Classes of groups in lattice framework

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We are studying groups by analyzing their lattices of weak congruences. For an algebra $\mathcal{A}$ these are congruences on subalgebras considered as relations on $\mathcal{A}$. When the algebra is a group G, the algebraic lattice $\mathsf{Wcon}(G)$ encompasses the lattices of subgroups and normal subgroups for every subgroup of G, up to isomorphism.

Subgroup lattices have been used to characterize various classes of groups to some extent, but weak congruence lattices provide much more information. We have obtained characterizations in this context, which are discussed in the papers listed here.

In this presentation, we expand our framework by incorporating systems of subgroups into weak congruence lattices. This advancement enhances our understanding of Kurosh-Chernikov classes of groups and offers a new characterization within this setting.

Furthermore, we address the Algebra of Group Theoretical Classes, which refers to group theoretical properties and the corresponding closure operations on these classes. We demonstrate that these closure operations can be expressed in terms of lattices.

As a consequence, we are able to formulate results of the following type.

A group G belongs to the class $\mathfrak{P}$ if and only if the lattice $\mathsf{Wcon}(G)$ satisfies the lattice theoretic properties $\mathsf{L}_{\mathfrak{P}}$.

If the above holds, we say that $\mathfrak{P}$ is an **L-class** of groups.

Theorem. *Let $\mathfrak{P}$ be an L-class of groups. A group G is a residually $\mathfrak{P}$-group (it belongs to the class $\mathbf{R}\mathfrak{P}$) if and only if the lattice $\mathsf{Wcon}(G)$ fulfils:*

($*$) For each $\Delta_X \in \mathsf{C}(\downarrow\Delta)$, $\Delta_X \neq \{(e,e)\}$, there is $\Delta_N \blacktriangleleft \Delta$, such that $\Delta_N \wedge \Delta_X < \Delta_X$ and the interval $[N^2, G^2]$, as the lattice with normal elements determined by $N^2 \vee \Delta$, satisfies the lattice theoretic properties $\mathsf{L}_{\mathfrak{P}}$.

We prove that most of the known classes of groups are L-classes. Conversely, we start with a lattice property and analyze (algebraic properties of) the corresponding L-class of groups.

Finally, we prove that Birkhoff's theorem for L-classes of groups can be formulated with purely lattice-theoretic arguments.

Theorem. *An* L*-class* $\mathfrak{P}$ *of groups is a variety if and only if the following hold:*

(i) *if* G *is a* $\mathfrak{P}$*-group, then in the lattice* $\mathsf{Wcon}(G)$ *for every* $\Delta_H \in \downarrow\Delta$*, such that* $\Delta_H \blacktriangleleft \Delta$ *(normal in lattice sense), the interval* $[H^2, G^2]$*, which is a lattice with normal element determined by* $H^2 \vee \Delta$*, satisfies* $\mathsf{L}_{\mathfrak{P}}$*, i.e., the lattice properties determining the class* $\mathfrak{P}$ *and*

(ii) *f* G *is a group, such that the lattice* $\mathsf{Wcon}(G)$ *satisfies* $(*)$*, then* G *belongs to* $\mathfrak{P}$*.*

References

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