Varieties of commutative BCK-algebras: Free algebras

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Free ŁBCK–algebras

The case of free ŁBCK–algebras in varieties $\mathrm{V}(S_m)$, $m \in \mathbb{N}$, was treated in [Figallo et al., 2004].

Let $X = \{x_1, x_2, \ldots, x_q\}$ be a set of generators.
For free ŁBCK–algebras $F_{\mathrm{V}(S_m)}(q)$ the following holds:

- The maximal elements of $F_{\mathrm{V}(S_m)}(q)$ are the free generators.
- Every interval $[0, x_i]$, $1 \leq i \leq q$ is a direct product of chains.
- $F_{\mathrm{V}(S_m)}(q)$ is symmetric, i.e. $[0, x_i] \cong [0, x_j]$ for every i, $j \in \{1, 2, \ldots, q\}$
- $$|F_{\mathrm{V}(S_m)}| = \sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} |[0, \bigwedge_{i=1}^{j} x_i]|.$$

Free cBCK-algebras

Let T be finite subdirectly irreducible cBCK-algebra.

- The maximal elements of $F_{\mathrm{V}(T)}(q)$ are the free generators.
  Consequence of $x \ominus y \leq x$.
- Every interval $[0, x_i]$, $1 \leq i \leq q$, is a direct product of chains.
  The interval $[0, x_i]$ is bounded cBCK-algebra
- $[0, x_i] \cong [0, x_j]$ for every i, $j \in \{1, 2, \ldots q\}$.
  By the universal property: Let $f\colon X \to F_{\mathrm{V}(T)}(q)$ be such that $f(X) = X$ and $x_i \mapsto x_j$, $x_j \mapsto x_i$. Then $\overline{f}\colon F_{\mathrm{V}(T)}(q) \to F_{\mathrm{V}(T)}(q)$ is isomorphism.
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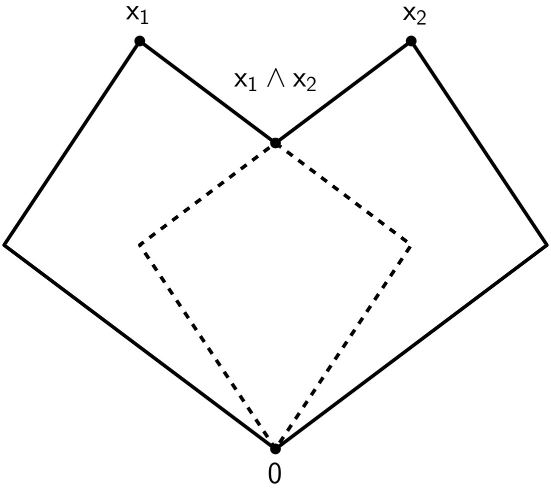
$$|F_{\mathrm{V}(T)}| = \sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} |[0, \bigwedge_{i=1}^{j} x_i]|.$$
  Application of inclusion-exclusion principle.

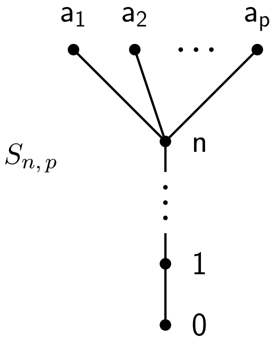


Free cBCK-algebra

The key is to find a direct decomposition of every interval $[0, \bigwedge_{i=1}^{j} x_i]$ to directly irreducible algebras = chains.

This was done in [Figallo et al., 2004] for free agebras in $\mathrm{V}(S_n)$.

Our goal is to find this decomposition for free algebras in $\mathrm{V}(S_{n,p})$.



Free algebras in $\mathrm{V}(S_{n,p})$: intervals $[0,x]$

Let's denote $\mathscr{V} = \mathrm{V}(S_{n,p})$ and $\mathscr{W} = \mathrm{V}(S_{n+1})$.

1. Let $h\colon F_{\mathscr{V}}(q) \to F_{\mathscr{W}}(q)$ be the homomorphism extending $id\colon X \to F_{\mathscr{W}}(q)$.
2. Let Θ denote the congruence determined by h. We have $F_{\mathscr{V}}(q)/\Theta \cong F_{\mathscr{W}}(q)$ and for any $x \in X$:

   $$[0,x]/\Theta \cong [0,x]_L,$$

   where the index L refers to interval of $F_{\mathscr{W}}(q)$.
3. Since cBCK-algebras are distributive and $[0,x]$ is direct product of simple algebras, the lattice of congruences on $[0,x]$ is Boolean. Moreover, bounded cBCK-algebras are permutable.

   Therefore, there is a unique complement Φ of Θ such that Θ and Φ permute. Thus Θ and Φ are factor congruences and

   $$[0,x] \cong [0,x]/\Phi \times [0,x]_L.$$

The above works for every $[0, \bigwedge_{i=1}^{j} x_i]$, $q \geq 2$, $1 \leq j \leq q$.

In the case of $q = 1$, we have $F_{\mathscr{V}}(q) \cong S_1$.

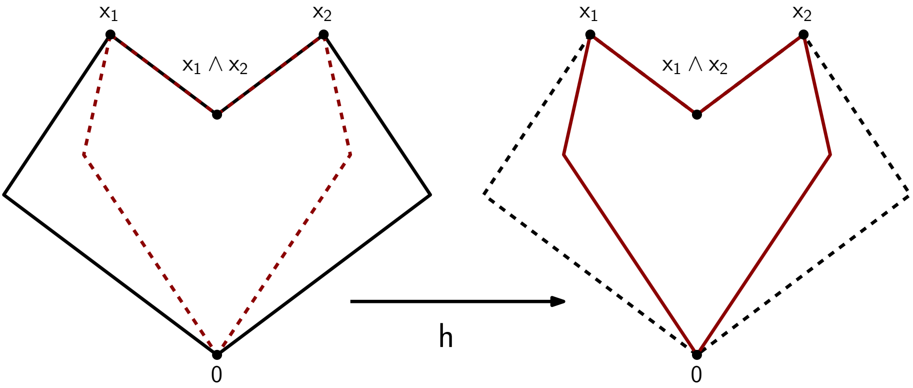
Lemma

Let $\mathscr{V} = \mathrm{V}(S_{n,p})$ *and* $\mathscr{W} = \mathrm{V}(S_{n+1})$*. Let* $[0, \bigwedge_{i=1}^{j} x_i]$ *be an interval in* $F_{\mathscr{V}}(q)$ *and* $[0, \bigwedge_{i=1}^{j} x_i]_L$ *be an interval in* $F_{\mathscr{W}}(q)$*. Then,* Θ *and* Φ *are factor congruences on the interval* $[0, \bigwedge_{i=1}^{j} x_i]$*. Thus*

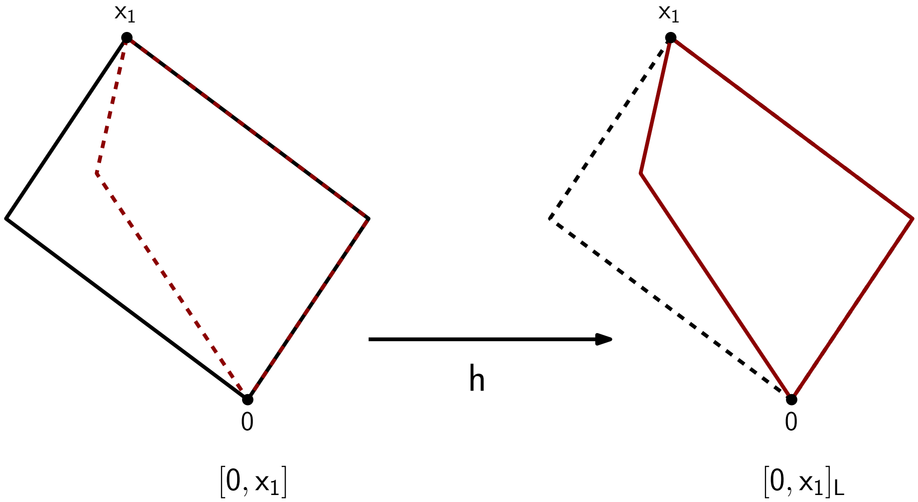
$$[0, \bigwedge_{i=1}^{j} x_i] \cong [0, \bigwedge_{i=1}^{j} x_i]/\Phi \times [0, \bigwedge_{i=1}^{j} x_i]_L.$$

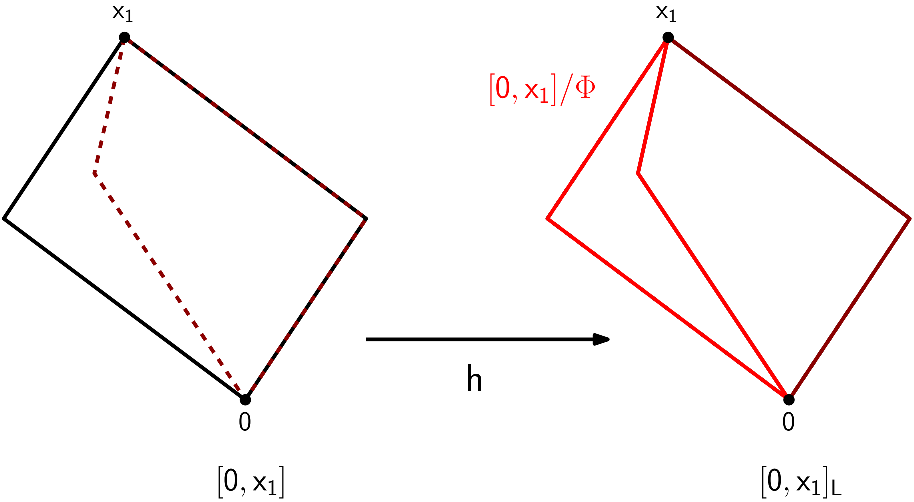
Since the decompositions of the intervals $[0, \bigwedge_{i=1}^{j} x_i]_L$ are known, it remains to investigate $[0, \bigwedge_{i=1}^{j} x_i]/\Phi$.

Free algebras in $\mathrm{V}(S_{n,p})$



Free cBCK-algebra **Free ŁBCK-algebra**


x1
0
h
x1
0
[0, x1]
[0, x1]L



x_1
0
$[0, x_1]$
h
x_1
$[0, x_1]/\Phi$
0
$[0, x_1]_L$

General construction of free algebras

Let $X \neq \emptyset$ be a set and A be an algebra. Then a valuation is mapping $v\colon X \to A$. Let $A(v)$ denote the subalgebra of A generated by $v(X)$.

$F_{\mathrm{V}(A)}(X)$ of is the subalgebra of $\prod_{v \in A^X} A(v)$ generated by the image of X.

A homomorphism $h\colon A(v) \to A(v')$ respects the labelling iff $h(v(x)) = v'(x)$ for every $x \in X$.

Two valuations are equivalent $v \sim v'$ iff there exist homomorphisms $h\colon A(v) \to A(v')$ and $h'\colon A(v') \to A(v)$ such that both respect the labelling.

Let $E(V)$ denote any maximal set of non-equivalent valuations. The free algebra is the subalgebra of $\prod_{v \in E(V)} A(v)$ generated by the image of X.

Equivalence of valuations $v\colon X \to S_{n,p}$

Proposition

Let $v,\ v'\colon X \to S_{n,p}$ *such that* $A(v)$, $A(v')$ *are not chains. Then* $v \sim v'$ *iff there exists an isomorphism* $h\colon A(v) \to A(v')$ *which respects the labelling.*

Let $m(A)$ denote the set of maximal elements of A.

Corollary

Let $v,\ v'\colon X \to S_{n,p}$ *such that* $A(v)$, $A(v')$ *are not chains. Then* $v \sim v'$ *iff there exists a bijection* $\varphi\colon m(A(v)) \to m(A(v'))$ *respecting the labelling and for each* $v(x) \notin m(A(v))$ *we have* $v(x) = v'(x)$.

Description of $[0, \bigwedge_{i=1}^{j} x_i]/\Phi$

Proposition

Let $X = \{x_1, \ldots, x_q\}$ be a set of generators and $q \geq 2$, $1 \leq j \leq q$. Let $[0, \bigwedge_{i=1}^{j} x_i]$ be an interval of the free algebra $F_{\mathscr{V}}(q)$. Then there are index sets K_2, K_3, …, K_p such that the following holds:

$$[0, \bigwedge_{i=1}^{j} x_i]/\Phi \cong \prod_{k_2 \in K_2} [0, \bigwedge_{i=1}^{j} v_{k_2}(x_i)] \times \cdots \times \prod_{k_p \in K_p} [0, \bigwedge_{i=1}^{j} v_{k_p}(x_i)],$$

where for every $l \in \{2, 3, \ldots, p\}$ the set $\{v_{k_l} \mid k_l \in K_l\}$ is a maximal set of mutually non-equivalent valuations satisfying $A(v_{k_l}) \cong S_{n,l}$ and $[0, \bigwedge_{i=1}^{j} v_{k_l}(x_i)]$ is an interval of $A(v_{k_l})$.

Description of free algebras

Let

$$\beta(m, q, j)$$

denote the power of S_m in the direct decomposition of the interval $[0, \bigwedge_{i=1}^{j} x_i]_L$ and let

$$\gamma_l(m, q, j)$$

denote the power of S_m in $\prod_{k_l \in K_l} [0, \bigwedge_{i=1}^{j} v_{k_l}(x_i)]$.

Corollary

Let $\mathscr{V} = \mathrm{V}(S_{n,\,p})$. *Then we have*

$$|F_{\mathscr{V}}(q)| = \sum_{j=1}^{q} (-1)^{j+1} \binom{q}{j} \prod_{m=1}^{n+1} (m+1)^{\beta(m,q,j) + \sum_{l=2}^{p} \gamma_l(m,q,j)}.$$

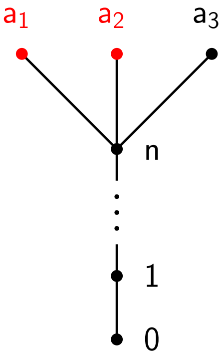
Proposition

Let $2 \leq q < p$, $\mathscr{U} = S_{n,p}$ and $\mathscr{V} = S_{n,q}$. Then the following is true

$$F_{\mathscr{U}}(q) \cong F_{\mathscr{V}}(q).$$

For the calculation of γ_l, it is sufficient to assume $2 \leq p \leq q$.

$S_{n,3}$,
$q = 2$



Main theorem

Let $2 \leq l \leq p \leq q$ and $1 \leq j \leq q$. Then the following holds for $\gamma_l(m, q, j)$, $1 \leq m \leq n+1$.

1. Case $m = n + 1$:
   1. Subcase $q > l + j - 1$:
   $$\gamma_l(n+1, q, j) = \frac{1}{l!} \sum_{i=0}^{l-1} (-1)^i \binom{l}{l-i} (n+1+l-i)^{q-j} (l-i).$$
   2. Subcase $q = l + j - 1$:
   $$\gamma_l(n+1, q, j) = \frac{1}{l!} \sum_{i=0}^{l} (-1)^i \binom{l}{l-i} (n+1+l-i)^{l}.$$
   3. Subcase $q < l + j - 1$:
   $$\gamma_l(n+1, q, j) = 0.$$

Main theorem

2. Case $m = n$:
   1. Subcase $q > l + j - 1$:

$$\gamma_l(n,q,j) = \frac{1}{l!}\Bigg[\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{q-j}[(n+1+l-i)^j - l + i] - \sum_{k=1}^{j}\binom{j}{k}[(n+1)^k - 1]\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{q-k}\Bigg].$$

   2. Subcase $q = l + j - 1$:

$$\gamma_l(n,q,j) = \frac{1}{l!}\Bigg[\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{l}[(n+1+l-i)^{j-1} - 1] - \sum_{k=1}^{j-1}\binom{j}{k}[(n+1)^k - 1]\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{q-k}\Bigg].$$

Main Theorem

3. Subcase $q < l + j - 1$:

$$\gamma_l(n,q,j) = \frac{1}{l!}\Bigg[\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^q - \sum_{k=1}^{\min\{q-l,j\}}\binom{j}{k}[(n+1)^k-1]\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{q-k}\Bigg].$$

3. Case $1 \leq m < n$:

$$\gamma_l(m,q,j) = \frac{1}{l!}\sum_{k=1}^{\min\{q-l,j\}}\binom{j}{k}\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^{q-k}.$$

Proving the main theorem

1. Valuations of the property $P \longleftrightarrow$ words of property P'

   Let V_l denote the set of valuations such that $A(v) \cong S_{n,l}$. Then the corresponding set of words is the set of q-long words over alphabet $S_{n,p}$ such that they contain exactly l maximal elements.

2. Calculate the number of $|V_l|$. Using word representation, it is easy to find that

$$|V_l| = \binom{p}{l} \sum_{\substack{k_1, k_2, \dots k_l \geq 1, \\ k_1 + k_2 + \dots + k_{n+l+1} = q}} \mathrm{P}(k_1, \dots, k_{n+l+1}) = \binom{p}{l} \sum_{i=0}^{l} (-1)^i \binom{l}{l-i} (n+1+l-i)^q,$$

   where P is permutation with repetition.

4. Calculate the number of $|V_l/\sim|$. Equivalent valuations correspond to injections $\{1,\dots l\} \to \{1,\dots,p\}$.

$$|V_l/\sim| = \frac{1}{l!}\sum_{i=0}^{l}(-1)^i\binom{l}{l-i}(n+1+l-i)^q.$$

5. For every case of $\gamma_l(m,q,j)$ consider V_γ – valuations of the case.

   Example: For the first case of $m = n+1$ the $v \in V_\gamma$ are valuations such that $[0, \bigwedge_{i=1}^{j} v(x_i)] \cong S_{n+1}$ and $A(v) \cong S_{n,l}$.
   In other words, $v \in V_\gamma$ are q-long words over the alphabet $S_{n,p}$ such that
   - the first j letters are the same maximal element and
   - they contain exactly l maximal elements.

6. Calculate $|V_\gamma/\sim|$ – use the previous result for $|V_l/\sim|$ and few clever ideas.

Examples

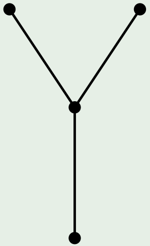
Example 1

For $S_{1,\,2}$, $q = 2$, we have

$$\beta(2,2,1) = 1, \quad \beta(1,2,1) = 3, \quad \beta(1,2,2) = 3,$$
$$\gamma_2(2,2,1) = 1, \quad \gamma_2(1,2,2) = 1,$$

others values of γ_2 and β are zero. Thus

$$[0, x_1] \cong {S_1}^3 \times S_2^{1+1},$$
$$[0, x_1 \wedge x_2] \cong {S_1}^{3+1},$$
$$|F_{\mathrm{V}(S_{1,\,2})}(2)| = 128.$$



$S_{1,\,2}$

Example 2

For $S_{1,\,2}$, $q = 3$

$$[0, x_1] \cong {S_1}^{9+1} \times {S_2}^{5+7},$$
$$[0, x_1 \wedge x_2] \cong {S_1}^{9+7} \times {S_2}^{1+1},$$
$$[0, x_1 \wedge x_2 \wedge x_3] \cong {S_1}^{7+8},$$
$$|F_{\mathrm{V}(S_{1,\,2})}(3)| = 1\,630\,850\,048.$$

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