Towards weak bases of minimal relational clones on all finite sets

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Introduction

What are weak bases good for?

- tool for reductions ($\leq$) between various types of computational problems in complexity theory
- obtaining special complexity reductions where other methods fail (e. g., incompatibility with $\exists$) or are too coarse, for example:
  - unique satisfiability
  - surjective satisfiability
  - inverse satisfiability
  - counting problems under parsimonious reductions
  - optimisation problems

weak base

$$P_1 \xrightarrow[\leq]{} P_2$$

'P_1 at most as hard as P_2'

Basic notions

Clones and relational clones

Clone = set of (total) finitary functions F

- closed under composition (substitution) $\mathbf{x} \mapsto f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x}))$
- containing all projection operations $(x_1, \ldots, x_n) \overset{e_i}{\mapsto} x_i$, $1 \leq i \leq n \in \mathbb{N}_+$

$\boxed{\circ, e_i}$

Relational clone = set of finitary relations Q

- containing equality relation $\Delta_A = \{(x, x) \mid x \in A\}$
- closed under pp-definable relations (by a formula $\exists z_1 \cdots z_t \colon \bigwedge_{i=1}^{\ell} \varrho_i(y_{i,1}, \ldots, y_{i,m_i})$)

$\boxed{\exists, \wedge, =}$

Preservation (compatibility)

$$f \rhd \varrho \iff \forall \mathbf{r}_1, \ldots, \mathbf{r}_n \in \varrho\colon \quad f \circ (\mathbf{r}_1, \ldots, \mathbf{r}_n) \in \varrho$$

$Q \mapsto \operatorname{Pol} Q$ (polymorphisms, compatible functions = clone)

$F \mapsto \operatorname{Inv} F$ (invariant (compatible) relations = rel. clone)

Strong partial clones and weak systems with eq.

Strong partial clone = set of partial finitary functions F

- closed under composition (substitution)
- containing all projection operations
- closed under domain restriction: $f \subseteq g \in F \implies f \in F$ $\boxed{\circ, e_i, \restriction}$

Weak system with equality = set of finitary relations Q

- containing equality relation $\Delta_A = \{(x,x) \mid x \in A\}$
- closed under conjunctively definable relations (by a formula $\bigwedge_{i=1}^{\ell} \varrho_i(y_{i,1}, \ldots, y_{i,m_i})$) $\boxed{\wedge, =}$

Preservation (compatibility)

$f \rhd \varrho \iff \forall \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n \in \varrho\colon \quad f \circ (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) \in \varrho$ if defined

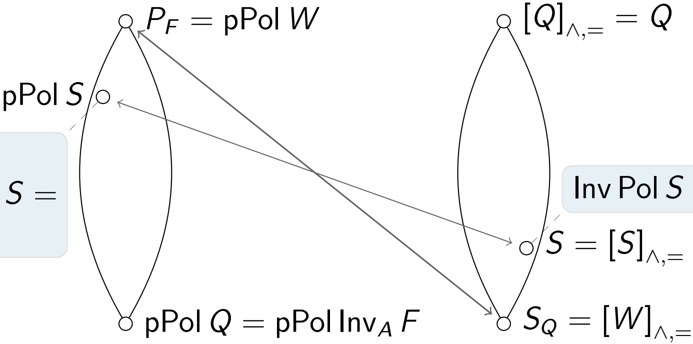
$Q \mapsto \mathrm{pPol}\, Q$ (partial polymorphisms = strong partial clone)

$F \mapsto \mathrm{Inv}\, F$ (invariant relations = weak system with equality)

Weak bases of a relational clone Q / clone F

interval of strong partial clones covering $F = \operatorname{Pol}_A Q \leq \mathrm{O}_A$

weak systems $\ni \Delta_A$ generating $Q = \operatorname{Inv}_A F$



$F = \mathrm{O}_A \cap \operatorname{pPol} S = \operatorname{Pol}_A S$

$\operatorname{Inv} \operatorname{Pol} S = Q$

Weak bases of a relational clone Q / clone F

interval of strong partial clones covering $F = \mathrm{Pol}_A Q \leq \mathrm{O}_A$

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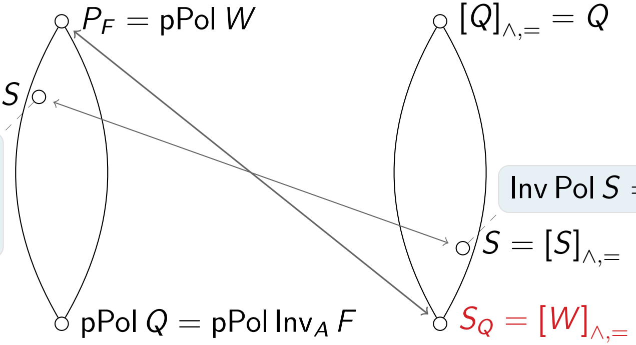

$F = \mathrm{O}_A \cap \mathrm{pPol}\, S = \mathrm{Pol}_A S$

$\mathrm{Inv}\,\mathrm{Pol}\, S = Q$

weak base of Q / F: a finite $W \subseteq Q$ with $[W]_{\wedge,=} = S_Q$ / $\mathrm{pPol}\, W$ the largest strong partial clone P with $\mathrm{O}_A \cap P = F$

Reduced weak base relations

$\varrho \subseteq A^m$ weak base relation $\iff$ $\{\varrho\}$ weak base

Fictitious coordinates

- m-th coordinate fictitious $\iff$ $\exists \tilde{\varrho} \subseteq A^{m-1}$: $\varrho = \tilde{\varrho} \times A$
- ϱ aficitious $\iff$ no fictitious coordinates,
  i.e. $\varrho \neq \tilde{\varrho} \times A$ up to permutation of arguments

Redundant pairs

- $1 \leq i < j \leq m$ redundant pair $\iff$ $\forall \mathbf{x}{=}(x_1, \ldots, x_m) \in \varrho \colon x_i = x_j$
- ϱ irredundant $\iff$ no redundant pairs

Reduced weak base relation $\varrho \subseteq A^m$

- ϱ afictitious (no fictitious coordinates)
- ϱ irredundant (no redundant pairs)
- identification of any coord's $1 \leq i < j \leq m$ in ϱ loses weak base

Tools

n-th graphic of a clone

For $F \subseteq \mathrm{O}_A$, $\varrho \subseteq A^m$, $m \in \mathbb{N}_+$

$\Gamma_F(\varrho)$: the least F-invariant relation containing ϱ, subalg. closure

Given $n \in \mathbb{N}_+$, set $m := |A^n|$; fix a bijection $\beta\colon m = |A^n| \longrightarrow A^n$

n-th graphic of a clone $F \leq \mathrm{O}_A$

representation of n-ary part $F^{(n)}$ as a relation of arity m (value tuples)
$\Gamma_F(\chi_n) = \left\{ f \circ \beta \,\middle|\, f \in F^{(n)} \right\}$

Example: $A = \{0,1,2\}$, $n = 2$, $m = 3^2 = 9$ $F^{(2)} = \{f_1, \ldots, f_s\}$

$$\beta\colon \begin{array}{l} 0 \mapsto x_0 = (0,0) \\ 1 \mapsto x_1 = (0,1) \\ 2 \mapsto x_2 = (0,2) \\ 3 \mapsto x_3 = (1,0) \\ 4 \mapsto x_4 = (1,1) \\ 5 \mapsto x_5 = (1,2) \\ 6 \mapsto x_6 = (2,0) \\ 7 \mapsto x_7 = (2,1) \\ 8 \mapsto x_8 = (2,2) \end{array} \Longrightarrow \Gamma_F(\chi_n) = \left\{ \begin{pmatrix} f_1(x_0) \\ f_1(x_1) \\ f_1(x_2) \\ f_1(x_3) \\ f_1(x_4) \\ f_1(x_5) \\ f_1(x_6) \\ f_1(x_7) \\ f_1(x_8) \end{pmatrix}, \ldots, \begin{pmatrix} f_s(x_0) \\ f_s(x_1) \\ f_s(x_2) \\ f_s(x_3) \\ f_s(x_4) \\ f_s(x_5) \\ f_s(x_6) \\ f_s(x_7) \\ f_s(x_8) \end{pmatrix} \right\}$$

Basic tool

Getting weak bases from sizes of cores

Core of a clone $F \leq \mathrm{O}_A$

$\equiv$ a relation $\varrho \in \mathrm{R}_A$ with $F = \mathrm{Pol}_A\{\Gamma_F(\varrho)\}$ $\quad |\varrho|$: a core size of F

aka a (finite) generating set for a single generator of a relational clone

Basic tool:

Theorem (Schnoor & Schnoor)

clone $F \leq \mathrm{O}_A$ has a core of size $n \in \mathbb{N}_+$

$\implies \Gamma_F(\chi_n)$ weak base relation of F

Will be our starting point!

Main tool

Getting new weak bases from old ones

Main tool:

$W \subseteq \mathrm{R}_A$ weak base of $F \leq \mathrm{O}_A$

$W' \subseteq [W]_{\wedge,=}$ and $\mathrm{Pol}_A\, W' \subseteq F \implies W'$ weak base of F

Note:

- $W' \subseteq [W]_{\wedge,=} \subseteq [W]_{\mathrm{R}_A} = \mathrm{Inv}_A \,\mathrm{Pol}_A\, W$
  $$\implies \mathrm{Pol}_A\, W' \supseteq \mathrm{Pol}_A\, W \stackrel{\text{wb}}{=} F$$
- $\mathrm{Pol}_A\, W' \subseteq F$ ensures that $\mathrm{Pol}_A\, W' = F$, i.e.,
- W' is not too simple (sufficiently rich)

Background tool

Characterisation of maximal clones

maximal clone $\equiv$ co-atom in the clone lattice $\leftrightarrow$ minimal relational clone

Theorem (I. Rosenberg)

$F \leq \mathrm{O}_A$ is maximal iff $\exists \varrho \in \mathrm{R}_A \setminus \mathrm{Inv}_A \mathrm{O}_A$: $\quad F = \mathrm{Pol}_A\{\varrho\}$ and

1. $\varrho = \leq$ partial order with top and bottom
2. $\varrho = s^\bullet = \{(x, s(x)) \mid x \in A\}$ for $s \in \mathrm{Sym}(A)$ with only cycles of prime length p, no fixed points
3. $\varrho = \varrho_{\mathrm{G}} = \{(x, y, u, v) \in A^4 \mid x + y = u + v\}$ for an elementary Abelian p-group $\langle A; +, 0\rangle$, p prime
4. $\Delta_A \subsetneq \varrho \subsetneq A^2$ non-trivial equivalence relation
5. $\varrho \subsetneq A^m$ non-trivial central relation where $1 \leq m < |A|$
6. $\varrho \subsetneq A^h$ h-universal relation where $3 \leq h < |A|$ (i.e., $\exists 1 \leq m \leq \log_h|A| \, \exists$ surj. $\varphi\colon A \twoheadrightarrow h^m$:
$$\varrho = \left\{ \mathbf{a} \in A^h \;\middle|\; \varphi \circ \mathbf{a} \in \eta \right\} \wedge \langle h^m; \eta\rangle = \langle h; \iota_h\rangle^m \wedge$$
$$\iota_h = \left\{ \mathbf{x} \in A^h \;\middle|\; |\mathrm{im}\, \mathbf{x}| < h \right\} \text{)}$$

Towards results

Two sorts of maximal clones

2 Cases for a maximal clone $F \leq \mathrm{O}_A$

$2 \leq k = |A| < \aleph_0$

$F \supseteq \mathrm{O}_A^{(1)}$ $k = 2 \implies F = L$ clone of (affine) linear functions

$k \geq 3 \implies F = U_{k-1} = \mathrm{Pol}_A\{\iota_k\}$ Słupecki's clone

(all non-surjective ops. or ess. permutations)

$F \not\supseteq \mathrm{O}_A^{(1)}$ all other maximal clones

Maximal clones of the first sort (of type 6)

$3 \leq k = |A| < \aleph_0$

$\mathrm{O}_A^{(1)} \subseteq F$, i.e., $F = U_{k-1} = \mathrm{Pol}_A\{\iota_k\}$ Słupecki's clone

$\exists f_1 \neq f_2 \in \mathrm{O}_A^{(1)} \subseteq F$: $\quad \iota_k = \Gamma_{U_{k-1}}(\{f_1 \circ \beta, f_2 \circ \beta\})$

Basic tool (Schnoor & Schnoor):

$\implies \{f_1 \circ \beta, f_2 \circ \beta\}$ is a core $\implies \Gamma_{U_{k-1}}(\chi_2)$ is irr. weak base rel.

Simplification with the main tool

$\iota_k = \{(x_1, \ldots, x_k) \mid (x_1, \ldots, x_k, \ldots, x_k) \in \Gamma_{U_{k-1}}(\chi_2)\} \in [\{\Gamma_{U_{k-1}}(\chi_2)\}]_{\wedge,=}$
and $\mathrm{Pol}_A\{\iota_k\} = F$
$\implies \iota_k$ reduced weak base relation

Maximal clones of the second sort ($F \not\supseteq \mathrm{O}_A^{(1)}$)

$3 \leq k = |A| < \aleph_0$

$\chi_1 = \{\mathrm{id}_A \circ \beta\}$

$U_{k-1} \neq F \leq \mathrm{O}_A$

F has core size 1, thus $\Gamma_F(\chi_1)$ irr. weak base rel. for F

- $F \neq U_{k-1} \implies F^{(1)} \subsetneq \mathrm{O}_A^{(1)}$

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- for $f \in \mathrm{O}_A^{(1)}$:
  $f \in F^{(1)} \iff f \circ \beta \in \Gamma_F(\chi_1) \iff f \in \mathrm{Pol}_A\{\Gamma_F(\chi_1)\}$

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- $\mathrm{Pol}_A^{(1)}\{\Gamma_F(\chi_1)\} = F^{(1)} \subsetneq \mathrm{O}_A^{(1)}$.

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- $\implies F \subseteq \mathrm{Pol}_A\{\mathit{\Gamma}_F(\chi_1)\} \subsetneq \mathrm{O}_A$
- $F = \mathrm{Pol}_A\{\mathit{\Gamma}_F(\chi_1)\}$ by maximality of F

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- $F = \mathrm{Pol}_A\{\mathit{\Gamma}_F(\chi_1)\}$ by maximality of F
- $\chi_1 = \{\mathrm{id}_A \circ \beta\}$ core of F with 1 element

Maximal clones of the second sort ($F \not\supseteq \mathrm{O}_A^{(1)}$)

$3 \leq k = |A| < \aleph_0$

$\chi_1 = \{\mathrm{id}_A \circ \beta\}$

$U_{k-1} \neq F \leq \mathrm{O}_A$

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- $F \neq U_{k-1} \implies F^{(1)} \subsetneq \mathrm{O}_A^{(1)}$
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- $F = \mathrm{Pol}_A\{\Gamma_F(\chi_1)\}$ by maximality of F
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- basic tool (Schnoor & Schnoor): $\Gamma_F(\chi_1)$ weak base relation

Maximal clones of the second sort ($F \not\supseteq \mathrm{O}_A^{(1)}$)

$3 \leq k = |A| < \aleph_0$ $\chi_1 = \{\mathrm{id}_A \circ \beta\}$

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F has core size 1, thus $\Gamma_F(\chi_1)$ irr. weak base rel. for F

- $F \neq U_{k-1} \implies F^{(1)} \subsetneq \mathrm{O}_A^{(1)}$
- for $f \in \mathrm{O}_A^{(1)}$:
  $f \in F^{(1)} \iff f \circ \beta \in \Gamma_F(\chi_1) \iff f \in \mathrm{Pol}_A\{\Gamma_F(\chi_1)\}$
- $\mathrm{Pol}_A^{(1)}\{\Gamma_F(\chi_1)\} = F^{(1)} \subsetneq \mathrm{O}_A^{(1)}$.
- $\implies F \subseteq \mathrm{Pol}_A\{\Gamma_F(\chi_1)\} \subsetneq \mathrm{O}_A$
- $F = \mathrm{Pol}_A\{\Gamma_F(\chi_1)\}$ by maximality of F
- $\chi_1 = \{\mathrm{id}_A \circ \beta\}$ core of F with 1 element
- basic tool (Schnoor & Schnoor): $\Gamma_F(\chi_1)$ weak base relation

Type 3: affine $F = L_{\mathrm{G}}$ for $\mathrm{G} = \langle A; +, 0\rangle$

$\Gamma_{L_{\mathrm{G}}}(\chi_1)$ reduced weak base relation for L_{G}

Further simplification using our main tool

Type 1: bounded orders

$\mathit{\Gamma}_F(\chi_1) \rightsquigarrow \leq$ reduced weak base relation

Type 2: graphs of prime permutations s

$\mathit{\Gamma}_F(\chi_1) \rightsquigarrow \{(a, s(a), s^2(a), \ldots, s^{p-1}(a)) \mid a \in A\}$ red. weak base rel.

Type 4: equivalence relations θ

$\mathit{\Gamma}_F(\chi_1) \rightsquigarrow \theta$ reduced weak base relation

Type 5: central relations $\varrho_a \subsetneq A^m$, $1 \leq m < |A|$

$\mathit{\Gamma}_F(\chi_1) \rightsquigarrow \varrho_a$ reduced weak base relation

Type 6: h-universal relations $\varrho' \subsetneq A^h$, $3 \leq h < |A|$, $\mathrm{Pol}_A\{\varrho'\} \neq U_{k-1}$:

$\mathit{\Gamma}_F(\chi_1) \rightsquigarrow \varrho'$ reduced weak base relation

Example: clone $\mathrm{Pol}_A\{\leq\}$ of monotone operations

$F = \mathrm{Pol}_A\{\leq\}$ with $\forall x \in A\colon 0 \leq x \leq 1$

- $\Gamma_F(\chi_1)$ irredundant weak base relation
- identify arguments:
  $\varrho := \{(x,y) \in A^2 \mid (x,y,\ldots,y) \in \Gamma_F(\chi_1)\} \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$
  (identified indices depend on a suitable choice of β)
- prove: $\varrho = \leq$
- hence: $\leq \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$ and clearly $\mathrm{Pol}_A\{\leq\} = F$
- main tool $\implies$ $W' = \{\leq\}$ weak base

Example: clone $\mathrm{Pol}_A\{\theta\}$ of θ-compatible op's

$F = \mathrm{Pol}_A\{\theta\}$

- $\Gamma_F(\chi_1)$ irredundant weak base relation
- identify arguments:
  $\varrho := \{(x, y) \in A^2 \mid (x, y, \ldots, y) \in \Gamma_F(\chi_1)\} \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$
  (identified indices depend on a suitable choice of β)
- prove: $\varrho = \theta$
- hence: $\theta \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$ and clearly $\mathrm{Pol}_A\{\theta\} = F$
- main tool $\implies$ $W' = \{\theta\}$ weak base

Example: clone $\mathrm{Pol}_A\{s^\bullet\}$ of s-self-dual op's

$F = \mathrm{Pol}_A\{s^\bullet\}$, $s \in \mathrm{Sym}(A)$ with d cycles of length p

- $\Gamma_F(\chi_1)$ irredundant weak base relation
- identify arguments:

$$\varrho := \{(x_1, x_2, \ldots, x_p) \in A^p \mid (x_1, \overset{1}{\ldots}, x_p, x_1, \overset{2}{\ldots}, x_p, \ldots, x_1, \overset{d}{\ldots}, x_p) \in \Gamma_F(\chi_1)\} \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$$

(identified indices depend on a suitable choice of β)

- prove: $\varrho = \{(a, s(a), \ldots, s^{p-1}(a)) \mid a \in A\}$
- hence: $\{(a, s(a), \ldots, s^{p-1}(a)) \mid a \in A\} \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$
  and even $\mathrm{Pol}_A\{\varrho\} = F$
- main tool $\Longrightarrow$ $\{(a, s(a), \ldots, s^{p-1}(a)) \mid a \in A\}$ weak base rel.

Example: clone $\mathrm{Pol}_A\{\varrho_a\}$ of ϱ_a-preserving op's

$F = \mathrm{Pol}_A\{\varrho_a\}$, $\varrho_a \subsetneq A^m$ with central element a

- $\Gamma_F(\chi_1)$ irredundant weak base relation
- identify arguments:
  $$\varrho := \{(x_1,\dots,x_m) \in A^m \mid (x_1,\dots,x_m,x_m,\dots,x_m) \in \Gamma_F(\chi_1)\} \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$$
  (identified indices depend on a suitable choice of β)
- prove: $\varrho = \varrho_a$
- hence: $\varrho_a \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$ and clearly $\mathrm{Pol}_A\{\varrho_a\} = F$
- main tool $\implies$ ϱ_a weak base relation

Example: clone $\mathrm{Pol}_A\{\varrho'\} \neq U_{k-1}$ of ϱ'-preserving op's, not Słupecki's clone

$U_{k-1} \neq F = \mathrm{Pol}_A\{\varrho'\}$, $\varrho' = (\varphi\circ)^{-1}\left[\iota_h^{\otimes m}\right] \subsetneq A^h$ h-universal

- $\Gamma_F(\chi_1)$ irredundant weak base relation
- identify arguments:
  $\varrho := \left\{ (x_1, \ldots, x_h) \in A^h \;\middle|\; (x_1, \ldots, x_h, x_h, \ldots, x_h) \in \Gamma_F(\chi_1) \right\}$
  $\in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$
  (identified indices depend on a suitable choice of β)
- prove: $\varrho = \varrho'$ (using $F \neq U_{k-1}$)
- hence: $\varrho' \in [\{\Gamma_F(\chi_1)\}]_{\wedge,=}$ and clearly $\mathrm{Pol}_A\{\varrho'\} = F$
- main tool $\Longrightarrow$ ϱ' weak base relation

Summary

Theorem (for $3 \leq |A| < \aleph_0$)

$F = \mathrm{Pol}_A\{\varrho\} \leq \mathrm{O}_A$ maximal clone, ϱ a Rosenberg rel.

- F affine linear op's $\implies$ $\Gamma_F(\chi_1)$ reduced weak base rel.
- F s-self-dual op's
  $\implies$ $\{(a, s(a), \ldots, s^{p-1}(a)) \mid a \in A\}$ reduced weak base rel.
- other F $\implies$ ϱ reduced weak base rel.

Example: The set $A = \{0, 1, 2\}$

18 maximal clones have the following reduced weak base relations:

- L clone of affine operations w.r.t. $\langle \mathbb{Z}_3; +, 0 \rangle$:
  $\varrho = \Gamma_L(\chi_1) = \left\{ \begin{smallmatrix} 012001122 \\ 012120201 \\ 012212010 \end{smallmatrix} \right\}$
- $s = (0\,1\,2)$ cyclic shift: $F = \mathrm{Pol}_A\{s^\bullet\}$ self-dual operations
  $\varrho = \left\{ \begin{smallmatrix} 012 \\ 120 \\ 201 \end{smallmatrix} \right\}$
- all other 16 maximal clones $F = \mathrm{Pol}_A\{\varrho\}$:
  ϱ as in Rosenberg's theorem.
  - 3 clones of monotone operations
  - 3 clones of partition preserving operations
  - $3 + 3$ clones of subset preserving operations
  - 3 clones of operations preserving binary central relations
  - 1 clone preserving $\iota_3 = U_2$ (Słupecki's clone)

Final remarks / next steps

- $F = \mathrm{Pol}_A\{\varrho\} = \mathrm{Pol}_A\{\mathit{\Gamma}_F(\chi_n)\}$ with $n \leq 2$ maximal clone ϱ from Rosenberg's theorem
- $\mathrm{Inv}_A\, F = [\{\varrho\}]_{\mathrm{R}_A} = [\{\mathit{\Gamma}_F(\chi_n)\}]_{\mathrm{R}_A}$ minimal relational clone
- any non-trivial $\sigma \in \mathrm{Inv}_A\, F = [\{\mathit{\Gamma}_F(\chi_n)\}]_{\mathrm{R}_A}$ satisfies
  - $\mathrm{Inv}_A\, F = [\{\sigma\}]_{\mathrm{R}_A}$, i.e., $\mathrm{Pol}_A\{\sigma\} = F$
  - $\sigma \in [\{\mathit{\Gamma}_F(\chi_n)\}]_{\exists,\wedge,=}$

  is a potential weak base relation (also ϱ is a candidate), depending on $\sigma \in [\{\mathit{\Gamma}_F(\chi_n)\}]_{\wedge,=}$ (by the main tool)

Future work

complexity perspective:
weak bases for other (e.g., minimal) clones also interesting
also more challenging: finite relatedness problem