# Completeness and topologizability of semigroups

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# Markov Problem

**Definition**

Let $\mathcal{C}$ be a class of topological semigroups containing all discrete semigroups. A semigroup $X$ is called

- $\mathcal{C}$-nontopologizable if the only topology $\tau$ such that $(X, \tau) \in \mathcal{C}$ is discrete;
- projectively $\mathcal{C}$-nontopologizable if each homomorphic image of $X$ is $\mathcal{C}$-nontopologizable.

By TG we denote the class of all Hausdorff topological groups. Investigation of nontopologizable groups started with the following problem of Markov.

**Problem (Markov)**

Does there exist a TG-nontopologizable group?

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- A group polynomial on a group $G$ is a function $f\colon G \to G$ of the form $f(x) = a_0 x^{\epsilon_1} a_1 \cdots x^{\epsilon_n} a_n$ for some elements $a_0, \dots, a_n \in G$ and $\epsilon_i \in \{-1, 1\}$, $i \leq n$;
- a semigroup polynomial on a semigroup $X$ is a function $f\colon X \to X$ of the form $f(x) = a_0 x a_1 \cdots x a_n$ for some elements $a_0, \dots, a_n \in X^1$.

Nontopologizability of groups and semigroups can be described in terms of corresponding Zariski topologies. For a group $G$ its

- group Zariski topology $\mathfrak{Z}_G^{\pm}$ is generated by the subbase consisting of the sets $\{x \in G : f(x) \neq e_G\}$, where $f$ is a group polynomial on $G$.

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Further by T$_1$S (T$_2$S, resp.) we denote the classes of $T_1$ (Hausdorff, resp.) topological semigroups.

The following theorems give an inner characterization of nontopologizability of countable semigroups and groups.

**Theorem (Markov)**

A countable group $G$ is TG-nontopologizable if and only if the group Zariski topology $\mathfrak{Z}_G^{\pm}$ is discrete.

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The following results solve Markov problem.

**Theorem (Shelah)**

Assuming CH there exists a TG-nontopologizable group.

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There exists a TG-nontopologizable group.

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# Introduction (Completeness)

In many cases, completeness properties of various objects of General Topology and Topological Algebra can be characterized externally as closedness in ambient objects. For example:

**Fact 1**

A metric space $X$ is complete if and only if $X$ is closed in any metric space containing $X$ as a subspace.

**Fact 2**

A uniform space $X$ is complete if and only if $X$ is closed in any uniform space containing $X$ as a uniform subspace.

**Fact 3**

A Tychonoff space $X$ is compact if and only if $X$ is closed in any Tychonoff space containing $X$ as a subspace.

**Fact 4**

A topological group $G$ is Raikov complete if and only if $G$ is closed in any topological group containing $G$ both as a subgroup and a subspace.

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- $\mathcal{C}$-closed if for any isomorphism $h\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ such that $h[X]$ is discrete, the image $h[X]$ is closed in $Y$;
- injectively $\mathcal{C}$-closed if for any injective homomorphism $i\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $i[X]$ is closed in $Y$;
- absolutely $\mathcal{C}$-closed if for any homomorphism $h\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ the image $h[X]$ is closed in $Y$.

We consider the classes:

- $\mathsf{T_zS}$ of Tychonoff zero-dimensional topological semigroups;
- $\mathsf{T_2S}$ of Hausdorff topological semigroups;
- $\mathsf{T_1S}$ of $T_1$ topological semigroups.

Since $\mathsf{T_zS} \subseteq \mathsf{T_2S} \subseteq \mathsf{T_1S}$, the following implications hold:

$$\begin{tikzcd}
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Let $\mathcal{C}$ be a class of topological semigroups. A discrete semigroup $X$ is called

- $\mathcal{C}$-closed if for any isomorphism $h\colon X \to Y$ to a topological semigroup $Y \in \mathcal{C}$ such that $h[X]$ is discrete, the image $h[X]$ is closed in $Y$;
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# Results

The next results reveal a connection between nontopologizability and completeness of countable groups.

**Theorem (Banakh, B.)**

For a countable group $G$ the following conditions are equivalent:

- $G$ is injectively $\mathsf{T}_1$S-closed;
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Each countable bounded simple TG-nontopologizable group $G$ is absolutely $\mathsf{T}_1$S-closed. In particular, so is the mentioned before group of Klyachko, Olshanskii and Osin.

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A semigroup $X$ is called

- singular if there exists an infinite subset $A \subset X$ such that $AA$ is a singleton;
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- group-finite if every subgroup of $X$ is finite;
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Each semilattice $E$ carries a natural partial order defined by $a \leq b$ iff $ab = a$.

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- bounded if there exists $n \in \mathbb{N}$ such that $x^n$ is an idempotent for every $x \in X$;
- group-finite if every subgroup of $X$ is finite;
- group-bounded if every subgroup of $X$ is bounded.

Each semilattice $E$ carries a natural partial order defined by $a \leq b$ iff $ab = a$.

### Definition

A semilattice $E$ is called chain-finite if all chains in $(E, \leq)$ are finite.

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For a semigroup $X$ by $E(X)$ we denote the set of all idempotents of $X$. If $X$ is commutative, then $E(X)$ is a semilattice.

The following result characterizes commutative $\mathcal{C}$-closed semigroups.

**Theorem (Banakh, B.)**

For a commutative semigroup $X$ the following conditions are equivalent:

- $X$ is $\mathsf{T_zS}$-closed;
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Thank You for attention!

# Papers

- T. Banakh, S. Bardyla: "Categorically closed countable semigroups", Forum Mathematicum, **35**:3 (2023), 689-711.
- T. Banakh, S. Bardyla: "Characterizing categorically closed commutative semigroups", Journal of Algebra **591** (2022), 84–110.
- T. Banakh, S. Bardyla: "Characterizing chain-compact and chain-finite topological semilattices", Semigroup Forum, **98** (2019), no.2, 234–250.
- T. Banakh, S. Bardyla: "Absolutely closed semigroups", (submitted) arXiv:2207.12778.