Kites and representations of pseudo MV-algebras

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Residuated lattices

We work within the general framework of *residuated lattices*, that is, algebras $\mathbf{A} = (A; \wedge, \vee, \cdot, \backslash, /, 1)$ such that

(RL1) $(A; \wedge, \vee)$ is a lattice,

(RL2) $(A; \cdot, 1)$ is a monoid,

(RL3) the equivalences

$$y \leq x \backslash z \quad \Leftrightarrow \quad xy \leq z \quad \Leftrightarrow \quad x \leq z \,/\, y$$

hold for all $x, y, z \in A$.

FL-algebras

Expansions of residuated lattices by an additional constant 0 are known as *FL-algebras* and it is there only to make it possible to define *negations*, that is, the operations

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An FL-algebra $\mathbf{A}$ is *integral* if 1 is the largest element of A; it is *0-bounded* if 0 is the smallest element of A. Integral 0-bounded FL-algebras are known as FL_w-algebras (w for *weakening*), so according to our conventions $\mathsf{FL_w}$ will stand for the variety of FL_w-algebras.

ℓ-groups and some derived residuated lattices

Definition

A *lattice ordered group* (ℓ-group) is an algebra $\mathbf{L} = (L; \wedge, \vee, \cdot, ^{-1}, e)$ where $(L; \wedge, \vee)$ is a lattice, $(L; \cdot, ^{-1}, e)$ is a group and

$$x(y \wedge z)w = (xyw) \wedge (xzw),$$
$$x(y \vee z)w = (xyw) \vee (xzw)$$

hold for any $x, y, z, w \in L$.

ℓ-groups and some derived residuated lattices

For our purposes here, it will suffice to recall that any ℓ-group $\mathbf{L}$ is completely determined by the residuation structure of its negative cone $L^- = \{x \in L : x \leq e\}$. Namely, defining the algebra

$$\mathbf{L}^- = (L^-; \wedge, \vee, \cdot, \backslash, /, e)$$

where e, $\wedge$, $\vee$ and $\cdot$ are inherited from $\mathbf{L}$, and

$$x \,/\, y := (xy^{-1}) \wedge e, \qquad y \,\backslash\, x := (y^{-1}x) \wedge e$$

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we obtain an integral residuated lattice satisfying the identities

$$xy \,/\, y = x = y \,\backslash\, yx \tag{Can}$$

$$x \,/ (y \,\backslash\, x) = x \vee y = (x \,/\, y) \,\backslash\, x. \tag{Łuk}$$

The first of these is equivalent over residuated lattices to the usual cancellation laws

$$zx = zy \Rightarrow x = y \qquad \text{and} \qquad xz = yz \Rightarrow x = y.$$

The second amounts to a non-commutative rendering of the *Łukasiewicz axiom* $(x \to y) \to y = (y \to x) \to x$.

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Residuated lattices satisfying (Can) are known as *cancellative*, whereas those satisfying $(\mathrm{Łuk})$ are called *integral generalised MV-algebras (IGMV-algebras)*.

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Residuated lattices satisfying (Can) are known as *cancellative*, whereas those satisfying $(\mathrm{Łuk})$ are called *integral generalised MV-algebras (IGMV-algebras)*.
Members of the variety CanIGMV of *cancellative integral generalised MV-algebras* are (up to isomorphism) precisely the negative cones of ℓ-groups. Namely, there exists functors

$$^{-}\colon \mathsf{LG} \to \mathsf{CanIGMV} \text{ and } \ell\colon \mathsf{CanIGMV} \to \mathsf{LG}$$

such that $\ell(\mathbf{A})^{-} = \mathbf{A}$ for any $\mathbf{A} \in \mathsf{CanIGMV}$ and $\ell(\mathbf{L}^{-}) \cong \mathbf{L}$ for any $\mathbf{L} \in \mathsf{LG}$. These functors establish a categorical equivalence between LG and CanIGMV. In particular, the subvariety lattices of LG and CanIGMV are isomorphic.

Pseudo MV-algebras

Pseudo MV-algebras were originally defined and studied by Georgescu and Iorgulescu, as algebras $(A; \oplus, ^-, ^\sim, 0, 1)$ satisfying the identities:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = x$,

(A3) $x \oplus 1 = 1$,

(A4) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A5) $(x \oplus y^\sim)^- \oplus x^- = y \oplus (x^- \oplus y)^\sim$,

(A6) $x \oplus (y^- \oplus x)^\sim = y \oplus (x^- \oplus y)^\sim$

(A7) $x^{-\sim} = x$,

(A8) $0^- = 1$.

The identities $0 \oplus x = x$ and $1 \oplus x = 1$ follow, as well as $1^- = 0 = 1^\sim$, and $x^{\sim -} = x$.

Pseudo MV-algebras

In any pseudo MV-algebra defined by (A1)–(A8), the lattice operations, multiplication and residuals are defined by

- $x \vee y := x \oplus (y \odot x^{\sim})$ and $x \wedge y := (x^{-} \oplus y) \odot x$,
- $x \cdot y := (x^{-} \oplus y^{-})^{\sim}$,
- $x \backslash y := y \oplus x^{\sim}$ and $y \,/\, x := x^{-} \oplus y$.

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Then pseudo MV-algebras are (equivalently) bounded IGMV-algebras and the variety $\Psi\mathsf{MV}$ of *pseudo MV-algebras* is a subvariety of $\mathsf{FL_w}$

Perfect FL_w-algebras

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Definition

An FL_w-algebra $\mathbf{A}$ is perfect if there is a homomorphism $h_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{2}$ such that for any $x \in h_{\mathbf{A}}^{-1}(0)$ and any $y \in h_{\mathbf{A}}^{-1}(1)$ the inequality $x \leq y$ holds.

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To spare notation, we put $F_{\mathbf{A}} := h_{\mathbf{A}}^{-1}(1)$ and $J_{\mathbf{A}} := h_{\mathbf{A}}^{-1}(0)$, whenever h is clear from context. Clearly, $F_{\mathbf{A}}$ is a maximal normal filter and $J_{\mathbf{A}} = A \setminus F_{\mathbf{A}}$ is a lattice ideal.

Lemma

Let $\mathbf{A}$ be a perfect FL_w-algebra. Then the homomorphism $f_{\mathbf{A}} \colon \mathbf{A} \to \mathbf{2}$ is unique. Hence $\mathbf{A}$ has a unique maximal normal filter.

Perfect FL_w-algebras

A normal filter is a set $F \subseteq A$ such that (i) F is a lattice filter with $1 \in F$, (ii) $a, b \in F$ implies $ab \in F$, (iii) $a \in F, b \in A$ implies $\lambda_b(a), \rho_b(a) \in F$. In integral FL-algebras, the conjugates simplify to $\lambda_b(a) = b \backslash ab$ and $\rho_b(a) = ba \,/\, b$.

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Definition

A conjugation polynomial $\boldsymbol{\alpha}$ over $\mathbf{A}$ is any unary polynomial $(\gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n})(x)$ where $\gamma \in \{\lambda, \rho\}$ and $a_i \in A$ for $1 \leq i \leq n$. We write $\mathrm{cPol}(\mathbf{A})$ for the set of all conjugation polynomials over $\mathbf{A}$. For an element $u \in A$, an iterated conjugate of u is $\boldsymbol{\alpha}(u)$ for some $\boldsymbol{\alpha} \in \mathrm{cPol}(\mathbf{A})$.

Perfect FL_w-algebras

For any class $\mathcal{K}$ of FL_w-algebras, we denote the subclass of all its perfect members by $\mathcal{K}_{pf}$. We say that a variety $\mathcal{V}$ of FL_w-algebras is *perfectly generated* if it is generated by its perfect members, that is, if $\mathcal{V} = V(\mathcal{V}_{pf})$.

Theorem

A subvariety $\mathcal{V}$ of $\mathsf{FL_w}$ is perfectly generated if and only if $\mathcal{V}$ is nontrivial and satisfies the following identities:

$$\boldsymbol{\alpha}(x\,/\,x^-) \vee \boldsymbol{\beta}(x^-\,/\,x) = 1, \tag{1}$$

$$\boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-))^- \leq \boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-)), \tag{2}$$

$$x \wedge x^- \leq y \vee y^- \tag{3}$$

for every $\mathbf{A} \in \mathcal{V}$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{cPol}(\mathbf{A})$.

Perfect FL_w-algebras

Corollary

Any nontrivial subvariety of a perfectly generated variety is also perfectly generated.

Since the trivial variety is not perfectly generated, the variety BA of Boolean algebras is the smallest perfectly generated variety. Indeed, perfectly generated varieties form a lattice ideal in the lattice $\Lambda^+(\mathsf{FL_w})$, as we will now show.

Theorem

Perfectly generated varieties form an ideal in $\Lambda^+(\mathsf{FL_w})$.

Kites

Definition

Let $\mathbf{L}$ *be an* ℓ*-group and* $\lambda\colon \mathbf{L} \to \mathbf{L}$ *be an automorphism. We define the algebra*

$$\mathcal{K}(\mathbf{L}, \lambda) := (L^- \uplus L^+; \wedge, \vee, \odot, \backslash, /, 0, 1)$$

where $L^- \uplus L^+$ *is a disjoint union,* $0 := e \in L^+$, $1 := e \in L^-$, *and the other operations are given by*

$$x \wedge y := \begin{cases} x \wedge y \in L^- & \text{if } x, y \in L^-, \\ x \in L^+ & \text{if } x \in L^+, y \in L^- \\ y \in L^+ & \text{if } x \in L^-, y \in L^+, \\ x \wedge y \in L^+ & \text{if } x, y \in L^+, \end{cases}$$

$$x \vee y := \begin{cases} x \vee y \in L^- & \text{if } x, y \in L^-, \\ y \in L^- & \text{if } x \in L^+, y \in L^- \\ x \in L^- & \text{if } x \in L^-, y \in L^+, \\ x \vee y \in L^+ & \text{if } x, y \in L^+, \end{cases}$$

$$x \odot y := \begin{cases} x \cdot y \in L^- & \text{if } x, y \in L^-, \\ \lambda(x) \cdot y \vee e \in L^+ & \text{if } x \in L^-, y \in L^+ \\ x \cdot y \vee e \in L^+ & \text{if } x \in L^+, y \in L^-, \\ e \in L^+ & \text{if } x, y \in L^+, \end{cases}$$

$$x \setminus y := \begin{cases} x^{-1} \cdot y \wedge e \in L^- & \text{if } x, y \in L^-, \\ e \in L^- & \text{if } x \in L^+, y \in L^- \\ \lambda(x)^{-1} \cdot y \vee e \in L^+ & \text{if } x \in L^-, y \in L^+, \\ x^{-1} \cdot y \wedge e \in L^- & \text{if } x, y \in L^+, \end{cases}$$

$$y \,/\, x := \begin{cases} y \cdot x^{-1} \wedge e \in L^- & \text{if } x, y \in L^-, \\ e \in L^- & \text{if } x \in L^+, y \in L^- \\ y \cdot x^{-1} \vee e \in L^+ & \text{if } x \in L^-, y \in L^+, \\ \lambda^{-1}(y \cdot x^{-1}) \wedge e \in L^- & \text{if } x, y \in L^+, \end{cases}$$

Kites

Remark

The negations $x^- := 0 \,/\, x$ *and* $x^\sim := x \setminus 0$ *in* $\mathcal{K}(\mathbf{L}, \lambda)$ *are given by*

$$x^- = \begin{cases} x^{-1} \in L^+ & \text{if } x \in L^-, \\ \lambda^{-1}(x)^{-1} \in L^- & \text{if } x \in L^+. \end{cases}$$

$$x^\sim = \begin{cases} \lambda(x)^{-1} \in L^+ & \text{if } x \in L^-, \\ x^{-1} \in L^- & \text{if } x \in L^+. \end{cases}$$

Theorem

Let $\mathbf{L}$ *be an* ℓ*-group and* $\lambda \colon \mathbf{L} \to \mathbf{L}$ *an automorphism. Then* $\mathcal{K}(\mathbf{L}, \lambda)$ *is a perfect pseudo MV-algebra with* $J_{\mathcal{K}(\mathbf{L},\lambda)} = L^+$ *and* $F_{\mathcal{K}(\mathbf{L},\lambda)} = L^-$*. Moreover, if* $\mathcal{K}(\mathbf{L}, \lambda)$ *is a perfect MV-algebra, then* $\mathbf{L}$ *is Abelian and* λ *is the identity.*

Kites

Now, in any perfect pseudo MV-algebra $\mathbf{A}$ the normal filter $F_{\mathbf{A}}$ is the universe of a cancellative IGMV-algebra $\mathbf{F_A}$. Since pseudo MV-algebras satisfy the identities

$$(x \wedge y)^{\sim\sim} = x^{\sim\sim} \wedge y^{\sim\sim}$$
$$(x \vee y)^{\sim\sim} = x^{\sim\sim} \vee y^{\sim\sim}$$
$$(x \cdot y)^{\sim\sim} = x^{\sim\sim} \cdot y^{\sim\sim}$$
$$x^{-\sim\sim} = x^{\sim\sim-}$$

the map $-^{\sim\sim}$ is an automorphism of $\mathbf{F_A}$. Applying the functor ℓ we lift $-^{\sim\sim}$ to an automorphism

$$\ell^{\approx} \colon \ell(\mathbf{F_A}) \to \ell(\mathbf{F_A})$$

defined, obviously, as $\ell^{\approx}(-) := \ell(-^{\sim\sim})$.

Theorem

Let $\mathbf{A}$ be a perfect pseudo MV-algebra. Then $\mathbf{A} \cong \mathcal{K}(\ell(\mathbf{F_A}), \ell^{\approx})$.

Kites

Definition

We define LGA *to be the category of* ℓ*-groups with a distinguished automorphism. The objects are algebras* $(\mathbf{L}, \lambda)$ *where* $\mathbf{L}$ *is an* ℓ*-group and* λ *is an automorphism of* $\mathbf{L}$*. The morphisms are* ℓ*-group homomorphisms commuting with the distinguished automorphism.*

Theorem

The categories pfΨMV *of perfect pseudo MV-algebras, and* LGA *of* ℓ*-groups with a distinguished automorphism, are equivalent.*

Corollary (Di Nola, Lettieri)

Let AbLG *be the category of Abelian* ℓ*-groups with homomorphisms and let* pfMV *be the category of perfect MV-algebras with homomorphisms. Then* AbLG *and* pfMV *are equivalent.*

Definition

A monounary algebra $\mathbf{B} = (B; \beta)$ *where* β *is a bijection on* B *will be called a B-cycle.*

Remark

B-cycles are not a variety, but as we will often need β^{-1}*, we could have equivalently defined B-cycles as a variety of bi-unary algebras* (B, β, δ) *satisfying* $\beta(\delta(x)) = x = \delta(\beta(x))$*, and write* β^{-1} *for* δ*.*

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Definition

Let $\mathbf{B} = (B; \beta)$ *be a B-cycle and* $\mathbf{L}$ *and* ℓ*-group. A kite over* $\mathbf{B}$ *and* $\mathbf{L}$ *is the algebra*

$$\mathcal{K}_{\mathbf{B}}(\mathbf{L}) := \mathcal{K}(\mathbf{L}^B, \lambda)$$

where $\lambda\colon \mathbf{L}^B \to \mathbf{L}^B$ *is the automorphism given by* $\lambda(x(i)) = x(\beta(i))$ *for any* $i \in B$*.*

Varieties generated by kites

Throughout the section $\mathbb{D}$ will stand for the lattice $(\mathbb{N}; |)$ of natural numbers under the divisibility ordering. All order theoretic notions: minima, maxima, suprema, etc., will be taken with respect to this ordering.
For any bijection λ on a nonempty set B, we define the *dimension* of λ as follows:

$$dim(\lambda) := \min^{\mathbb{D}}\{n \in \mathbb{N} : \lambda^n = id_B\}.$$

For a B-cycle $\mathbf{B} = (B; \lambda)$, we put $dim(\mathbf{B}) := dim(\lambda)$ and call it the dimension of $\mathbf{B}$.

Lemma

We have $\Lambda^+(\mathrm{BC}) \cong \mathbb{D}$ and $\Lambda(\mathrm{BC}) \cong \mathbf{1} \oplus \mathbb{D}$, that is, the ordinal sum of the trivial lattice $\mathbf{1}$ and $\mathbb{D}$.

Varieties generated by kites

We will write C_n for the variety defined by $\lambda^n(x) = x$, so in particular $\mathsf{BC} = \mathsf{C}_0$. Moreover, for each n, the lattice of subvarieties of C_n is isomorphic to the lattice of all divisors of n.
For pseudo MV-algebras we will abbreviate the term operation $(-^{\sim\sim})$ by $(-^{\approx})$, and put:

$$(-^{\approx}) = (-^{1\times\approx}) := (-^{\sim\sim}),\ (-^{(n+1)\times\approx}) := (-^{n\times\approx})^{\approx}.$$

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For any pseudo MV-algebra $\mathbf{A}$, the operation $-^{\approx}$ is a bijection on A, so for any $\mathbf{A}$ we define the dimension of $\mathbf{A}$ to be $dim(-^{\approx})$. This is essential for the rest of the section, so we state it formally.

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Definition

Let $\mathbf{A} \in \mathsf{PMV}$ *and* $\mathcal{V} \in \Psi\mathsf{MV}$. *Then*

1. $dim(\mathbf{A}) := dim(-^{\approx})$,
2. $dim(\mathcal{V}) := \min^{\mathbb{D}}\{dim(\mathbf{A}) \mid n : \text{ for all } \mathbf{A} \in \mathcal{V}\}$,
3. $\mathsf{P\Psi MV}_n := \mathsf{P\Psi MV} \cap \mathrm{Mod}\{\lambda^n(x) = x\}$, *for any* $n \in \mathbb{D}$.

Varieties generated by kites

- All finite subdirectly irreducible cycles are of the form $\mathbf{Z}_n := (\{0, 1, \ldots, n-1\}; \lambda_n)$, where $\lambda_n(m) := m + 1 \pmod{n}$ for any $m \in Z_n$. To spare notation, for any ℓ-group $\mathbf{L} \in \mathsf{LG}$ we will write $\mathcal{K}_n(\mathbf{L})$ to denote the kite $\mathcal{K}_{\mathbf{Z}_n}(\mathbf{L})$.

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- We introduce an notation $\mathbf{B} \times S = (B \times S, \lambda')$ for the B-cycle $\mathbf{B} = (B, \lambda)$ and the set S where $\lambda'(b, s) = (\lambda(b), s)$.

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Lemma

Let $\mathbf{L}$ be an ℓ-group and let $\mathbf{B}$ be a B-cycle. Then

$$\mathcal{K}_{\mathbf{B}}(\mathbf{L}) \in ISP(\mathcal{K}_{dim(\mathbf{B})}(\mathbf{L})).$$

Varieties generated by kites

Definition

We define two pairs of maps

$$\psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}),\ \textit{where } \psi(\mathcal{V}) = V\{\mathbf{F_A} : \mathbf{A} \in \mathcal{V}_{pf}\},$$
$$\Psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanIGMV}) \times \mathbb{D},\ \textit{where } \Psi(\mathcal{V}) = (\psi(\mathcal{V}), \textit{dim}(\mathcal{V})),$$

for any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *and*

$$\delta\colon \Lambda(\mathsf{CanIGMV}) \to \Lambda(\mathsf{P\Psi MV}),\ \textit{where } \delta(\mathcal{V}) = V\{\mathbf{A} \in \mathsf{pf\Psi MV} : \mathbf{F_A} \in \mathcal{V}\}$$
$$\Delta\colon \Lambda(\mathsf{CanIGMV}) \times \mathbb{D} \to \Lambda(\mathsf{P\Psi MV}),\ \textit{where } \Delta(\mathcal{V}, n) = \delta(\mathcal{V}) \cap \mathsf{P\Psi MV}_n,$$

for any $\mathcal{V} \in \Lambda(\mathsf{CanIGMV})$ *and* $n \in \mathbb{D}$*.*

Varieties generated by kites

Lemma

Let $\mathcal{V} \in \Lambda(\mathsf{CanIGMV})$, let E be an equational base for $\mathcal{V}$, and let $\mathbf{A} \in \mathsf{P\Psi MV}$. The following are equivalent.

1. $\mathbf{A} \in \delta(\mathcal{V})$,
2. $\mathbf{A} \models t(x_1 \vee x_1^-, \ldots, x_k \vee x_k^-) = 1$ *for all terms t in the language of residuated lattices, such that* $\mathcal{V} \models t(x_1, \ldots, x_k) = 1$.
3. $\mathbf{A} \models t_\varepsilon(x_1 \vee x_1^-, \ldots, x_k \vee x_k^-) = 1$ *for all equations* $\varepsilon(x_1, \ldots, x_k) \in E$.

Varieties generated by kites

Lemma
The equality $\mathcal{V} = \psi\delta(\mathcal{V})$ *holds for any* $\mathcal{V} \in \Lambda(\mathsf{CanIGMV})$.

Lemma
For any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *we have* $\mathcal{V} \leq \delta\psi(\mathcal{V})$.

Lemma
For any $\mathcal{V} \in \Lambda^{+}(\mathsf{CanIGMV})$ *and any* $n \in \mathbb{D}$, *we have*

$$(\mathcal{V}, n) = \Psi\Delta(\mathcal{V}, n).$$

Lemma
For any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *we have*

$$\mathcal{V} \subseteq \Delta\Psi(\mathcal{V}).$$

Main Theorem

Theorem

Let $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$. *The following are equivalent.*

1. $\mathcal{V}$ *is generated by kites.*
2. $\mathcal{V} = \Delta\Psi(\mathcal{V})$.
3. $\mathcal{V} = \Delta(\mathcal{W}, n)$ *for some* $\mathcal{W} \in \Lambda(\mathsf{CanIGMV})$ *and some* $n \in \mathbb{D}$.

Theorem

Varieties generated by kites form a complete sublattice of $\Lambda(\mathsf{P\Psi MV})$ *with* $\mathsf{P\Psi MV}$ *being its largest, and* BA *its smallest element.*

Theorem

Let $\mathbb{K}$ *be the lattice of subvarieties of* $\mathsf{P\Psi MV}$ *generated by kites.*

$$\mathbb{K} \cong \mathbf{1} \oplus \big(\Lambda^{+}(\mathsf{CanIGMV}) \times \mathbb{D}\big) \cong \mathbf{1} \oplus \big(\Lambda^{+}(\mathsf{LG}) \times \mathbb{D}\big)$$

where $\mathbf{1}$ *is the trivial lattice and* $\oplus$ *is the operation of ordinal sum.*

Thank you for your attention!