The (in)comparability orthoset of a finite poset

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In his PhD. thesis [1], Dacey explored the notion of "abstract orthogonality", by means of sets equipped with a symmetric, irreflexive relation $\perp$. He named these structures *orthogonality spaces*, nowadays called *orthosets*. Every orthoset has an orthocomplementation operator $X \mapsto X^{\perp}$ defined on the set of all its subsets. Dacey proved that $X \mapsto X^{\perp\perp}$ is a closure operator and that the set of all closed subsets of an orthoset forms a complete ortholattice, which we call *the logic of an orthoset*. Moreover, Dacey gave a characterization of orthosets such that their logic is an orthomodular lattice. The orthosets of this type are nowadays called *Dacey spaces*.

In [3], we constructed an orthoset $(Q^{+}(P), \perp)$ from every poset P. The elements of $Q^{+}(P)$ are pairs (a, b) of elements of P with $a < b$, which we called *quotients*.

Theorem 1. *[3, Theorem 4.10] For every finite bounded poset P, P is a lattice if and only if $Q^{+}(P)$ is Dacey.*

We are continuing this line of research in a natural direction. To every poset P we associate two orthosets on the underlying set of P, namely *the strict comparability orthoset* $C(P)$ and *the incomparability orthoset* $I(P)$.

In $C(P)$, two elements $a, b \in P$ are orthogonal iff $a > b$ or $b > a$. In $I(P)$, $a, b \in P$ are orthogonal iff a, b are incomparable.

Theorem 2. *Let P be a finite poset. Then the incomparability orthoset of P is Dacey if and only if P is* N-free *in the sense of [2]*

Similarly, the characterization of finite posets for which their strict comparability orthoset is Dacey can be given in terms of their order structure.

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References

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