ there exists some } b \in B \text{ with } a \leq b, \\
A \leq_2 B & \text{ if for every } b \in B \text{ there exists some } a \in A \text{ with } a \leq b, \\
A \sqsubseteq B & \text{ if there exists some } a \in A \text{ and some } b \in B \text{ with } a \leq b, \\
A \approx_1 B & \text{ if both } A \leq_1 B \text{ and } B \leq_1 A, \\
A \approx_2 B & \text{ if both } A \leq_2 B \text{ and } B \leq_2 A.
\end{aligned}
$$

For every set A we denote the set of its non-empty subsets by $\mathcal{P}_+ A$.

Effect algebras

The concept of an effect algebra was introduced in 1989 by R. Giuntini and H. Greuling [15] under a different name. The name effect algebra was used the first time by D. J. Foulis and M. K. Bennett [14], see e.g. also [13].

Recall from [13] that an *effect algebra* is a partial algebra $(E, +, 0, 1)$ of type $(2, 0, 0)$ satisfying the following conditions:

- (E1) If $a, b \in E$ and $a + b$ is defined then so is $b + a$ and both coincide,
- (E2) if $a, b, c \in E$ and $a + b$ and $(a + b) + c$ are defined then so are $b + c$ and $a + (b + c)$ and $(a + b) + c = a + (b + c)$,
- (E3) for each $a \in E$ there exists a unique $b \in E$ with $a + b = 1$; in the sequel, this element b will be denoted by a' and called the *supplement* of a,
- (E4) $a + 1$ is defined only if $a = 0$.

Because of (E3), $'$ is a unary operation on E and we will write effect algebras in the form $(E, +, ', 0, 1)$.
In the following let $\mathbf{E} = (E, +, ', 0, 1)$ be an effect algebra, $a, b \in E$ and $A, B \subseteq E$.
On E we introduce a binary relation $\leq$ as follows:

$$a \leq b \text{ if there exists some } c \in E \text{ with } a + c = b.$$

As shown e.g. in [13], $(E, \leq, 0, 1)$ is a bounded poset. If $(E, \leq)$ is even a lattice then $\mathbf{E}$ is called a *lattice effect algebra*.

Example 1

If $E := \{0, a, b, c, d, a', b', c', 1\}$ and a partial binary operation $+$ and a unary operation $'$ on E are defined by:

$+$	0	a	b	c	d	c'	b'	a'	1
0	0	a	b	c	d	c'	b'	a'	1
a	a	$-$	c'	b'	$-$	$-$	$-$	1	$-$
b	b	c'	d	a'	b'	$-$	1	$-$	$-$
c	c	b'	a'	$-$	$-$	1	$-$	$-$	$-$
d	d	$-$	b'	$-$	1	$-$	$-$	$-$	$-$
c'	c'	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$
b'	b'	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$
a'	a'	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$
1	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

x	x'
0	1
a	a'
b	b'
c	c'
d	d
c'	c
b'	b
a'	a
$1'$	0

then $(E, +, ', 0, 1)$ is a non-lattice effect algebra whose induced poset is depicted in Figure 1:

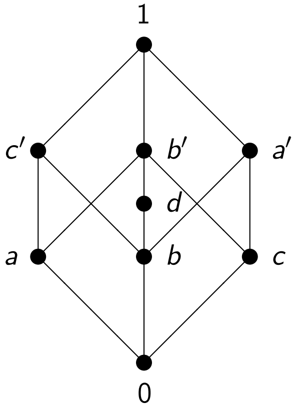

Fig. 1

A non-lattice effect algebra

The elements a and b are called *orthogonal* to each other (shortly $a \perp b$) if $a \leq b'$. It can be shown that $a + b$ is defined if and only if $a \perp b$. We define a partial binary operation $\odot$ on E by $a \odot b := (a' + b')'$. It is evident that $a \odot b$ is defined if and only if $a' \perp b'$.

Example 2

The operation table of $\odot$ corresponding to the effect algebra from Example 1 looks as follows:

$\odot$	0	a	b	c	d	c'	b'	a'	1
0	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0
a	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	a
b	$-$	$-$	$-$	$-$	$-$	$-$	0	$-$	b
c	$-$	$-$	$-$	$-$	$-$	0	$-$	$-$	c
d	$-$	$-$	$-$	$-$	0	$-$	b	$-$	d
c'	$-$	$-$	$-$	0	$-$	$-$	a	b	c'
b'	$-$	$-$	0	$-$	b	a	d	c	b'
a'	$-$	0	$-$	$-$	$-$	b	c	$-$	a'
1	0	a	b	c	d	c'	b'	a'	1

We say that

- $A + B$ is defined if so is $a + b$ for all $a \in A$ and all $b \in B$,
- $A \odot B$ is defined if so is $a \odot b$ for all $a \in A$ and all $b \in B$,
- $A \leq B$ if $a \leq b$ for all $a \in A$ and all $b \in B$.

If $A + B$ is defined we put $A + B := \{a + b \mid a \in A, b \in B\}$. If $A \odot B$ is defined we put $A \odot B := \{a \odot b \mid a \in A, b \in B\}$.
The following result is well-known, see e.g. [13] or [14].

Lemma 3

If $(E,+,',0,1)$ *is an effect algebra and* $a,b,c\in E$ *then*

- (i) $(E,\leq,',0,1)$ *is a bounded poset with an antitone involution,*
- (ii) $a+0=a\odot 1=a$*,*
- (iii) $a+a'=1$ *and* $a\odot a'=0$*,*
- (iv) $a,b\leq a+b$ *and* $a\odot b\leq a,b$*,*
- (v) *if* $a\leq b$ *then* $a=b\odot(a+b')=\big(b'+(a+b')'\big)'$*,*
- (vi) *if* $a\leq b$ *then* $b=a+(a'\odot b)=a+(a+b')'$*,*
- (vii) *if* $a\leq b$ *and* $b+c$ *is defined so is* $a+c$ *and we have* $a+c\leq b+c$*,*
- (viii) *if* $a\leq b$ *and* $a\odot c$ *is defined so is* $b\odot c$ *and we have* $a\odot c\leq b\odot c$*,*
- (ix) *if* $a+b$ *and* $a+c$ *are defined and* $a+b=a+c$ *then* $b=c$*,*
- (x) *if* $a\odot b$ *and* $a\odot c$ *are defined and* $a\odot b=a\odot c$ *then* $b=c$*.*

Logical connectives in effect algebras

As stated in [14], effect algebras serve as a model for unsharp quantum logic. Hence there is the question, what are the logical connectives within this logic. Usually, the partial operations $+$ and $\odot$ are considered as disjunction and conjunction, respectively. In every logic the most productive connective, however, is implication. Using this connective it is possible to derive new propositions from given ones by certain derivation rules (e.g. Modus Ponens or substitution rule). The question arises how to introduce the connective implication in the quantum logic based on an effect algebra. For lattice-ordered effect algebras this problem was already solved by the authors. Now we will investigate effect algebras that need not be lattice-ordered. It is worth noticing that the present authors together with R. Halaš derived a Gentzen system for the connective implication in lattice effect algebras and also for the non-lattice case, but the implication treated there differs essentially from that we will investigate now. The main difference is that now we are going to connect our implication with conjunction via a certain kind of adjointness.

Let $\mathbf{E} = (E, +, ', 0, 1)$ be an effect algebra. Define the following binary operator on E:

$$b \to c := \text{Max}\{x \in E \mid x \odot b \text{ is defined and } x \odot b \leq c\}$$

$(b, c \in E)$. This is our "unsharp" implication in the logic based on $\mathbf{E}$. The denotation "unsharp" expresses the fact that the result of $b \to c$ need not be an element of E (as it was the case for the implication introduced in [2] and [6]), but may be a subset of E. The elements of $b \to c$ form an antichain, it means we cannot prefer one with respect to another by their order. Moreover, $b \to c$ is defined for all $b, c \in E$.

Example 4

The "operation table" of $\to$ corresponding to the effect algebra from Example 1 looks as follows:

$\to$	0	a	b	c	d	c'	b'	a'	1
0	1	1	1	1	1	1	1	1	1
a	a'	1	a'	a'	a'	1	1	a'	1
b	b'	b'	1	b'	1	1	1	1	1
c	c'	c'	c'	1	c'	c'	1	1	1
d	d	d	b'	d	1	b'	1	b'	1
c'	c	b'	a'	c	a'	1	$\{a', b'\}$	a'	1
b'	b	c'	d	a'	b'	$\{d, c'\}$	1	$\{d, a'\}$	1
a'	a	a	c'	b'	c'	c'	$\{b', c'\}$	1	1
1	0	a	b	c	d	c'	b'	a'	1

The next result shows that our unsharp implication still shares some important properties asked usually in any non-classical logic.

Lemma 5

Let $(E, +, ', 0, 1)$ be an effect algebra satisfying the ACC and $a, b, c \in E$. Then the following holds:

1. $a' \leq_1 a \to b$ *and* $a \to b \neq \emptyset$,
2. $a \to 0 = a'$ *and* $1 \to a = a$,
3. $a \to b = 1$ *if and only if* $a \leq b$,
4. *if* $a \leq b$ *then* $c \to a \leq_1 c \to b$.

At first we show some kind of adjointness between $\odot$ and $\rightarrow$.

Theorem 6

Let $(E, +, ', 0, 1)$ be an effect algebra satisfying the ACC and $a, b, c \in E$ and assume $a \odot b$ to be defined. Then

$$a \odot b \leq c \text{ if and only if } a \leq_1 b \rightarrow c.$$

Adjointness shown in Theorem 6 yields an important derivation rule valid in this logic. Namely,

$$a \rightarrow b \leq_1 a \rightarrow b$$

implies by adjointness

$$a \odot (a \rightarrow b) = (a \rightarrow b) \odot a \leq b$$

(provided $a \odot (a \rightarrow b)$ is defined) saying properly that the value of b cannot be less than the value of the conjunction of a and $a \rightarrow b$ (provided this conjunction is defined), which is just the derivation rule Modus Ponens.

Lemma 7

Let $(E, +, ', 0, 1)$ be an effect algebra satisfying the ACC and $a, b \in E$. Then the following holds:

1. *If $(a \to b) \odot a$ is defined then $(a \to b) \odot a \leq b$.*
2. *If $a \odot b$ is defined then $a \leq_1 b \to (a \odot b)$.*

Hence, if $(a \to b) \odot a$ is defined, evidently $(a \to b) \odot a \leq a$, thus, together with Lemma 7, we obtain

$$(a \to b) \odot a \leq_1 \operatorname{Max} L(a, b)$$

which is *divisibility*.

There is also another possibility how to define the connective implication in an effect algebra $(E, +, ', 0, 1)$, namely as the following partial binary operation:

$$b \rightsquigarrow c := b' + c$$

$(b, c \in E)$. It is evident that $b \rightsquigarrow c$ is defined if and only if $c \leq b$.

Example 8

The operation table of $\rightsquigarrow$ corresponding to the effect algebra from Example 1 looks as follows:

$\rightsquigarrow$	0	a	b	c	d	c'	b'	a'	1
0	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
a	a'	1	$-$	$-$	$-$	$-$	$-$	$-$	$-$
b	b'	$-$	1	$-$	$-$	$-$	$-$	$-$	$-$
c	c'	$-$	$-$	1	$-$	$-$	$-$	$-$	$-$
d	d	$-$	b'	$-$	1	$-$	$-$	$-$	$-$
c'	c	b'	a'	$-$	$-$	1	$-$	$-$	$-$
b'	b	c'	d	a'	b'	$-$	1	$-$	$-$
a'	a	$-$	c'	b'	$-$	$-$	$-$	1	$-$
1	0	a	b	c	d	c'	b'	a'	1

Also this implication which is a partial operation satisfies several important properties of implication known in non-classical logics.

Lemma 9

Let $(E, +, ', 0, 1)$ be an effect algebra and $a, b, c \in E$. Then the following holds:

1. *if $a \rightsquigarrow b$ is defined then $a' \leq_1 a \rightsquigarrow b$,*
2. *$a \rightsquigarrow 0 = a'$ and $1 \rightsquigarrow a = a'$,*
3. *$a \rightsquigarrow b = 1$ if and only if $a = b$,*
4. *if $a \leq b$ and $c \rightsquigarrow b$ is defined then $c \rightsquigarrow a$ is defined and $c \rightsquigarrow a \leq c \rightsquigarrow b$.*

The following theorem shows the relationship between $\rightarrow$ and $\rightsquigarrow$.Also for the implication $\rightsquigarrow$ we can show some kind of adjointness which is, however, only partial.

Theorem 10

Let $(E, +, ', 0, 1)$ be an effect algebra and $a, b, c \in E$ and assume $a \odot b$ and $b \rightsquigarrow c$ to be defined. Then

$$a \odot b \leq c \text{ if and only if } a \leq b \rightsquigarrow c.$$

There is a further possibility how to define the connective implication in an effect algebra $\mathbf{E} = (E, +, ', 0, 1)$ that need not be lattice-ordered, namely

$$b \Rightarrow c := b' + \operatorname{Max} L(b, c)$$

($b, c \in E$, cf. [2]). If $\mathbf{E}$ satisfies the ACC then $b \Rightarrow c \neq \emptyset$. This kind of implication is, of course, again an unsharp one since the result of $a \Rightarrow b$ need not be an element of the corresponding effect algebra, but may be a subset of it. On the other hand, we see that $\Rightarrow$ is an everywhere defined binary operator on E.

Example 11

The "operation table" of $\Rightarrow$ corresponding to the effect algebra from Example 1 looks as follows:

$\Rightarrow$	0	a	b	c	d	c'	b'	a'	1
0	1	1	1	1	1	1	1	1	1
a	a'	1	a'	a'	a'	1	1	a'	1
b	b'	b'	1	b'	1	1	1	1	1
c	c'	c'	c'	1	c'	c'	1	1	1
d	d	d	b'	d	1	b'	1	b'	1
c'	c	b'	a'	c	a'	1	$\{a', b'\}$	a'	1
b'	b	c'	d	a'	b'	$\{d, c'\}$	1	$\{d, a'\}$	1
a'	a	a	c'	b'	c'	c'	$\{b', c'\}$	1	1
1	0	a	b	c	d	c'	b'	a'	1

The properties of the implication $\Rightarrow$ are very natural, see the following result.

Lemma 12

Let $(E, +, ', 0, 1)$ *be an effect algebra satisfying the* ACC *and* $a, b, c \in E$. *Then the following holds:*

1. $a' \leq a \Rightarrow b \neq \emptyset$,
2. $a \Rightarrow 0 = a'$ *and* $1 \Rightarrow a = a$,
3. $a \Rightarrow b = 1$ *if and only if* $a \leq b$,
4. *if* $a \leq b$ *then* $c \Rightarrow a \leq_1 c \Rightarrow b$.

Now we can compare all three implications considered here.

Theorem 13

Let $(E, +, ', 0, 1)$ *be an effect algebra and* $a, b \in E$*. Then the following holds:*

1. $a \Rightarrow b \leq_1 a \rightarrow b$.
2. *If* $a \rightsquigarrow b$ *is defined then* $a \rightarrow b = a \Rightarrow b = a \rightsquigarrow b$,

There is the question how to define a binary operator $\otimes$ on an effect algebra $\mathbf{E} = (E, +, ', 0, 1)$ such that $\otimes$ and $\Rightarrow$ form an adjoint pair. For this purpose we define

$$a \otimes b := \operatorname{Min} U(a, b') \odot b$$

$(a, b \in E)$. It is easy to see that

$$\begin{aligned} a \otimes b &= (b \Rightarrow a')', \\ a \Rightarrow b &= (b' \otimes a)' \end{aligned}$$

for all $a, b \in E$. For subsets A, B of E we define

$$\begin{aligned} A \otimes B &:= \{a \otimes b \mid a \in A, b \in B\}, \\ A \Rightarrow B &:= \{a \Rightarrow b \mid a \in A, b \in B\}. \end{aligned}$$

Example 14

The "operation table" of $\otimes$ corresponding to the effect algebra from Example 1 looks as follows:

$\otimes$	0	a	b	c	d	c'	b'	a'	1
0	0	0	0	0	0	0	0	0	0
a	0	a	0	0	b	a	$\{d, a'\}$	0	a
b	0	0	0	0	0	$\{a, b\}$	0	$\{b, c\}$	b
c	0	0	0	c	b	0	$\{c, d\}$	c	c
d	0	a	0	c	0	a	b	c	d
c'	0	a	b	0	d	c'	a	b	c'
b'	0	a	0	c	b	a	d	c	b'
a'	0	0	b	c	d	b	c	a'	a'
1	0	a	b	c	d	c'	b'	a'	1

Analogous to Lemma 12 we obtain

Lemma 15

Let $(E, +, ', 0, 1)$ *be an effect algebra satisfying the* DCC *and* $a, b, c \in E$. *Then the following holds:*

1. $\emptyset \neq a \otimes b \leq b$,
2. $a \otimes 1 = 1 \otimes a = a$,
3. $a \otimes b = 0$ *if and only if* $a \perp b$,
4. *if* $a \leq b$ *then* $a \otimes c \leq_2 b \otimes c$.

We can show that also the operators $\otimes$ and $\Rightarrow$ form an adjoint pair.

Theorem 16

Let $(E, +, ', 0, 1)$ be an effect algebra satisfying both the ACC and the DCC and $a, b, c \in E$. Then the following holds:

1. $a \otimes b \sqsubseteq c$ *if and only if* $a \sqsubseteq b \Rightarrow c$ (adjointness),
2. $(a \Rightarrow b) \otimes a = \operatorname{Max} L(a, b)$ (divisibility).

The previous result can be generalized for subsets of E.

Lemma 17

Let $(E, +, ', 0, 1)$ be an effect algebra satisfying both the ACC and the DCC and $A, B, C \in \mathcal{P}_+A$. Then $A \otimes B \sqsubseteq C$ is equivalent to $A \sqsubseteq B \Rightarrow C$ (adjointness)

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The end!

Thanks for your attention!!