STU SvF

Department of Mathematics and Descriptive Geometry

BASIC HYDROLOGICAL MODELING IN SOFTWARE NATURASAT

Bc. Ivana Piačková

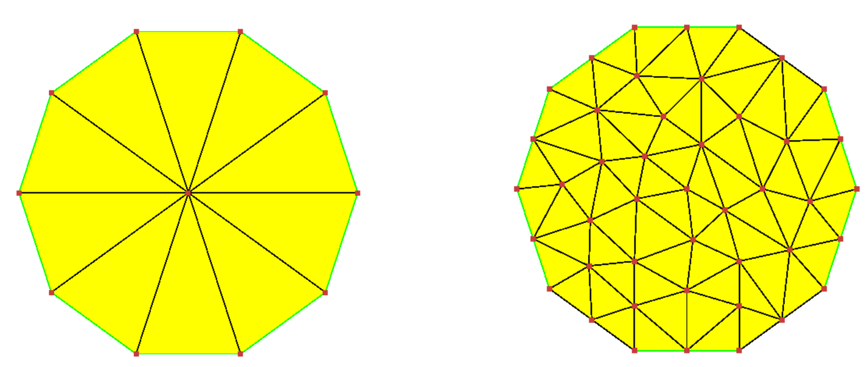
Motivation

Wetlands belong to the most productive and biodiverse ecosystems in the world. However, they are disappearing at an alarming rate. The solution is to analyse wetland conditions and, if needed, apply restoration mechanisms.

Our goal is to help in the analysis by implementing basic hydrological modeling into the environmental software NaturaSat. We focus on one of wetland's conditions – water level – which we determine by solving the Laplace equation on a computational domain bounded by distributaries. Dirichlet boundary conditions are specified on the boundary. The Laplace equation is solved using the complementary volume method on an irregular triangular mesh.

Triangulation

The computational domain is defined by a segmentation polygon, obtained through semi-automatic or automatic segmentation in the NaturaSat software. To create a triangular mesh of the domain, we use Delaunay triangulation via the CGAL library because its good properties guarantee efficiency and easy construction of co-volumes. For denser triangular grid, we adjust the size criterion S for the triangles of the triangulation using the mesh refinement method.



a) Default triangulation *b) Mesh with S = 0.5*

Triangulation of a discretized unit circle and its mesh with a specified size criterion

Complementary volume method for the Laplace equation

The boundary problem consisting of the Laplace equation and Dirichlet boundary conditions can be formulated as:

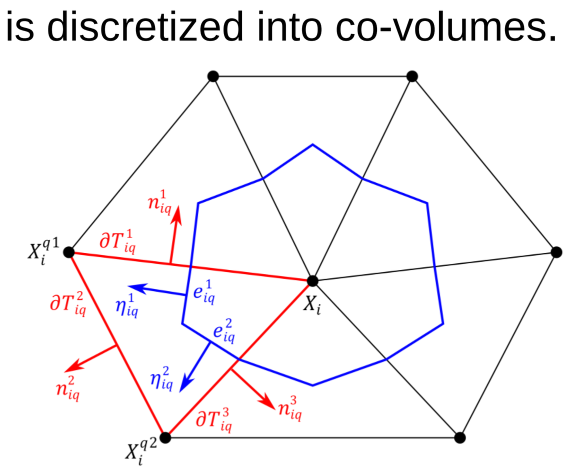
$$\Delta u(x) = 0 \quad x \in \Omega$$

$$u(x) = u_0(x) \quad x \in \partial\Omega$$

It can be solved using complementary volume method.

Spatial discretization

The computational domain is discretized into co-volumes. For each node X_i of the triangular grid, we define a co-volume V_i as a polygon with vertices at the centers of mass of the surrounding triangles T_{iq}, $q = 1, \dots, Q_i$ and at the midpoints of the triangles' edges running from the vertex X_i.



The co-volume V_i with highlighted geometric features corresponding to the triangle T_{iq}

Numerical scheme

Deriving from the Laplace equation, we get a numerical scheme for a co-volume V_i as:

$$\sum_{q=1}^{Q_i} \sum_{j=1}^{2} m\left(e_{iq}^j\right) \vec{P}_{T_{iq}} \cdot \vec{\eta}_{iq}^j = 0\,,$$

where the vector $\vec{P}_{T_{iq}}$ represents an approximation of Δu on the triangle T_{iq} and is computed using a formula:

$$\vec{P}_{T_{iq}} = \frac{1}{m(T_{iq})}\left(\frac{u_i+u_{q1}}{2} d_{iq}^1 \vec{n}_{iq}^1 + \frac{u_{q1}+u_{q2}}{2} d_{iq}^2 \vec{n}_{iq}^2 + \frac{u_i+u_{q2}}{2} d_{iq}^3 \vec{n}_{iq}^3\right).$$

For $i = 1, \dots, N$ we get a system of linear equations. The system matrix has non-zero diagonal coefficient a_i:

$$a_i = \sum_{q=1}^{Q_i} \frac{1}{2m(T_{iq})} \sum_{j=1}^{2} m\left(e_{iq}^j\right) \vec{\eta}_{iq}^j \cdot \left(d_{iq}^1 \vec{n}_{iq}^1 + d_{iq}^3 \vec{n}_{iq}^3\right)$$

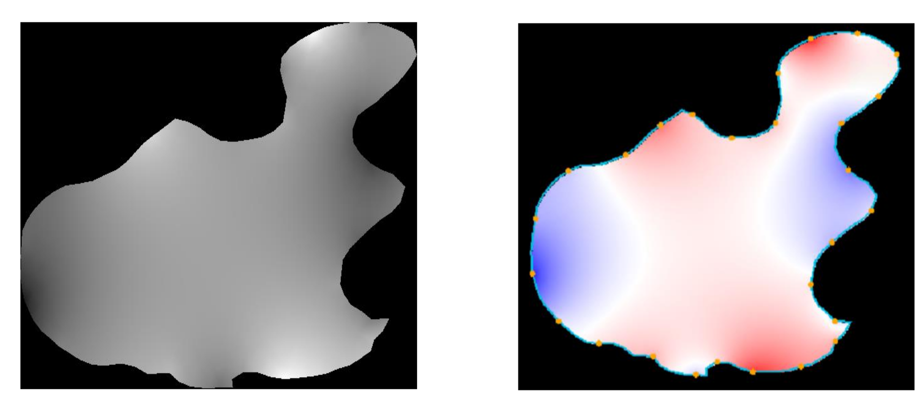
and off-diagonal coefficients a_{iq} corresponding to the surrounding co-volumes:

$$a_{iq} = \frac{1}{2m(T_{iq})} \sum_{j=1}^{2} m\left(e_{iq}^j\right) \vec{\eta}_{iq}^j \cdot \left(d_{iq}^1 \vec{n}_{iq}^1 + d_{iq}^2 \vec{n}_{iq}^2\right) + \frac{1}{2m(T_{ik})} \sum_{j=1}^{2} m\left(e_{ik}^j\right) \vec{\eta}_{ik}^j \cdot \left(d_{ik}^3 \vec{n}_{ik}^3 + d_{ik}^2 \vec{n}_{ik}^2\right)$$

After including the boundary conditions, we solve the system using BiCGSTAB method in EIGEN library.

Visualization

To visualize the solution, we use an interpolation onto a regular grid. Firstly, we define a regular grid that covers the entire computational region. Then, for each pixel of the grid lying inside one of the triangles of the triangular grid, we interpolate the solution at the triangle's vertices to this point using the barycentric interpolation. This leads to a grayscale image, which can be converted into a colour image by applying a linear colour gradient.



The visualization of the solution

Numerical experiments

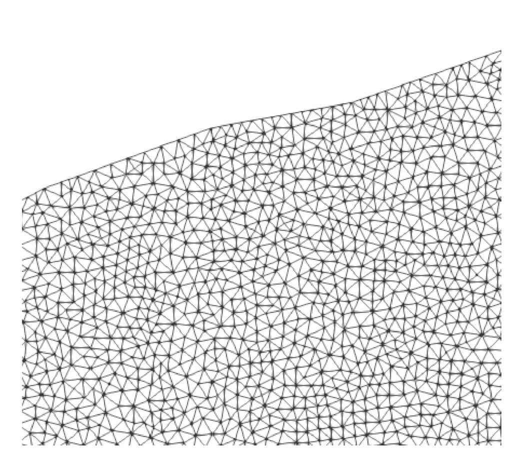
We tested the implementation on both synthetic and real experiments. Initially, we applied it to a specific Poisson equation with a known exact solution to determine the experimental order of convergence (EOC). By computing the L^2 norm error of the numerical solution while refining the triangular grid, we found that the method is of the second order of the convergence.

n	Error	EOC
2	0.00195312000	
4	0.00045168300	2.112400
8	0.00011233800	2.007460
16	0.00002807700	2.000390
32	0.00000701919	2.000010
64	0.00000175480	1.999998

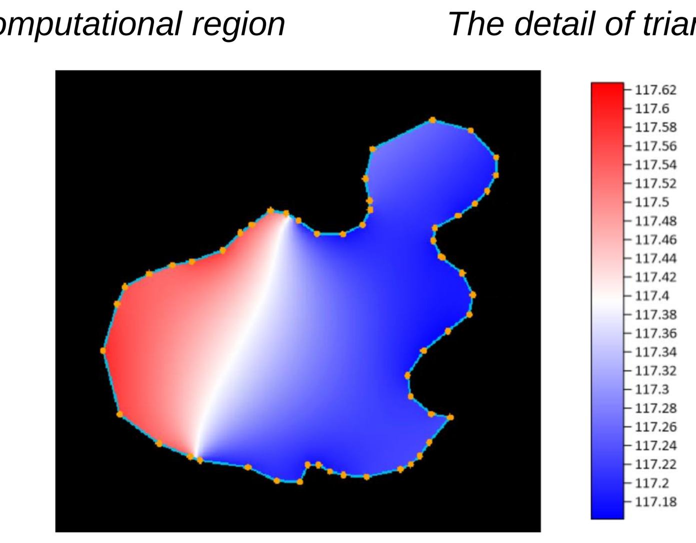
Table of L^2 norm errors and EOCs for progressively denser grid

In real experiments, we focused on a specific region located in CHKO Dunajské luhy near the municipality of Bodíky. This region is bounded by two distributaries. The boundary conditions are acquired from the Digital terrain model (DTM) provided by Geodetický a kartografický ústav Bratislava.

The computational region



The detail of triangulation

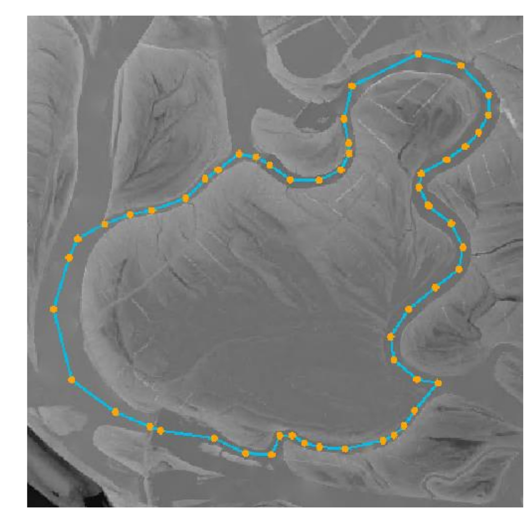


The solution

The solution is in the range [117.161, 117.628] m. The highest water level occurs in the west, while the lowest in the east of the region. The abrupt change in water level is caused by the presence of weirs in the distributaries, which lowers the water level by a few tens of centimeters.

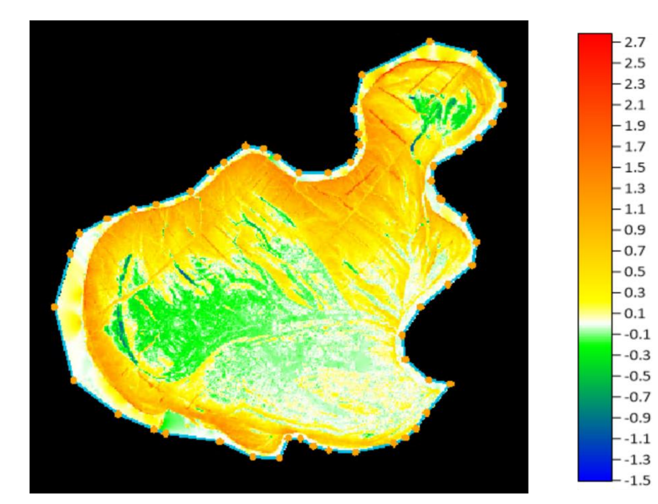
The comparison of the solution with DTM

For assessing wetlands conditions, the water level needs to be compared with the DTM. The DTM provides the ground elevation, but for water bodies it represents the water surface elevation at the time of scanning. The solution is computed at the same resolution of $0.5m \times 0.5m$ as the DTM, so the comparison is calculated in each pixel of the domain as follows:



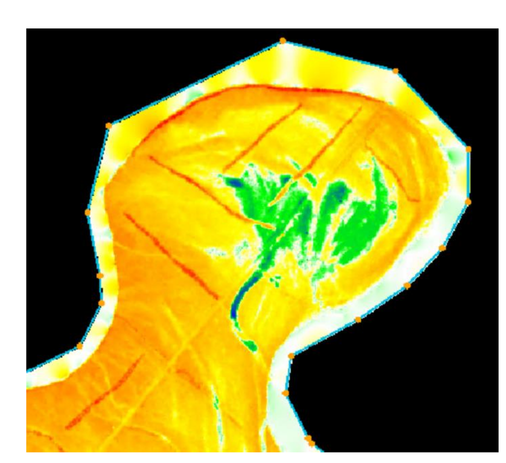
The digital terrain model

$$comparison = DTM - solution$$



The comparison of the solution and the DTM

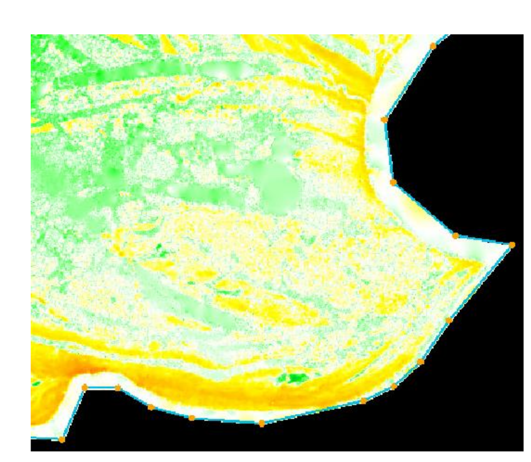
The range of the comparison is [−1.508, 2.782] meters. Minimal values indicating potentially flooded areas are observed in the north and west of the region. In both locations, there seems to be a land depression, which could be potentially flooded. However, due to vegetation cover, verification using orthophoto images was not possible.



Detail of the comparison with minimal (blue) values

Orthophoto image for the detail of the region

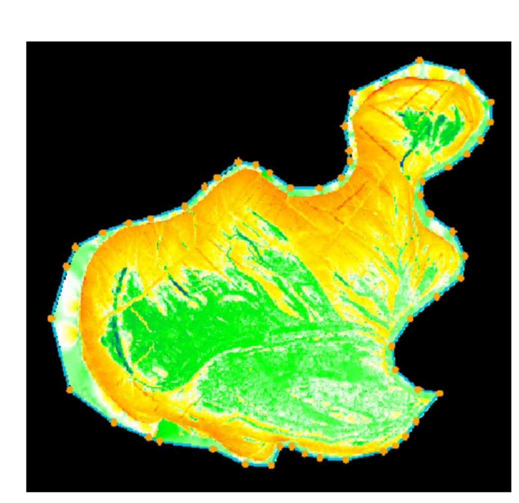
In the southeastern part of the region, a lake is visible in the orthophoto images. As expected, the comparison values in this area are close to zero, indicated by shades of light green, white, and light yellow.



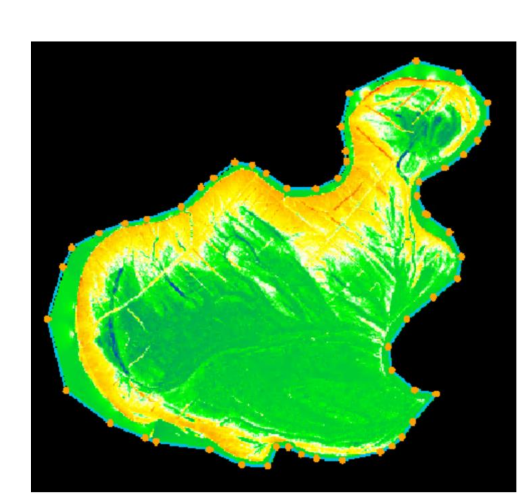
Detail of the comparison corresponding to a lake

Orthophoto image for the detail of the region

We also simulated a constant rise in water level in the surrounding distributaries to identify theoretically flooded areas. Such simulations can be used in water regulation strategies to find an optimal water level for maintaining existing wetlands or to create conditions for a formation of new ones. We observed, that with a 50 cm rise in water level, most of the region should be flooded.



10cm rise in water level



50cm rise in water level

Student: Bc. Ivana Piačková
Supervisor: Ing. Michal Kollár, PhD.
Consultant: Prof. RNDr. Karol Mikula, DrSc.

Academic year: 2023/2024
Study programme: Mathematical and Computational Modeling

MASTER'S THESIS
Basic hydrological modeling in software NaturaSat