A dagger kernel category of orthomodular lattices

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Dagger kernel categories have been introduced in [HeJa] as a simple setting in which one can study categorical quantum logic. The present paper continues the study of dagger kernel categories in relation to orthomodular lattices in the spirit of [Jac].

In particular, we show that the category of orthomodular lattices **OMLatLin** where morphisms are mappings having adjoints is a dagger kernel category. We describe finite dagger biproducts and free objects over finite sets in **OMLatLin**.

A meet semi-lattice $(X, \wedge\ 1)$ is called an *ortholattice* if it comes equipped with a function $(-)^{\perp}\colon X \to X$ satisfying:

- $x^{\perp\perp} = x$;
- $x \leq y$ implies $y^{\perp} \leq x^{\perp}$;
- $x \wedge x^{\perp} = 1^{\perp}$.

One can then define a bottom element as $0 = 1 \wedge 1^{\perp} = 1^{\perp}$ and join by $x \vee y = (x^{\perp} \wedge y^{\perp})^{\perp}$, satisfying $x \vee x^{\perp} = 1$. We write $x \perp y$ if and only if $x \leq y^{\perp}$.

Such an ortholattice is called *orthomodular* if $x \leq y$ implies $y = x \vee (x^{\perp} \wedge y)$.

Definition 1. A *dagger* on a category $\mathcal{C}$ is a functor ${}^{\star}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ that is involutive and the identity on objects. A category equipped with a dagger is called a *dagger category.*

Let $\mathcal{C}$ be a dagger category. A morphism $f\colon A \to B$ is called a *dagger monomorphism* if $f^{\star} \circ f = \mathrm{id}_A$, and f is called a *dagger isomorphism* if $f^{\star} \circ f = \mathrm{id}_A$ and $f \circ f^{\star} = \mathrm{id}_B$.

We now introduce a new way of organising orthomodular lattices into a dagger category.

Definition 2. The category **OMLatLin** has orthomodular lattices as objects. A morphism $f\colon X \to Y$ in **OMLatLin** is a function $f\colon X \to Y$ between the underlying sets such that there is a function $h\colon Y \to X$ and, for any $x \in X$ and $y \in Y$,

$$f(x) \perp y \text{ if and only if } x \perp h(y).$$

We say that h is an *adjoint* of a *linear map* f. It is clear that adjointness is a symmetric property: if a map f possesses an adjoint h, then f is also an adjoint of h.

Moreover, a map $f\colon X \to X$ is called *self-adjoint* if f is an adjoint of itself.

The identity morphism on X is the self-adjoint identity map $\mathrm{id}\colon X \to X$. Composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ is given by usual composition of maps.

Lemma 3. *Let $f\colon X \to Y$ be a morphism of orthomodular lattices. Then* $\downarrow f^*(1)^\perp = \{x \in X : f(x) = 0\}$ *is an orthomodular lattice.*

OMLatLin has a *zero object* $\underline{0}$; this means that there is, for any orthomodular lattice X, a unique morphism $\underline{0} \to X$ and hence also a unique morphism $X \to \underline{0}$. The zero object $\underline{0}$ will be one-element orthomodular lattice $\{0\}$.

For objects X and Y, we denote by $0_{X,Y} = X \to \underline{0} \to Y$ the morphism uniquely factoring through $\underline{0}$.

Definition 4. For a morphism $f\colon A \to B$ in a dagger category with zero morphisms, we say that a morphism $k\colon K \to A$ is a *weak dagger kernel* of f if $fk = 0_{K,B}$, and if $m\colon M \to A$ satisfies $fm = 0_{M,B}$ then $kk^*m = m$.

A *dagger kernel category* is a dagger category with a zero object, hence zero morphisms, where each morphism f has a weak dagger kernel k (called *dagger kernel*) that additionally satisfies $k^*k = 1_K$.

Theorem 5. *The category* **OMLatLin** *is a dagger kernel category. The dagger kernel of a morphism $f\colon X \to Y$ is $k\colon \downarrow k \to X$, where $k = f^*(1)^\perp \in X$.*

Corollary 6. *Every morphism $f\colon X \to Y$ in* **OMLatLin** *has a factorisation me where $m = f(1)\colon \downarrow f(1) \to Y$ and $e = f|^{\downarrow f(1)}\colon X \to \downarrow f(1)$.*

By a *dagger biproduct* of objects A, B in a dagger category $\mathcal{C}$ with a zero object, we mean a coproduct $A \xrightarrow{\iota_A} A \oplus B \xleftarrow{\iota_B} B$ such that ι_A, ι_B are dagger monomorphisms and $\iota_B{}^\star \circ \iota_A = 0_{A,B}$. The dagger biproduct of an arbitrary set of objects is defined in the expected way.

Proposition 7. *The category* **OMLatLin** *has arbitrary finite dagger biproducts $\bigoplus$. Explicitly, $\bigoplus_{i\in I} X_i$ is the cartesian product of orthomodular lattices X_i, $i \in I$, I finite.*

The coprojections $\kappa_j\colon X_j \to \bigoplus_{i\in I} X_i$ are defined by $(\kappa_j)(x) = x_{j=}$ with $x_{j=}(i) = \begin{cases} x & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$ *and $(\kappa_j)^*((x_i)_{i\in I}) = x_j$. The dual product structure is given by $p_j = (\kappa_j)^*$.*

Proposition 8. *A free object on a finite set A in* **OMLatLin** *is isomorphic to the finite Boolean algebra $\mathcal{P}A$.*

References

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