Structures with slow unlabelled growth

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Orbit growth function

Definition

$\mathfrak{A}$: structure.

- $o_n(\mathfrak{A}) := \#\{n\text{-orbits of } \mathfrak{A}\} = \#\{\text{orbits of } \mathrm{Aut}(\mathfrak{A}) \curvearrowright A^n\}$

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General question:
Given some upper bound $f : \omega \to \mathbb{R}$, determine all structures with $\ell_n(\mathfrak{A}) < f(n)$ or $u_n(\mathfrak{A}) < f(n)$.

ω-categoricity

Observation

TFAE for all $n \in \omega$.

1. $o_n(\mathfrak{A}) < \infty$.
2. $\ell_n(\mathfrak{A}) < \infty$.
3. $u_n(\mathfrak{A}) < \infty$.

Note: $u_n \leq \ell_n \leq n! u_n, o_n(G) = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} \ell_k(G)$.

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$\mathfrak{A}$ is ω-categorical ($\mathrm{Aut}(\mathfrak{A})$ is oligomorphic) if $o_n(\mathfrak{A}) < \infty$ for all $n \in \omega$.

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In this talk: all structures are countable and ω-categorical.

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- $\ell_n < cn^{dn}$ with $d < 1 \leftrightarrow \mathfrak{A}$ is cellular (Bodirsky, B. '21)
- $u_n < c^n$ for *all/some* $c > 1$: Topic of today's talk, examples later.

Correspondence to groups

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$\mathfrak{A}$ and $\mathfrak{B}$ are bidefinable if $\exists e$ bijection $A \to B$ s.t. e and e^{-1} preserve definable relations.

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Fact

A permutation group G is of the form $G = \mathrm{Aut}(\mathfrak{A})$ iff $G \in \mathrm{Sym}(A)$ is closed (in the topology of pointwise convergence).

Correspondence to groups

ω-categorical structures up to bidefinability $\leftrightarrow$ closed oligomorphic groups

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Alternative formulation of the question:
Given some upper bound $f\colon \omega \to \mathbb{R}$, determine all closed groups with $\ell_n(G) < f(n)$ or $u_n(G) < f(n)$.

Stability

Definition

A formula $\phi(\bar{x}, \bar{y})$ has the order property (in $\mathfrak{A}$) if for all $\exists(\bar{a}_j, \bar{b}_j : j \in \omega)$ such that $\mathfrak{A} \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$. ("$\phi$ defines a half-graph")

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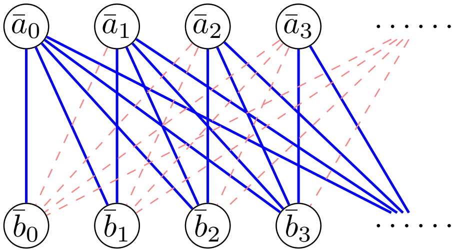

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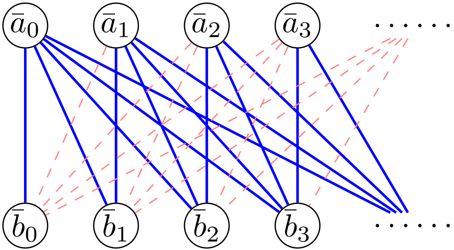

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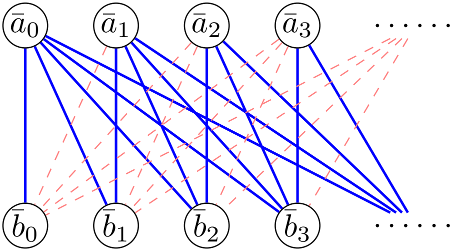

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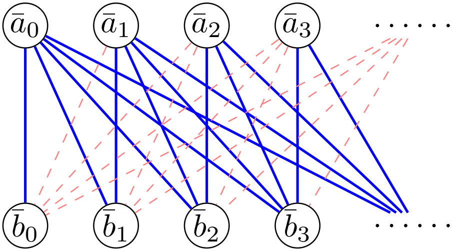

- Stable structures: pure set, unary structures, vector spaces,...
- Unstable structures: $(\mathbb{Q}, <)$, random graph, infinite Boolean algebras,...

Hereditarily cellular structures

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Theorem (Braunfeld)

$\mathfrak{A}$: ω-categorical stable structure. Then one of the following holds.

1. $\exists c \, (u_n(\mathfrak{A}) < c^n)$ *and* $\mathfrak{A} \in \mathcal{M}$.
2. $\exists c > 1 \, (u_n(\mathfrak{A}) > c^n)$ *and* $\mathfrak{A} \notin \mathcal{M}$.

Hereditarily cellular structures

Recursive description

Theorem (B. '21+Lachlan '92)

$\mathcal{M}$ is the smallest class of structures which contains all finite structures, and is closed under taking

1. *finite disjoint unions,*
2. *infinite copies,*
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We call these structures hereditarily cellular.

Theorem (B. '21+Lachlan '92)

$\{\mathrm{Aut}(\mathfrak{A}) : \mathfrak{A} \in \mathcal{M}\}$ *is the smallest class of groups which contains all groups with finite degree, and is closed under taking*

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Clarification:

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Remark: These groups can be described explicitly.

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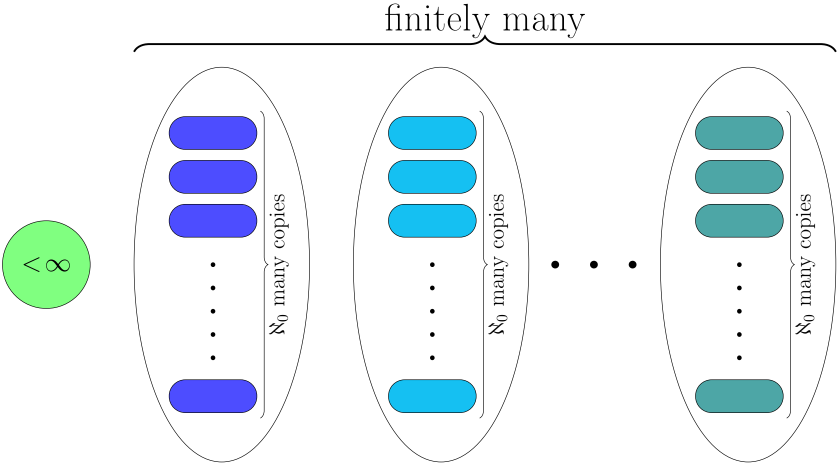

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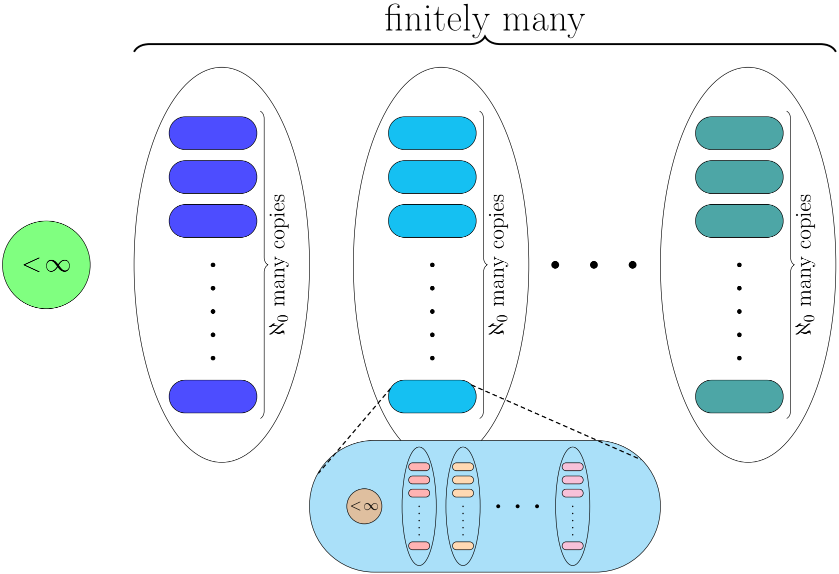

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Remark: $u_n \sim 2^{n-1}/n$ can be attained (by $S(2)$, the *local order*)

Structures with subexponential unlabelled growth

Recursive characterization

Theorem (Baby version, B. '23+)

$\mathcal{S}_0 = \{\mathfrak{A} : u_n(\mathfrak{A}) < c^n$ for all $c\}$ is the smallest class of structures which contain is closed under taking

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Remark: The automorphism groups can be described explicitly.

Structures with at most $(2-\varepsilon)^n$ unlabelled growth

Recursive characterization

Theorem (Full version, B. '23+)

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and contains

- *all finite structures,*
- *"some" finite covers of $(\mathbb{Q}, <)$.*

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Corollaries (B.)

- Every structure in $\mathcal{S}$ has finitely many reducts up to interdefinability.
- $\mathcal{S}$ has countably many structures up to bidefinability.
- Every structure in $\mathcal{S}$ is finitely homogenizable.
- Every structure in $\mathcal{S}$ has a first-order interpretation in $(\mathbb{Q}, <)$.

Theorem (B.)

Let $\mathfrak{A} \in \mathcal{S}$. Then one of the following holds.

1. $u_n(\mathfrak{A})$ *is slower than exponential, and* $u_n(\mathfrak{A}) = u_n(\mathfrak{A}^*)$ *for some* $\mathfrak{A}^* \in \mathcal{M}$. *(c.f. Braunfeld '22).*
2. $c_1\gamma_d^n < u_n(\mathfrak{A}) < c_2(\gamma_d + \varepsilon)^n$ *for some* $2 \le d \in \omega$ *where* γ_d *is the largest real root of the polynomial* $x^d - x^{d-1} - \cdots - 1 = 0$.

$\gamma_2 \approx 1.618$
$\gamma_3 \approx 1.839$
$\gamma_d \nearrow 2$

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Conjecture

There are countably many structures $\mathfrak{A}$ with $u_n(\mathfrak{A}) < c^n$ up to bidefinability.

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What can we say about structures with at most c^n unlabelled growth?

Conjecture (Braunfeld)

$\{\mathfrak{A} : u_n(\mathfrak{A}) < c^n$ for some $c\}$ is closed under ω-categorical unary expansions.

Conjecture

If $u_n(\mathfrak{A}) < c^n$ then $\mathfrak{A}$ is finitely homogenizable.

Conjecture

There are countably many structures $\mathfrak{A}$ with $u_n(\mathfrak{A}) < c^n$ up to bidefinability.

Questions

- What happens with $c = 2$?
- What is the next gap (primitive example is known for $c \approx 2.483$)?