On Congruences of Weakly Dicomplemented Lattices*

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Weakly dicomplemented lattices arise as abstractions of concept algebras, introduced by Rudolf Wille when modelling negation on concept lattices [2]; they are algebras $(L,\wedge,\vee,^{\Delta},^{\nabla},0,1)$ of type $(2,2,2,2,0,0)$ formed of bounded lattices $(L,\wedge,\vee,0,1)$ endowed with two unary operations: $^{\Delta}$, called *weak complementation*, and $^{\nabla}$, called *dual weak complementation*, together forming the *weak dicomplementation* $(^{\Delta},^{\nabla})$, both of which are order--reversing and that also satisfy, for all $x,y\in L$: $x^{\Delta\Delta}\leq x\leq x^{\nabla\nabla}$ and $(x\wedge y)\vee(x\wedge y^{\Delta})=x=(x\vee y)\wedge(x\vee y^{\nabla})$, where $\leq$ is the lattice order of L. Their bounded lattice reducts endowed with the weak complementation, $(L,\wedge,\vee,^{\Delta},0,1)$, are called *weakly complemented lattices*, and their bounded lattice reducts endowed with the dual weak complementation, $(L,\wedge,\vee,^{\nabla},0,1)$, are called *dual weakly complemented lattices*.

For instance, any Boolean algebra is both a weakly complemented lattice and a dual weakly complemented lattice. Furthermore, any bounded lattice can be endowed with the *trivial weak dicomplementation*, formed of the *trivial weak complementation*, that sends 1 to 0 and all other elements to 1, and the *trivial dual weak complementation*, that sends 0 to 1 and all other elements to 0.

A *context* is a triple (J,M,I), where J and M are sets and $I\subseteq J\times M$. A *subcontext* of (J,M,I) is a context $(H,N,I\cap(H\times N))$, with $H\subseteq J$ and $N\subseteq M$. For every $(A,B)\in\mathcal{P}(J)\times\mathcal{P}(M)$, we denote by: $A'=\{m\in M:(\forall\, a\in A)\,(a\,I\,m)\}$ and $B'=\{j\in J:(\forall\, b\in B)\,(j\,I\,b)\}$. The *concept algebra associated to the context* (J,M,I) is the weakly dicomplemented (complete) lattice $(\mathcal{B}(J,M,I),\wedge,\vee,^{\Delta},^{\nabla},0,1)$, where: $\mathcal{B}(J,M,I)=\{(A,B)\in\mathcal{P}(J)\times\mathcal{P}(M):A'=B,B'=A\}$ is the set of the *formal concepts* associated to the context (J,M,I), $\wedge$ and $\vee$ are the lattice operations corresponding to the order $\subseteq\times\supseteq$ of $\mathcal{P}(J)\times\mathcal{P}(M)$ restricted to $\mathcal{B}(J,M,I)$, $0=(\emptyset'',M)$, $1=(J,J')$ and, for any $(A,B)\in\mathcal{B}(J,M,I)$, $(A,B)^{\Delta}=((J\setminus A)'',(J\setminus A)')$ and $(A,B)^{\nabla}=((M\setminus B)',(M\setminus B)'')$. A subcontext (H,N,E) of (J,M,I) is said to be *compatible* iff the map $\Pi_{J,M,H,N}:\mathcal{B}(J,M,I)\to\mathcal{B}(H,N,E)$, $\Pi_{J,M,H,N}(A,B)=(A\cap H,B\cap N)$ for all $(A,B)\in\mathcal{B}(J,M,I)$, is well defined,

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case in which this map is a surjective weakly dicomplemented lattice morphism.

Whenever J is a join--dense subset and M is a meet--dense subset of a complete lattice L, L can be endowed with the weak dicomplementation $({}^{\Delta J}, {}^{\nabla M})$ defined by $x^{\Delta J} = \bigvee(J \setminus (x]_L)$ and $x^{\nabla M} = \bigwedge(M \setminus [x)_L)$ for all $x \in L$, and the map $\varphi_{L,J,M} : L \to \mathcal{B}(J, M, \leq)$, defined by $\varphi_{L,J,M}(x) = (J \cap (x]_L, M \cap [x)_L)$ for all $x \in L$, is a weakly dicomplemented lattice isomorphism. In this case, it follows that, for any compatible subcontext $(H, N, \leq)$ of the context $(J, M, \leq)$, $\Pi_{J,M,H,N} \circ \varphi_{L,J,M} : L \to \mathcal{B}(H, N, \leq)$ is a surjective weakly dicomplemented lattice morphism, hence its kernel, that we denote by $\zeta_{L,H,N}$, is a lattice congruence of L that preserves the weak dicomplementation $({}^{\Delta J}, {}^{\nabla M})$, and the weakly dicomplemented lattices $L/\zeta_{L,H,N}$ and $\mathcal{B}(H, N, \leq)$ are isomorphic; we call $\zeta_{L,H,N}$ the *weakly dicomplemented lattice congruence induced by the compatible subcontext* $(H, N, \leq)$ *of* $(J, M, \leq)$. We say that a weak dicomplementation $({}^{\Delta}, {}^{\nabla})$ on L is *representable* iff ${}^{\Delta} = {}^{\Delta J}$ and ${}^{\nabla} = {}^{\nabla M}$ for some join--dense subset J and some meet--dense subset M of L.

In the paper [1], we study the existence of nontrivial and of representable (dual) weak complementations, along with the lattice congruences that preserve them, in different constructions of bounded lattices, then use this study to determine the finite (dual) weakly complemented lattices with the largest numbers of congruences, along with the structures of their congruence lattices. It turns out that, if $n \geq 7$ is a natural number, then the four largest numbers of congruences of the n--element (dual) weakly complemented lattices are: $2^{n-2} + 1$, $2^{n-3} + 1$, $5 \cdot 2^{n-6} + 1$ and $2^{n-4} + 1$, which yields the fact that, for any $n \geq 5$, the largest and second largest numbers of congruences of the n--element weakly dicomplemented lattices are $2^{n-3} + 1$ and $2^{n-4} + 1$. For smaller numbers of elements, several intermediate numbers of congruences appear between the elements of these sequences. While already published, this research has never been presented at a conference before.

In the same purely lattice--theoretical manner, we study compatible subcontexts and the congruences they induce in various types and constructions of lattices. Out of this part of our ongoing research, I will do my best to select the most interesting results that I can fit into my talk.

Keywords: congruence, (glued/ordinal, horizontal) sum of bounded lattices, (nontrivial, representable) (dual) weak (di)complementation, compatible subcontext.

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