

# Completeness and topologizability of semigroups

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- 1 Topologizability of groups and semigroups.
- 2 Completeness and unconditional closedness.
- 3 Results establishing a connection between completeness and nontopologizability of countable groups.
- 4 Results determining inner algebraic conditions which guarantee unconditional closedness of commutative semigroups.

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## Definition

Let  $\mathcal{C}$  be a class of topological semigroups containing all discrete semigroups. A semigroup  $X$  is called

- $\mathcal{C}$ -nontopologizable if the only topology  $\tau$  such that  $(X, \tau) \in \mathcal{C}$  is discrete;
- projectively  $\mathcal{C}$ -nontopologizable if each homomorphic image of  $X$  is  $\mathcal{C}$ -nontopologizable.

By TG we denote the class of all Hausdorff topological groups. Investigation of nontopologizable groups started with the following problem of Markov.

## Problem (Markov)

Does there exist a TG-nontopologizable group?

Despite Markov couldn't find an example of a TG-nontopologizable group, he found an inner characterization of countable TG-nontopologizable groups. However, to state it we need another portion of definitions.

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- A **group polynomial** on a group  $G$  is a function  $f: G \rightarrow G$  of the form  $f(x) = a_0 x^{\epsilon_1} a_1 \cdots x^{\epsilon_n} a_n$  for some elements  $a_0, \dots, a_n \in G$  and  $\epsilon_i \in \{-1, 1\}$ ,  $i \leq n$ ;
- a **semigroup polynomial** on a semigroup  $X$  is a function  $f: X \rightarrow X$  of the form  $f(x) = a_0 x a_1 \cdots x a_n$  for some elements  $a_0, \dots, a_n \in X^1$ .

Nontopologizability of groups and semigroups can be described in terms of corresponding Zariski topologies. For a group  $G$  its

- **group Zariski topology**  $\mathfrak{Z}_G^\pm$  is generated by the subbase consisting of the sets  $\{x \in G : f(x) \neq e_G\}$ , where  $f$  is a group polynomial on  $G$ .

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- **Zariski topology**  $\mathfrak{Z}_X$  is the topology on  $X$  generated by the subbase consisting of the sets  $\{x \in X : f(x) \neq b\}$  and  $\{x \in X : f(x) \neq g(x)\}$  where  $b \in X$  and  $f, g$  are semigroup polynomials on  $X$ ;
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Further by  $T_1S$  ( $T_2S$ , resp.) we denote the classes of  $T_1$  (Hausdorff, resp.) topological semigroups.

The following theorems give an inner characterization of nontopologizability of countable semigroups and groups.

## Theorem (Markov)

A countable group  $G$  is TG-nontopologizable if and only if the group Zariski topology  $\mathfrak{Z}_G^\pm$  is discrete.

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Assuming CH there exists a TG-nontopologizable group.

## Theorem (Hesse)

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# Introduction (Completeness)

In many cases, completeness properties of various objects of General Topology and Topological Algebra can be characterized externally as closedness in ambient objects. For example:

## Fact 1

A metric space  $X$  is complete if and only if  $X$  is closed in any metric space containing  $X$  as a subspace.

## Fact 2

A uniform space  $X$  is complete if and only if  $X$  is closed in any uniform space containing  $X$  as a uniform subspace.

## Fact 3

A Tychonoff space  $X$  is compact if and only if  $X$  is closed in any Tychonoff space containing  $X$  as a subspace.

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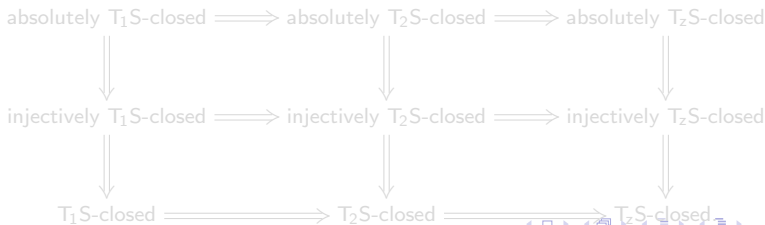
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- $\mathcal{C}$ -closed if for any isomorphism  $h: X \rightarrow Y$  to a topological semigroup  $Y \in \mathcal{C}$  such that  $h[X]$  is discrete, the image  $h[X]$  is closed in  $Y$ ;
- injectively  $\mathcal{C}$ -closed if for any injective homomorphism  $i: X \rightarrow Y$  to a topological semigroup  $Y \in \mathcal{C}$  the image  $i[X]$  is closed in  $Y$ ;
- absolutely  $\mathcal{C}$ -closed if for any homomorphism  $h: X \rightarrow Y$  to a topological semigroup  $Y \in \mathcal{C}$  the image  $h[X]$  is closed in  $Y$ .

We consider the classes:

- $T_zS$  of Tychonoff zero-dimensional topological semigroups;
- $T_2S$  of Hausdorff topological semigroups;
- $T_1S$  of  $T_1$  topological semigroups.

Since  $T_zS \subseteq T_2S \subseteq T_1S$ , the following implications hold:



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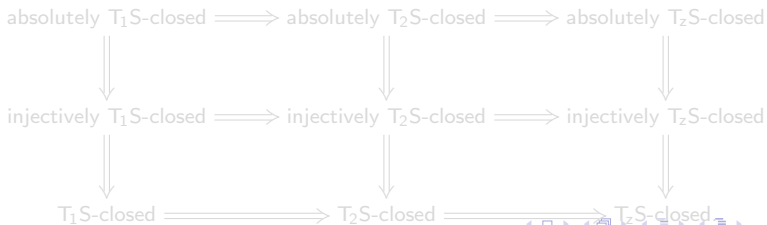
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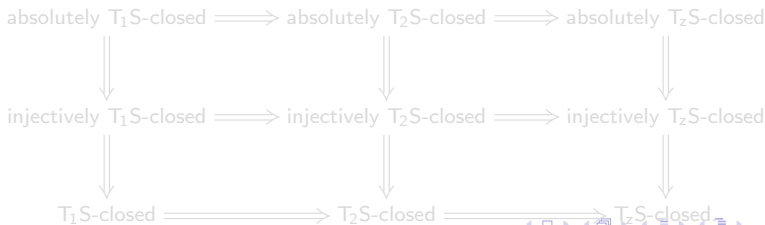
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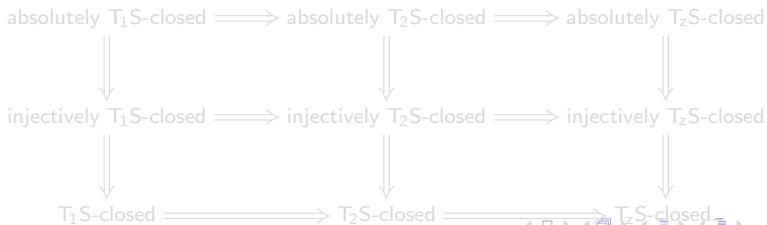
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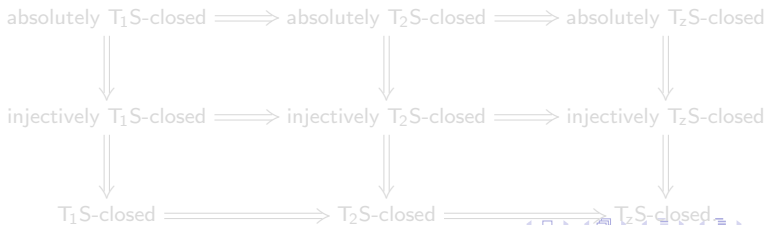
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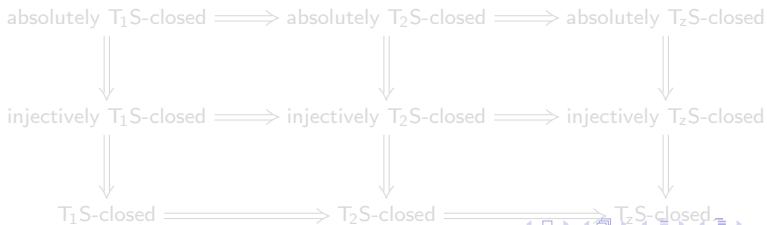
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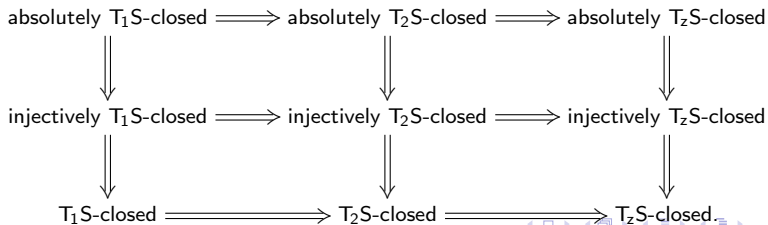
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The next results reveal a connection between nontopologizability and completeness of countable groups.

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- **periodic** if for every  $x \in X$  there exists  $n \in \mathbb{N}$  such that  $x^n$  is an idempotent;
- **bounded** if there exists  $n \in \mathbb{N}$  such that  $x^n$  is an idempotent for every  $x \in X$ ;
- **group-finite** if every subgroup of  $X$  is finite;
- **group-bounded** if every subgroup of  $X$  is bounded.

Each semilattice  $E$  carries a natural partial order defined by  $a \leq b$  iff  $ab = a$ .

## Definition

A semilattice  $E$  is called **chain-finite** if all chains in  $(E, \leq)$  are finite.

For a semigroup  $X$  by  $E(X)$  we denote the set of all idempotents of  $X$ . If  $X$  is commutative, then  $E(X)$  is a semilattice.

The following result characterizes commutative  $\mathcal{C}$ -closed semigroups.

## Theorem (Banakh, B.)

For a commutative semigroup  $X$  the following conditions are equivalent:

- $X$  is  $T_2S$ -closed;
- $X$  is  $T_1S$ -closed;
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For every class  $\mathcal{C}$  such that  $T_2S \subseteq \mathcal{C} \subseteq T_1S$ , every subsemigroup of a  $\mathcal{C}$ -closed commutative semigroup is  $\mathcal{C}$ -closed.

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A commutative semigroup  $X$  is absolutely  $T_1S$ -closed iff  $X$  is finite.

Remark

Commutativity is essential in previous results. Namely, the mentioned before group of Klyachko, Olshanskii and Osin is absolutely  $T_1S$ -closed.



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Thank You for attention!

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