SLOVAK UNIVERSITY OF TECHNOLOGY
Faculty of Civil Engineering

Registry number: SvF-5342-63956

Modeling of Fluid Flow Past Solid Objects via Complex Analysis

Bachelor Thesis

Study program: Mathematical and Computational Modelling
Study program number: 1114
Major: 9.1.9 Applied Mathematics
Department: Department of Mathematics and Constructive Geometry
Supervisor: prof. RNDr. Jozef Širáň, DrSc.

Bratislava 2018

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Faculty of Civil Engineering
Academic year: 2017/2018
Reg. No.: SvF-5342-63956

BACHELOR THESIS TOPIC

Student: **Martin Čavarga**
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Study programme: Mathematical and Computational Modeling
Study field: 9.1.9. Applied Mathematics
Thesis supervisor: prof. RNDr. Jozef Širáň, DrSc.
Workplace: Bratislava

Topic: **Modeling of Fluid Flow Past Solid Objects via Complex Analysis**

Language of thesis: English

Specification of Assignment:

The goal of this work is to model ideal laminar fluid flow past solid objects using the methods of complex analysis, and study changes in the flow caused by changing shape parameters through conformal mappings.

Length of thesis: 50

Assignment procedure from: 12. 02. 2018
Date of thesis submission: 03. 05. 2018

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Acknowledgments

I would like to thank my supervisor prof. RNDr. Jozef Širáň, DrSc. for expert assistance and consultation which he provided for my research when necessary.

Čestné prehlásenie

Vyhlasujem, že som bakalársku prácu vypracoval samostatne s použitím uvedenej odbornej literatúry a s odbornou pomcou vedúceho práce.

v Bratislave 3.5.2018

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Vlastnoručný podpis

Abstract

In this bachelor thesis we introduce the most fundamental concepts of complex analysis, such as: analytic functions, conformal mappings and the Riemann Mapping Theorem, which we apply in the modeling of fluid flow past solid objects. Although most modern engineering disciplines focus on finding numerical solutions to the equations of turbulent or laminar flow, we examine flow of ideal fluids which can be expressed via a complex potential $F = \phi + i\psi$ whose real and imaginary parts (potential and stream function) are conjugate harmonic, that is: they are both solutions to the Laplace equation which also satisfy the Cauchy-Riemann conditions. Conformal (angle-preserving) mappings are utilized to ensure incompressibility, irrotationality and zero-flux condition through the solid boundary of the analyzed fluid flow. After proving a relevant version of the Riemann Mapping Theorem, we also deduce that any boundary-value problems on disk domains can be conformally transformed onto arbitrary simply-connected regions. Special emphasis is given to flow past solids generated by the Joukovsky map, involving a simplified simulation of aircraft dynamics.

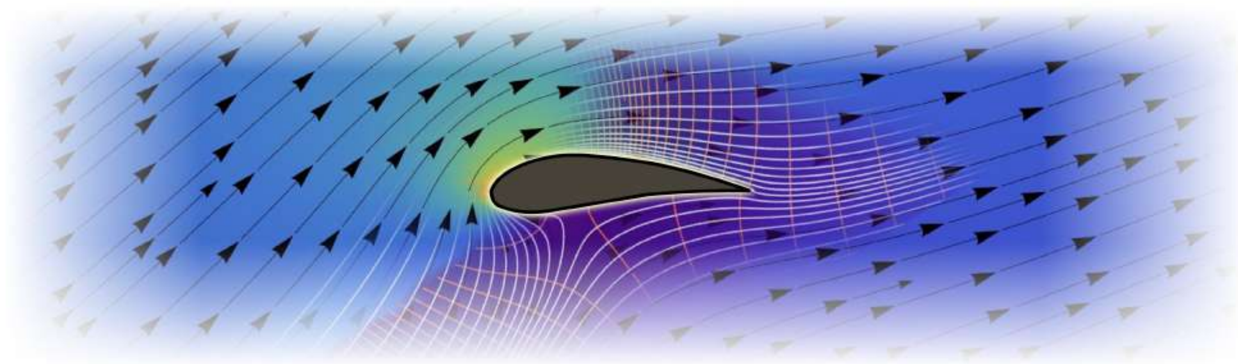
Keywords - complex analysis, Cauchy-Riemann equations, analytic function, harmonic function, conformal mapping, Riemann Mapping Theorem, fluid mechanics, potential flow, Joukovsky mapping, airfoil

Abstrakt

V tejto bakalárskej práci sú vysvetlené základné koncepty komplexnej analýzy, ako napríklad analytické funkcie, konformné zobrazenia, či Riemannova veta o zobrazení, ktoré používame pri modelovaní toku tekutín okolo pevných telies. Hoci sa väčšina moderných inžinierských disciplín sústredí na numerické riešenie rovníc prúdenia v laminárnom alebo turbulentnom režime, zameriavame sa na prúdenie ideálnych tekutín pomocou komplexného potenciálu $F = \phi + i\psi$ ktorého reálna a imaginárna časť (potenciál a prúdová funkcia) sú konjugovane harmonické, teda sú riešeniami Laplaceovej rovnice a tiež spĺňajú Cauchy-Riemannove podmienky. Konformné (uhly zachovávajúce) zobrazenia sú použité, aby boli zabezpečené podmienky nestlačiteľnosti, nevírovosti a nulového toku cez hranicu telesa pre skúmané prúdenie. Po dokázaní relevantnej verzie Riemannovej vety o zobrazení predpokladáme, že akékoľvek okrajové úlohy na kruhu možno konformne transformovať na ľubovoľnú jednoducho súvislú oblasť. Kladieme dôraz na tok tekutín okolo telies vytvorených pomocou Žukovského zobrazenia, pričom zahŕňame aj zjednodušenú simuláciu leteckej dynamiky.

Kľúčové slová - komplexná analýza, Cauchy-Riemannove rovnice, analytická funkcia, harmonická funkcia, konformné zobrazenie, Riemannova veta o zobrazení, mechanika tekutín, potenciálové prúdenie, Žukovského zobrazenie, profil krídla

Contents



Chapter 1

Introduction and Historical Background

Due to its remarkably complicated nature, fluid mechanics is a constantly developing field with a wide variety of unsolved problems. Yet much of what is described in this area has been derived and experimentally verified during 19th and up to the latter half of the 20th century, attributed to great names like Lagrange, Euler, Cauchy, Riemann, Reynolds, and Prandtl who were major contributors to the theory in its applied form (for engineering purposes) as well as the underlying mathematics (especially the theory of partial differential equations). Similarly to other topics, problems in fluid mechanics range from the simplest *ideal* type of flow, through *laminar* up to complicated *turbulent* systems introducing *viscous forces* and *compressibility.*

This work focuses on the simplest models of *incompressible* and *irrotational* with *zero viscosity* in *steady-state* conditions (time-independent). The reason for this simplification is its direct overlap with *complex function theory* which shows curious versatility in finding unique solutions to problems formulated on subsets of the complex plane $\mathbb{C}$. More precisely, types of flow classified as *ideal* can be described via a complex-valued function $F = \phi + \psi i$ where ϕ is the *potential* (the system's capacity to carry out work) and ψ the *stream function* whose level sets (*contours*) coincide with the *streamlines* (*trajectories*) of the flow.

Although the simplest models of flow have already been thoroughly studied, in most literature only a small subset of examples for (potential) flow past solid bodies has been shown, mostly involving simple bodies like the *cylinder* a *corner*, or a *tilted plate*. The goal of this work is to model ideal flow past more complicated solids, specifically various types of *airfoils*.

The structure of this thesis is adjusted to introduce the most fundamental concepts in *complex analysis*. It is assumed that the reader is already familiar with elementary properties of complex numbers like the *Euler's formuma*, or *de Moivre's theorem*, as well as elementary *calculus* with basics of *ordinary* and *partial differential equations*. This chapter provides brief history of complex numbers, and is followed by an introduction of *functions of complex variable* (illustrated with *phase portraits*), *complex differentiability*, theory of *analytic functions* and *complex integration*, in Chapter 2. Then *harmonic functions* and ideal flow are briefly introduced in Chapter 3, with first elementary examples of flow. Chapter 4 introduces the idea of *conformal mapping* and shows the underlying mathematical concept, the *Riemann Mapping Theorem*, and its implications. Finally Chapter 5 follows with main practical examples simulating ideal flow past solid objects and introduces additional properties like *circulation*, *drag*, and *lift*, all of which is evaluated and depicted on sequences of *stream* and *complex potential plots*. The final Chapter 6. contains a brief summary of the results and ideas for future research and development.

Additional theoretical background, including the proofs of important lemmas and theorems, is given in Appendices A, B, and C.

1.1 The Beginnings of Imaginary Quantities

Some mathematicians and historians on the subject say that the history of the square roots of negative numbers began as early as the first century in *Stereometria* by Heron of Alexandria who after giving a formula for the height of a frustum of a pyramid (pyramid section) with a side of the lower base a, upper base b and side edge c: $h = \sqrt{c^2 - 2(\frac{a-b}{2})^2}$ speculated that for $a = 28$, $b = 4$ and $c = 15$ the height is $\sqrt{225 - 288}$. He took $\sqrt{288}$ instead by replacing $\sqrt{-1}$ by 1, unable to observe the problem which he stated as impossible. It is not clear whether Heron's mistake was due to his ignorance or due to mistranslation of his work [7]. Heron's problem of the impossible pyramid had since been forgotten, until a similar issue arose later (in Renaissance).

Contrary to common belief, complex numbers did not arise from the need to solve quadratic equations, but rather cubic equations. Quadratic equations had been pioneered by Arab mathematicians like Al-Khwarizmi (*780 - †850) in *Algebra* who used geometric proofs, restricting himself to positive solutions. The methods of Algebra known to the Arabs were later introduced in Italy by the Latin translation of Al-Khwarizmi's work by Gerard of Cremona (*1114 - †1187), and by the work of Leonardo da Pisa (Fibonacci himself)(*1170 ∽ †1240). Being a talented merchant who learned a lot of mathematical secrets from Arab algebra, was presented to the Holy Roman Emperor Frederick II who held court in Sicily around 1255. One of the problems posed by Frederick's local mathematician was the solution of a cubic equation $x^3 + 2x^2 + 10x = 20$.

The general cubic equation $x^3 + ax^2 + bx + c = 0$ can be reduced to a simpler form $x^3 + px + q = 0$ by a change of variable $x' = x + a/3$. Such method appears for the first time in two anonymous Florentine manuscripts near the end of the 14th century. Generally, there had been three cases given for the reduced cubic equation:

$$(a): \quad x^3 + px = q \qquad (b): \quad x^3 = px + q \qquad (c): \quad x^3 + q = px$$

For positive coefficients, of course. The first to solve case (a) (and perhaps cases (b) and (c) as well) was Scipione del Ferro (*1465 - †1526), a professor of the University of Bologna. In his deathbed, del Ferro passed on the formula to his pupil Antonio Maria Fiore. Fiore challenged Niccolo Tartaglia (*1499 - †1557) to a mathematical contest. The night before the contest, Tartaglia re-discovered the formula and won. Tartaglia in turn told the formula (without proof) to Gerolamo Cardano (*1501 - †1576), who signed an oath to secrecy. From his knowledge of the formula, Cardano was able to re-construct the proof. Later, Cardano found out that del Ferro had the formula and verified this by interviewing relatives who gave him access to del Ferro's papers.

Cardano then proceeded to publish the formula for all three cases in his 1545 *Ars Magna* where he mentioned del Ferro as first author and Tartaglia as later independently obtaining the formula. A difficulty that was not present in case (a) was the possibility of having the square root of a negative number appear in the solution.

Substitute $x = u + v$ to case (b): $x^3 - px = u^3 + v^3 + 3uv(u + v) - p(u + v) = q$. Set $3uv = p$ to obtain $u^3 + v^3 = q$ and $u^3v^3 = (p/3)^3$. That is, the sum and the product of two cubes is known. This is used to form a quadratic equation which is readily solved

$$x = u + v = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$$

When the expression in the square root term gives a negative number, Cardano called it *casus irreducibilis*. He avoided discussing this case in Ars Magna, perhaps justifying it by the (incorrect) assumption of correspondence between casus irreducibilis and the lack of real positive solution to the cubic.

Cardano might have been the first to introduce complex numbers $a + \sqrt{-b}$ into algebra, but had doubts about it, saying (in Ars Magna) that for example the case for "dividing 10 into two parts, the product of which is 40" $(5 + \sqrt{-15})((5 - \sqrt{-15}))$ was clearly impossible.

Rafael Bombelli (*1526 - †1572) introduced the "notation" for $\sqrt{-1}$ in his *l'Algebra* (1572) and calls it "piu di meno". He followed Cardano in his discussion of cubics, considering equation $x^3 = 15x + 4$ for which the Cardano formula gives $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$. Bombelli observed that the cubic has $x = 4$ as a root, and proceeds to write $\sqrt[3]{2 + \sqrt{-121}} = a + b\sqrt{-1}$ and $\sqrt[3]{2 - \sqrt{-121}} = a - b\sqrt{-1}$. After algebraic manipulations he obtained $a = 2$ and $b = 1$. Thus $x = a + bi + a - bi = 4$, and commented: "At first the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof."

Enter Descartes (*1596 - †1650), the father of what will be known as *Cartesian geometry*. Pressured by his friends to publish his ideas, he wrote a treatise on science titled "*Discours de la method pour bien conduire sa raison et chercher la verite dans les sciences*" with there appendices *La Dioptrique*, *Les Meteores* and *La Geometrie*. Descartes associated the square roots of negative numbers with geometric impossiblity: "Moreover, the true roots as well as the false [roots] are not always real; but sometimes only imaginary [quantities]; that is to say, one can always imagine as many of them in each equation as I said."

Later John Wallis (*1616 - †1703) notes in his *Algebra* that negative numbers, so long viewed with suspicion by mathematicians, had a perfectly good physical explanation, based on a line with a zero mark, and positive numbers being on the right and negative numbers on the left of zero. Wallis also made some progress at giving a geometric interpretation of $\sqrt{-1}$.

Leaving France to seek religious refuge in London (at the age of eighteen), Abraham de Moivre (*1667 - †1754) befriended Isaac Newton. In 1698 he mentions that Newton knew, as early as 1676, of an expression: $(\cos\theta + \sqrt{-1}\sin\theta)^n = \cos(n\theta) + \sqrt{-1}\sin(n\theta)$. Apparently, Newton used this formula to compute the cube roots that appear in the irreducible case of Cardano's formulas. Moivre then used the formula (which will be named after him as *de Moivre's Theorem*) in his work.

Leonard Euler (*1707 - †1783) introduced notation $i = \sqrt{-1}$ and visualized complex numbers as points with cartesian coordinates, but did not give a satisfactory foundation for complex numbers. Euler used the formula $x + iy = r(\cos\theta + i\sin\theta)$ and visualized the roots of $z^n = 1$ as vertices of regular polygons. He defined the complex exponential and proved the identity $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$ [8]. In one of his posthumously published papers, he had some ideas about putting complex variables into integrals, separating real and imaginary parts, and decomposing the expressions into partial fractions. A similar approach by substitution might have been used by Johan Bernoulli (*1667 - †1748) in his attempts to simplify the quotients in certain integrals, with no actual justification, only examples.

A Norwegian mathematician, Caspar Wessel (*1745 - †1818) was the first to obtain and publish a suitable presentation of complex numbers. In 1797, Wessel presented his paper *On the Analytic Representation of Direction: An Attempt* to the Royal Danish Academy of Sciences. The paper was published in the Academy's Memoires of 1799. Its quality was judged to be so high that it was the first paper by a non-member of the Academy to be accepted for publication.

Wessel's approach used what we today call vectors. He uses the parallelogram law for geometric addition and he defined multiplication of vectors in terms of adding the polar angles and multiplying magnitudes.

Wessel's paper, written in Danish, went unnoticed until 1897, when it was unearthed by an antiquarian, and its significance recognized by the Danish mathematician Sophus Christian Juel (*1855 - †1935).

A Parisan bookkeeper Jean-Robert Argand (*1768 - †1822) then comes to the scene. It is not known whether he had formal mathematical training. In 1806 Argand produced a pamphlet, run by a private press in small print, without including his name on the title page. The title of the essay was "*Essay on Geometrical Interpretation of Imaginary Quantities*".

One copy ended up in the hands of Adrien-Marie Legendre (*1752 - †1833) who in turn mentioned it in a letter to Francois Francais, a professor of mathematics. When Francais died, his brother Jaques, who was a professor of military art and a mathematician, inherited his papers. He found Legendre's letter describing Argand's results, but Legendre failed to mention Agrand.

Jaques Francais published an 1813 article in the Annales de Mathematiques, giving the basics of complex numbers. In the last paragraph, he acknowledged his debt to Legendre's letter, and urged the unknown author to come forward. Argand found out and his reply appeared in the next issue of the journal.

William Rowan Hamilton (*1805 - †1865) defined, in an 1831 memoir, ordered pairs of real numbers (a, b) as a *couple*. He defined the addition and multiplication of couples as: $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ab - bd, bc + ad)$. In other words, he established an algebraic definition of what will be known as complex numbers.

Finally, Carl Friedrich Gauss (*1777 - †1855) himself introduced the term "*complex number*". There are indications that Gauss had been in possession of the geometric representation of complex numbers since 1796, but it went unpublished until 1831, when he submitted his ideas to the Royal society of Göttingen.

Gauss wrote that this subject has been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, and that it was largely unsuitable terminology which was to blame.

In an 1811 letter to Bessel, Gauss mentions the theorem that will be later known as *Cauchy's theorem* [8]. Now that complex numbers have found their place among mathematicians of the 19th century, it was a matter of time for someone to start developing complex function theory.

1.2 Complex Function Theory in the French and German Schools

Figure 1.1: Augustin Louis Cauchy.

Until the turn of the 19th century (since Newton's *Principia*), the majority of mathematicians grounded their study in real analysis with applications to physical phenomena. According to Niels Henrik Abel (*1802 - †1829), the only one working in pure mathematics was Augustin Louis Cauchy (*1789 - †1857). Poisson, Fourier, Ampere etc. occupied themselves with nothing but magnetism and other physical matters [6].

Cauchy was born in Paris and graduated from Ecole Polytechnique in 1807. In 1815, after a brief period as a civil engineer, he returned to his alma mater to teach. He composed his first contribution to the complex function theory in his *Memoire sur les integrales definies*, a memoir on definite integrals. Cauchy's first memoir was not published until 1827 [6], but until then he went on to publish over 200 more papers in this field. The culmination of his effort was *Memoire sur les integrales definies, prises entre des limites imaginaires* in 1825, in which he precisely formulates the meaning of the definite integral with complex limits and presents what will be known as the *Cauchy Integral Theorem* [9]. In this theorem he states that when integrating a certain class of complex-valued functions along a closed path in the complex plane, one gets zero.

By the end of 1820's, supported by other French mathematicians in his efforts, Cauchy had already laid theoretical foundations for complex analysis. Nonetheless, it took longer for the essential concepts to develop. Cauchy's integral theorem , for instance, was already present in his 1814 memoir, but only in the case of rectangles with sides parallel to the real and imaginary axes. For the generalization of this theorem to arbitrary closed curves, one must wait until 1846. Interestingly, Cauchy only required functions to be "continuous" and "finite" on a certain domain in his early work. From 1851 onwards he recognized the importance of the existence of a derivative independent of direction. He called functions with this property "*fonctions monogenes*" and showed that their real and imaginary parts (u and v) must satisfy a certain set of partial differential equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

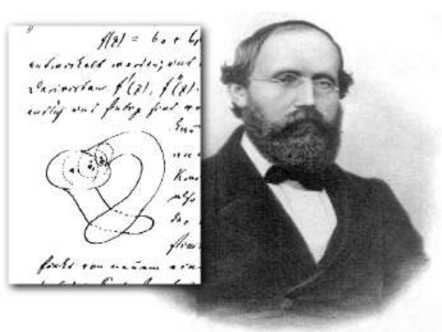


Figure 1.2: Bernhard Riemann and a sketch from his lecture notes depicting his treatment of analytic continuation.

French mathematicians who had recognized and supported Cauchy's work include, for example, Pierre Alphonse Laurent (*1813 - †1854) who discovered the eponymous *Laurent expansion* of an analytic function in the neighborhood of an isolated singularity, Joseph Liouville (*1809 - †1882) who formulated a variety of theorems in the theory of elliptic functions. The results achieved by these mathematicians were first assembled by Charles Auguste Briot (*1817 - †1882) and Jean-Claude Bouquet (*1819 - †1885) in a series of articles that was followed shortly after by an influential textbook on elliptic functions, which was then (after several editions) adopted in Italy and Germany [9].

Gauss' student, Bernhard Riemann (*1826 - †1866) is considered to be the second of the "founding fathers" of complex function theory. In his renowned dissertation , at University of Göttingen, he wrote his dissertation: *Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Größe* , taking an netirely new geometric approach to complex analysis and introduces what will be called *Riemann surfaces.*

There seems to be no record of anyone having used a similar device before. It was a fusion of two distinct and equally important ideas:

(1) A purely topological notion of covering surface, necessary to clarify the concept of multiply-correspondent mapping (a multiply-valued function)

(2) An abstract conception of the space of the variable; with a local structure defined by a uniformizing parameter.

In his dissertation, Riemann introduced another important concept (essential for throughout the course of this work):

"Two given simply-connected plane surfaces can always be related to each other in such way that each point of one surface corresponds to a point of the other, varying continously with that point with the corresponding smallest parts similar"

This was his first formulation of a theorem now called the *Riemann mapping theorem*. In short, the theorem states that any bounded simply-connected planar domain (with no "holes") has a map that can be used to transform it onto another such domain. [6]

Figure 1.3: Karl Weierstrass.

Much of the theory's current rigor stems from the numerous contributions of Karl Weierstrass (*1815 – †1897). Considered to be the third of the "founding fathers", Weierstrass was, at first, a student of finance and administration in Bonn ans then mathematics in Münster, where (already as a student) he composed three papers (in 1841-42). While Riemann freely used the methods of others, by contrast, Weierstrass seems to have admitted them into his theory usually only after a systematic reworking. Some say he hardly ever included anything of the numerous discoveries of Cauchy in his lectures. Yet still, he developed a variety of concepts in the theory, for example one finds a proof of the *Laurent Theorem*, independent of Cauchy and discovered before Laurent, the Cauchy estimates, the concept of uniform convergence, the definition of an *analytic function* by the means of *power series*, and the principle of analytic continuation.

The relationship between Riemann and Weierstrass was quite interesting. On a personal level, they seemed to have been friends, but professionally there was noticeable competition between them. For example, in 1857 Weierstrass published a paper on Jacobi's inversion problem but because he deemed it incomplete, he withdrew it, only to find out after a few weeks that Riemann had published a solution to the same problem which rested on entirely different foundations. Weierstrass apparently never quite got over having withdrawn his work from publication. After Riemann's death he often criticized Riemann's approaches on analysis as "insufficient". Proposing, for instance, his most notable example of a continuous, but nowhere differentiable function[1].

In comparison with the Weierstrassian function theory, built on strictly arithmetical foundations, the Riemannian theory, still operating in part with intuition and unproven limiting procedures, was in a truly difficult position. During its prime years, the Weierstrassian school took over nearly every position in Germany. Only with the works of Klein and the rehablitation of the Dirichlet Principle by Hilbert could Riemann's theory again gradually recover from the blow delivered by Weierstrass.

About 1900, after Edouard Goursat (*1858 – †1936) had shown the complete equivalence between the functions that are complex-differentiable in the Riemannian sense and the analogous class in Weierstrassian sense, the two rival approaches were again (bit by bit) unified, thanks to the efforts of Goursat, Bieberbach, Courant and others. [9]

[1] the Weierstrass function

Chapter 2

Complex Analysis

2.1 Complex Functions and Differentiability

The set of complex numbers $\mathbb{C}$ (i.e.: the *complex plane*) is homeomorphic[1] to the $\mathbb{R}^2$ plane. It can be thought of as a linear vector space under addition and scalar multiplication, since every complex number $z = x+yi$ can be thought of as a linear combination of 1 and the imaginary unit i with its coefficients being the *real* and *imaginary* part $\mathrm{Re}(z) = x$ and $\mathrm{Im}\{z\} = y$. It can also be equipped with a norm $|.| : \mathbb{C} \to \mathbb{R}_0^+ : z \mapsto |z| = \sqrt{x^2 + y^2}$, i.e.: a *complex modulus*, which can also be used as a metric: $d : (z_1, z_2) \mapsto |z_1 - z_2|$. It can also be shown that $(\mathbb{C}, |.|)$ is a complete metric space.

Let $\Omega \subseteq \mathbb{C}$. A map $f : \Omega \to \mathbb{C}$ can be defined on Ω. This map is called a *complex function* of a *single complex variable* (i.e.: a *complex-valued function*) and much like in the case of real-valued functions, properties like continuity and differentiability can be defined in terms of the metric of $\mathbb{C}$.

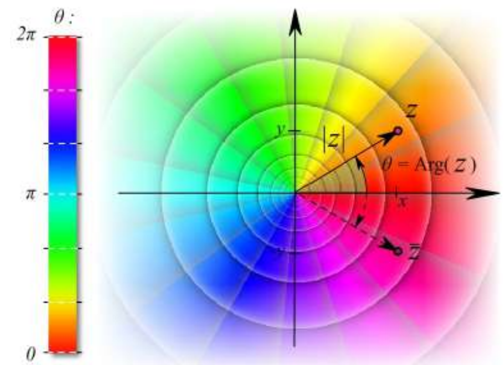


Figure 2.1: Complex number $z \in \mathbb{C}$, its *modulus* $|z|$, *argument* $\mathrm{Arg}(z)$ and *complex conjugate* $\overline{z} = x - yi$. The rainbow-colored background is a, so called, *enhanced phase portrait* of an identity map id: $z \mapsto z$, in which the color corresponds to the argument and concentric circular contours to the complex numbers with the same modulus.

In case of real-valued functions, one can easily visualize their graphs in $\mathbb{R}^2$, but since a complex-valued function $f : \Omega \to \mathbb{C}$ is, in general, a map between two-dimensional sets (homeomorphic to $\mathbb{R}^2$ or its open subsets), its graph $\{(z, f(z))|z \in \Omega\}$ would be a two-dimensional surface embedded in four-dimensional space, and thus impossible to fully visualize.

There are multiple ways to partially visualize complex functions. The first (and most straight-forward) is comparing a parametrization of the domain Ω (or a part of it) and its image $f[\Omega] \subseteq \mathbb{C}$, as shown in Fig.2.2 (left).

Another graphical representation is a *phase portrait*. Every complex number z can be written in its polar form: $z = re^{i\theta}$, where $r = |z|$ and $\theta = \mathrm{Arg}(z)$ is its *phase* or *argument*. A phase of a complex number can

[1]equivalence between topological spaces (see Def.A.0.5 in Appendix A)

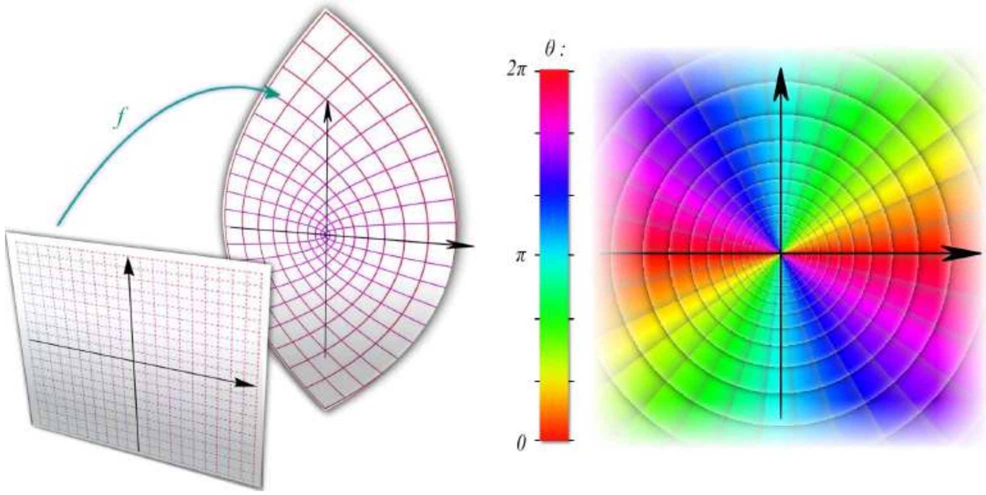


Figure 2.2: (left) A transformation of the $\mathbb{C}$-plane via $f : z \mapsto z^2$ map, and (right) its enhanced phase portrait.

be thought of as the angle between z as a vector and the real axis (see Fig.2.1). Then a point (x, y) can be given a particular color from the color wheel, each of which corresponds to an angle from 0 to 2π. In some cases it is suitable to see what point gets mapped to which phase rather than modulus (how far from the origin), but in case both polar coordinates need to be shown, one can use an *enhanced phase portrait* (see Fig.2.2 (right)), with contours corresponding to complex numbers with the same moduli.

Naturally, the image $f(z)$ of a complex number $z \in \Omega$ is also a complex number, and has its real and imaginary part: $f(z) = u(x, y) + v(x, y)i$. Functions u and v are the real and the imaginary part of $f = u + iv$. Take $f : z \mapsto z^2$, for example. Squaring $x + yi$ gives $\mathrm{Re}\{f(z)\} = u(x, y) = x^2 - y^2$ and $\mathrm{Im}\{f(z)\} = v(x, y) = 2xy$. $\boldsymbol{f} : (x, y) \mapsto (u(x, y), v(x, y))$ can also be thought of as a vector field in $\mathbb{R}^2$.

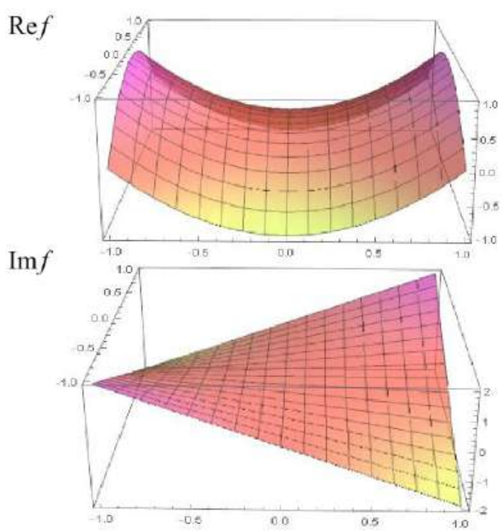


Figure 2.3: Real and imaginary parts of $f : z \mapsto z^2$.

Like any map from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$, f is *continuous* at z when for every neighborhood N of $f(z)$ there exists a neighborhood M of z, such that all points (complex numbers) in M get mapped into N, so $f[M] \subseteq N$, a *neighborhood* on the $\mathbb{C}$-plane, of course, being any subset of $\mathbb{C}$ containing z and a disk $\mathbb{D}_\delta(z)$ centered at z with radius $\delta > 0$. This is one of multiple ways to define continuity of f, but apparently the most intuitive[2]. In other words: continuity of f at any point $z \in \Omega$ means that a small neighborhood of z gets mapped onto a sufficiently small neighborhood of $f(z)$.

Differentiation of complex-valued functions may not be as geometrically intuitive as in the case of real-valued functions, but the property of *differentiability* (for reasons that will be stated later) is considerably far-reaching.

[2] this is continuity defined with respect to the "standard" or "Euclidean" topology on $\mathbb{C}$

Definition 2.1.1. The *derivative* of a complex-valued function $f : \Omega \to \mathbb{C}$ is a function $f' : \Omega \to \mathbb{C}$ such that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \tag{2.1}$$

where $\Delta z = z - w$, $w \in \Omega$.

Division of complex numbers z_1/z_2 gives a complex number with a difference argument with the modulus of the quotient of their moduli: $|z_1|/|z_2|\mathrm{e}^{i(\theta_1 - \theta_2)}$. In this sense, the limit expression (2.1) can be interpreted as the instantaneous rate of change of a complex vector with respect to a particular direction in $\mathbb{C}$ (that is: rotation and scaling in $\mathbb{C}$)[3].

Definition 2.1.2. $f : \Omega \to \mathbb{C}$ is *differentiable* (*holomorphic*[4]) when the limit (2.1) exists

Now of course, the existence of limit (2.1) depends on how the point $z \in \mathbb{C}$ is approached. Just like in the case of real-valued functions, there are multiple ways the limit can approach a given point. The real-valued case is very simple, approaching from only two possible sides, whereas in $\mathbb{C}$ one can approach from uncountably many directions. Intuitively, when (2.1) assumes the same value for all directions, it exists and f is differentiable.

Verifying differentiability of f by trying out all limit directions is impossible. Yet, there is a profound *necessary condition* that has to be met, in order for f to be differentiable.

Theorem 2.1.1. *(Cauchy-Riemann equations)*
Let $f : \Omega \to \mathbb{C}$ and $f = u + vi$ where u and v are the real and the imaginary part of f. Then f is differentiable if and only if u and v are continuously differentiable and satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2.2}$$

Proof. Take $\Delta z = \Delta x + \Delta y i$ and the limit (2.1). Now assuming that this limit exists (i.e.: that f is differentiable). Take the case when $\Delta y = 0$, which is when point z_0 is approached with constant imaginary part (see approaching z_0 from the right in Fig.2.4) and get

$$f'(z_0) = \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \right] =$$

$$= \lim_{\Delta x \to 0} \left[\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} \right] + i \lim_{\Delta x \to 0} \left[\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right]$$

Thus for any $z \in \Omega$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \tag{2.3}$$

And taking $\Delta x = 0$

$$f'(z_0) = \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \right] =$$

$$= -i \lim_{\Delta y \to 0} \left[\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \right] + \lim_{\Delta x \to 0} \left[\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \right]$$

[3] also called the *amplitwist* in some literature

[4] introduced by two of Cauchy's students, Briot (1817 − 1882) and Bouquet (1819 − 1895), and derived from the Greek *holos* meaning "entire", and *morphe* meaning "form" or "appearance".

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{2.4}$$

Combining (2.3) and (2.4) and comparing real and imaginary parts we obtain (2.2).

To show the converse, assume Δz is small enough, so that disk $D(z_0, |\Delta z|) \subseteq \Omega$. Then we can write differences

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]}{\Delta x + i\Delta y} \tag{2.5}$$

Take the difference of u and write

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)] + [u(x_0, y_0 + \Delta y) - u(x_0, y_0)] \tag{2.6}$$

Since it is assumed that the first partial derivatives of u and v exist and are continuous in Ω, by the *mean value theorem* there exists $x^* \in [x_0, x_0 + \Delta x]$ such that

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \frac{\partial u}{\partial x}(x^*, y_0 + \Delta y)$$

where

$$\frac{\partial u}{\partial x}(x^*, y_0 + \Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1 \tag{2.7}$$

and $\varepsilon_1 : D((x_0, y_0), |\Delta z|) \to \mathbb{R}$ is a function such that $\varepsilon_1 \to 0$ as $x^* \to x_0$ and $\Delta y \to 0$. So write

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y) = \Delta x \left[\frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1\right]$$

and similarly

$$u(x_0, y_0 + \Delta y) - u(x_0, y_0) = \Delta y \left[\frac{\partial u}{\partial y}(x_0, y_0) + \varepsilon_2\right]$$

And using forms similar to (2.7) in terms of (2.5) get

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\Delta x\left[\frac{\partial u}{\partial x} + \varepsilon_1 + i\frac{\partial v}{\partial x} + i\varepsilon_3\right] + \Delta y\left[\frac{\partial u}{\partial y} + \varepsilon_2 + i\frac{\partial v}{\partial y} + i\varepsilon_4\right]}{\Delta x + i\Delta y}$$

Where all partial derivatives are evaluated at (x_0, y_0) and where each $\varepsilon_i \to 0$ as $\Delta z \to 0$. Now using Cauchy-Riemann equations for the derivatives

$$\frac{\Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + i\Delta y\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]}{\Delta x + i\Delta y} + \frac{\lambda}{\Delta x + i\Delta y} \tag{2.8}$$

where $\lambda = \Delta x(\varepsilon_1 + i\varepsilon_3) + \Delta y(\varepsilon_2 + i\varepsilon_4)$. Now since

$$\left|\frac{\lambda}{\Delta x + i\Delta y}\right| \le \left|\frac{\Delta x}{\Delta x + i\Delta y}\right| |\varepsilon_1 + i\varepsilon_3| + \left|\frac{\Delta y}{\Delta x + i\Delta y}\right| |\varepsilon_2 + i\varepsilon_4| \le |\varepsilon_1 + i\varepsilon_3| + |\varepsilon_2 + i\varepsilon_4| \tag{2.9}$$

And the terms on the rightmost side of (2.9) approach zero as $\Delta z \to 0$, the term $\lambda/(\Delta x + i\Delta y)$ all the more approaches zero. Thus the limit of (2.5) exists if u and v are continuously differentiable and satisfy Cauchy-Riemann equations. □

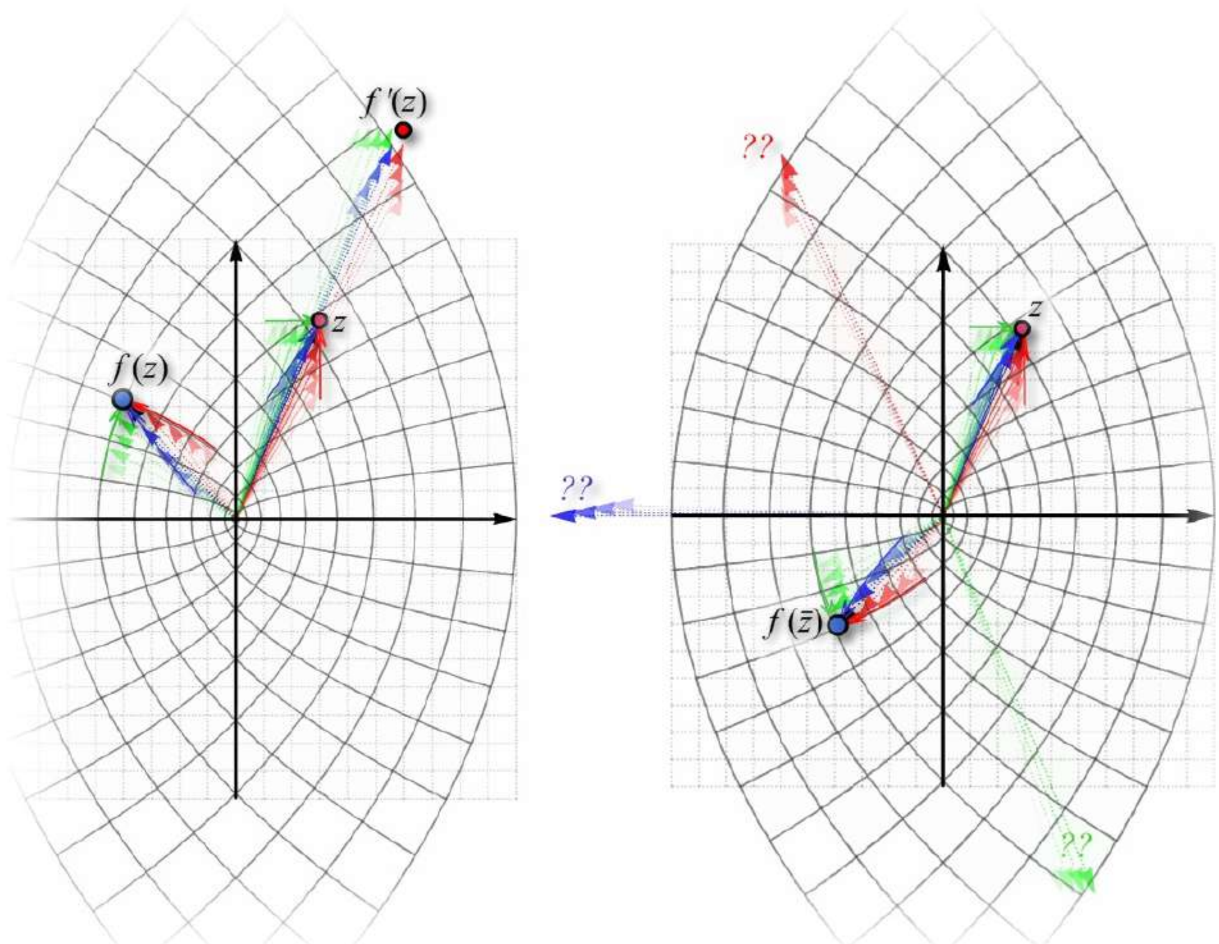


Figure 2.4: (left) A visualization (dashed grid is the set $[-1,1]\times[-i,i] \subset \mathbb{C}$) of how the ratio $\frac{f(z+\Delta z)-f(z)}{\Delta z}$ of function $f : z \mapsto z^2$ approaches its derivative $f'(z)$ at point z. Naturally, $f'(z) = 2z$ which can be seen on the picture as identity map with a scale factor of 2. (right) A similar function $f : z \mapsto \overline{z}^2$ whose difference ratio does not approach a single value from all directions and thus is not differentiable. (Visualized in *Wolfram Mathematica*$^{©}$).

It is easy to verify whether the real and imaginary part of $f : z \mapsto z^2$ satisfiy Cauchy-Riemann equations. As a counter-example, take $f : z \mapsto \overline{z}^2$. Its real and imaginary parts do not satisfy (2.2). In Fig.2.4 one cannot help but notice that the conjugate function changes the orientation of three directions of approach towards $z \in \mathbb{C}$, thus the difference ratio (2.1) will approach three different values. In fact, any conjugate-valued complex function has the same property and thus is not differentiable.

The Cauchy-Riemann equations (2.2) form a system of partial differential equations which give necessary conditions for a complex-valued function to be differentiable (holomorphic). This system can also be written as

$$\frac{\partial f}{\partial \overline{z}} = 0$$

which is also known as the *Wirtinger derivative*, and the $\partial/\partial\overline{z}$-operator is called the *Cauchy-Riemann operator*.

Methods of finding derivatives of holomorphic functions are the same as for real-valued functions. It can be shown that the same derivatives of elementary functions ($^2, \sqrt{}, \sin, \cos, ...$), as well as the same kinds of rules such as, the *product rule* , *quotient rule*, and *chain rule*, apply for holomorphic functions.

2.2 Analytic Functions

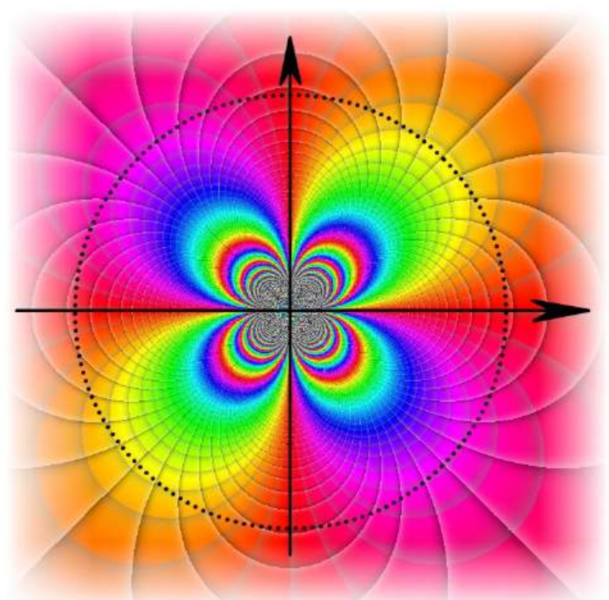


Figure 2.5: An enhanced phase portrait of $f(z) = \mathrm{e}^{-1/z^2}$ if $z \neq 0$ and $f(z) = 0$ if $z = 0$ which is a not analytic at 0 (with the unit circle (dotted)).

A remarkable feature of complex differentiation is that the existence of one complex derivative automatically implies the existence of infinitely many. Surprising as it may be, the result follows from the theory of *analytic* functions, and will be shown later after the introduction of complex integration. Often, a complex function f is said to be *analytic* on Ω if it is complex differentiable. The terms "holomorphic" and "analytic" are sometimes used interchangeably. Many mathematicians prefer the term "holomorphic" function, whereas "analytic" seems to be more widespread among physicists and engineers (and in older literature).

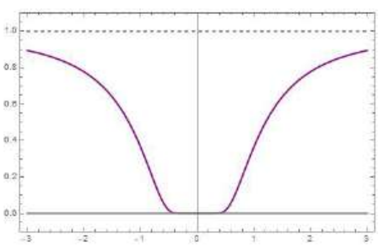
While real analysis certainly has its roots in the calculus of Newton and Leibniz, it can be said that the true spirit of analysis is the decomposition of arbitrary functions into fundamental units. The basic idea of an analytic function is that it can be broken down into elementary units - these units being the integer powers of variable z^n, $n \in \mathbb{N}$, a.k.a.: *power series*. The theory behind this seems, at first, a bit confusing because Taylor series might lead one to think that any C^∞-function (an infinitely differentiable, smooth function) can be expanded with power series. Yet, nothing could be further from the truth. Functions that have a power series expansion form a rather thin (but still suitably dense) subset of the C^∞-functions.

We know that for real-valued functions, it is true that if $f \in C^k(\Omega)$, $k \in \mathbb{N}$, then for any $x_0 \in \Omega$

$$f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i + \mathcal{O}(x^k)$$

What must be emphasized, is that the error term $\mathcal{O}(x^k)$ plays a vital role. In fact, the Taylor expansion converges to f if and only if $\mathcal{O}(x^k) \to 0$ as $k \to \infty$. The statement is, of course, a tautology, but still the heart of the matter.

It is a fact that "most" C^∞-functions are not analytic. Furthermore, even if the power series expansion does converge, it typically will not converge to its original function f. A frequently given example is



$f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0 \end{cases}$$

which is certainly a smooth function, but its Taylor expansion at 0 is $T_0^k(x) \equiv 0$ and the Taylor series converges to a function that is identically zero as well, which does not agree with f everywhere except at $x = 0$ (even though it is pretty close to 0 in its neighborhood). A similar example can be found for a function defined on $\Omega \subseteq \mathbb{C}$ (see Fig.2.5).

As it turns out, the fact that f can be expanded into power series is a strong property. Properly understood, analytic functions are a powerful and versatile tool for all analysts.

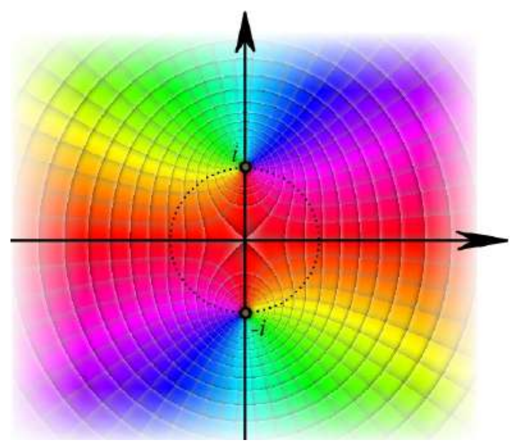


Figure 2.6: An enhanced phase portrait of $f(z) = \frac{1}{z^2+1}$ with poles of order $n = 1$ at i and $-i$.

Definition 2.2.1. $f : \Omega \to \mathbb{C}$ is called *analytic* at $z_0 \in \Omega$ when it has a power series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{2.10}$$

where $a_n \in \mathbb{C}$. A domain $\mathcal{D} \subseteq \Omega$ is called the *domain of convergence* if the power series (2.10) converges for all $z \in \mathcal{D}$.

Theorem 2.2.1. *Every analytic function is differentiable.*

Proof. By taking $a_0 = f(z_0)$ and the analyticity of f:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \implies$$

$$\frac{f(z) - f(z_0)}{z - z_0} = a_1 + (z - z_0) \sum_{k=2}^{\infty} a_k (z - z_0)^{k-2}$$

$$\implies f'(z_0) = a_1 \quad \text{as } z \to z_0$$

□

The converse is absolutely counter-intuitive and has many strange consequences.

In general, there are three possible options for the domain of convergence of a complex power series (2.10):

(a) The series converges for all $z \in \Omega$.

(b) The series converges inside a disk $\mathbb{D}_\rho(z_0)$ and diverges for all $z \in \Omega \backslash \mathbb{D}_\rho(z_0)$
(c) The series converges trivially at $z = z_0$.
The number $\rho > 0$ is called the *radius of convergence*. f is called an *entire* function if $\rho = \infty$.

A simple example of a power series expansion is $f_1(z) = \mathrm{e}^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ which is based at $z_0 = 0$. A straightforward application of the ratio test proves that the series converges for all $z \in \mathbb{C}$. On the other hand, the power series $f_2(z) = \frac{1}{z^2+1} = \sum_{n=0}^{\infty}(-1)^n z^{2n}$ converges inside the unit disk $\mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$ and diverges outside, when $|z| > 1$. Again, the convergence of f_2 is established through ratio test which is inconclusive at the unit disk.

Points where complex function f fails to be analytic are called a *singularities*. There are multiple ways in which they may occur for complex functions.

Definition 2.2.2. A singularity point z_0 of complex function $f : \Omega \to \mathbb{C}$ is called a *pole* of *order* $n \in \mathbb{N}$ if there is a function $h : \Omega \to \mathbb{C}$ such that $h(z) \neq 0$ for all $z \in \Omega$ and

$$f(z) = \frac{h(z)}{(z - z_0)^n} \tag{2.11}$$

Take $f(z) = 1/(z^2 + 1)$ from the previous example (see Fig.2.6 for phase portrait). Naturally, the denominator can be written as $(z^2 + 1) = (z + i)(z - i)$ which means that f has two 1st-order poles at $z = i$ and $z = -i$.

The origin of the name "pole" comes from the fact that when the image $f[\Omega]$ is mapped onto the, so called, *Riemann sphere*[5], by stereographic projection

$$\Pi : u + vi \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right) \tag{2.12}$$

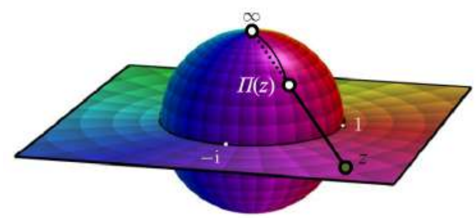


Figure 2.7: Projection onto the Riemann sphere.

the complex "poles" are actually points $z_\infty \in \mathbb{C}$ that are mapped by f onto $f(z_\infty)$ (which is a complex number with infinite modulus) and then mapped by Π (i.e.: by $\Pi \circ f$) onto the north pole of the Riemann sphere itself (see the ∞-point in Fig.2.7). On the other hand, *zeros* are points that get mapped onto the south pole of the Riemann sphere by $\Pi \circ f$. Their images $f(z_0)$ are all zeros on the complex plane, i.e.: $f(z_0) = 0$. In case of a complex polynomial function P, all the roots of $P(z) = 0$ are essentially its zeros.

More about power series and analytic functions can be found in Appendix B.

2.3 Complex Integration

Since a complex function f is typically not defined on an interval, but on a domain $\Omega \subseteq \mathbb{C}$, there is a variety of possible integrals. In principle, every (measurable) subset $G \subseteq \Omega$ could serve as domain of integration with respect to some complex measure. The integrals that often occur are:

$$\iint_G f(z)\mathrm{d}x\mathrm{d}y \ , \qquad \int_\Gamma f(z)|\mathrm{d}z| \ , \qquad \int_\Gamma f(z)\mathrm{d}z$$

[5]The classification of the singularities of an algebraic function as poles and critical points was carried out, for example, in the *Théorie des fonctions elliptiques of Briot and Bouquet* (1875). However, the term "pole," as pointed out by E. Neuenschwander (1978) was first used in this sense by K. Neumann in his Vorlesungen über Riemann's *Theorie der Abelschen Integrale* (1865) in connection with the fact that the point at infinity was depicted as the pole of the sphere in this book.[10]

The first integral is integrated over G with respect to the area measure, and since this can be done separately for the real and the imaginary parts of f, it easily reduces to two real integrals $\iint_G u(x,y)\mathrm{d}x\mathrm{d}y + i\iint_G v(x,y)\mathrm{d}x\mathrm{d}y$. The same holds for the second integral which integrates along the arc length of a path Γ (for more about paths in $\mathbb{C}$, see Appendix A).

For the purposes of this work, the third type of integral along paths is used, instead of line integrals along curves. This concept is a little simpler, more flexible, and it helps to keep distinction between notions like *path* γ its *trace* (or *image*) $[\gamma]$ and the generated *curve* Γ. With the appropriate definition of curves as equivalence classes of paths, all assertions made for path integrals are valid for curve (line, contour) integrals as well because they are independent of the chosen *parametrization*.

The integration of analytic functions along paths in their domains leads to intertwined topics, the first of which concerns reversing the operation of differentiation, i.e.: showing the existence of the anti-derivative. The solution to this problem is related to the *Cauchy integral formula* which will be shown later in this section.

The construction of the *primitive* (anti-derivative) involves finding whether it can be locally represented by function elements which can be then combined by *analytic continuation* to a primitive along paths. The global existence of a primitive then depends on the topology of the domain, i.e.: if Ω is multiply-connected, there may be conflicts between primitives along different paths. This problem is inherently responsible for the notion of *multiple-valued* complex functions, and thus leads to the concept of *Riemann surfaces*.

The representation by function elements is no different from finding a suitable power series expansion. Taylor series coefficients, for example, can be represented integrally:

Theorem 2.3.1. *Let f be analytic: $f(z) = \sum a_k(z-z_0)^k$ with the radius of convergence ρ. Then for any $r \in]0,\rho[$:*

$$a_k = \frac{1}{2\pi r^k}\int_0^{2\pi} f(z_0 + r e^{it})e^{-ikt}\ dt \quad k \in \mathbb{N}_0^+ \tag{2.13}$$

Proof. $(z-z_0)^n = r^n\mathrm{e}^{int}$. The uniform convergence of the power series on $\mathbb{D}_r(z_0)$ and

$$\int_0^{2\pi} \mathrm{e}^{ikt}\ \mathrm{d}t = \begin{cases} 2\pi & \text{if } \ k=0 \\ 0 & \text{otherwise} \end{cases}$$

gives

$$\int_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\mathrm{e}^{-ikt}\ \mathrm{d}t = \int_0^{2\pi}\sum_{n=0}^{\infty} a_n r^n \mathrm{e}^{i(n-k)t}\ \mathrm{d}t = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} \mathrm{e}^{i(n-k)t}\ \mathrm{d}t = 2\pi a_k r^k$$

□

Corollary 2.3.1. *(Mean Value Theorem) Let f satisfy the conditions of Theorem 2.3.1, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r e^{it})\ dt \tag{2.14}$$

A standard estimate for a continuous real-valued function $f : [a,b] \to \mathbb{R}$ says

$$\left|\int_a^b f(t)\mathrm{d}t\right| \le \int_a^b |f(t)|\mathrm{d}t \le (b-a)\max_{t\in[a,b]}|f(t)| \tag{2.15}$$

Then if $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}$ let $\int_a^b f(t)\mathrm{d}t = r\mathrm{e}^{i\theta}$ and $r = \left|\int_a^b f(t)\mathrm{d}t\right|$, $\theta \in \mathbb{R}$ where θ can be chosen arbitrarily when $r = 0$. Similarily to (2.15):

$$\left|\int_a^b f(t)\mathrm{d}t\right| = r = \mathrm{Re}\Big(\int_a^b \mathrm{e}^{-i\theta} f(t)\mathrm{d}t\Big) = \int_a^b \mathrm{Re}\big(\mathrm{e}^{-i\theta} f(t)\big)\mathrm{d}t \tag{2.16}$$

And since $\mathrm{Re}(z) \leq |z|$:

$$\int_a^b \mathrm{Re}\big(\mathrm{e}^{-i\theta} f(t)\big)\mathrm{d}t \leq \int_a^b \big|\mathrm{e}^{-i\theta} f(t)\big|\mathrm{d}t = \int_a^b |f(t)|\mathrm{d}t \leq (b-a)\max_{t\in[a,b]}|f(t)| \tag{2.17}$$

Definition 2.3.1. Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a path smooth on $[t_0, t_1], ..., [t_{n-1}, t_n]$ where $\alpha = t_0 < t_1 < ... < t_{n-1} < t_n = \beta$ and let $f : \Omega \to \mathbb{C}$ be continuous on $\gamma([\alpha, \beta]) \subset \Omega$. The *path integral* of f over γ is

$$\int_\gamma f(z)\ \mathrm{d}z = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(\gamma(t))\gamma'(t)\ \mathrm{d}t \tag{2.18}$$

By defining the integral of f along a path γ and not along its trace $[\gamma]$ complications which may occur when (parts) of the trace are run through several times by a given parametrization, are avoided. If one needs to define an integral along the oriented trace of a path (usually called *contour integral*) it is essential that the definition of the integral is invariant with respect to reparametrization. This implies, in particular, that a linear reparametrization $t \mapsto at + b$, for instance, does not change the integral's value. If the constant a is negative, however, the lower limit of the transformed parameter interval becomes larger than its upper limit, thus

$$\int_{\gamma^-} f(z)\mathrm{d}z = -\int_\gamma f(z)\mathrm{d}z \tag{2.19}$$

The invariance of the integral with respect to reparametrization can be shown directily using substitution rules:

Proposition 2.3.2. *Let γ be a piecewise-smooth path in $\Omega \subseteq \mathbb{C}$ and $\xi = \gamma \circ \phi$ its reparametrization. Then*

$$\int_\gamma f(z)dz = \int_\xi f(z)dz \tag{2.20}$$

Proof. Suppose ξ is smooth on $[\alpha, \beta]$, so $\phi : [\alpha, \beta] \to \mathbb{R}$ is an increasing continuous bijection. Then

$$\int_\xi f(z)\mathrm{d}z = \int_\alpha^\beta f(\xi(t))\xi'(t)\mathrm{d}t = \int_\alpha^\beta f((\gamma\circ\phi)(t))(\gamma\circ\phi)'(t)\mathrm{d}t = \int_\alpha^\beta ((f\circ\gamma)\gamma')(t)\phi'(t)\mathrm{d}t$$

$$= \int_{\phi(\alpha)}^{\phi(\beta)} ((f\circ\gamma)\gamma')(\tau)\mathrm{d}\tau = \int_\gamma f(z)\mathrm{d}z \tag{2.21}$$

Naturally, by (2.18) the result can be extended to a piecewise smooth path. □

A more general result follows from the Cauchy integral formula.

Lemma 2.3.3. *(Standard Integral Estimate) Let γ be a piecewise-smooth path of length $L(\gamma) = \int_\gamma |\gamma'(t)|dt$ and assume that $f : [\gamma] \to \mathbb{C}$ is continuous. If $M(f) = \max_{z\in[\gamma]}|f(z)|$ then*

$$\left|\int_\gamma f(z)dz\right| \leq L(\gamma)M(f) \tag{2.22}$$

Proof. By (2.17) and the assumption that γ is smooth:

$$\left|\int_{\gamma} f(z)\mathrm{d}z\right| = \left|\int_{\alpha}^{\beta} f(\gamma(t))\gamma'(t)\mathrm{d}t\right| \leq M(f)\int_{\alpha}^{\beta} |\gamma'(t)|\mathrm{d}t = L(\gamma)M(f) \tag{2.23}$$

And by (2.18) the result follows for a piecewise smooth path. □

Consequently if $\{f_n\}$ is a sequence of continuous functions that converges uniformly on $[\gamma]$ to f, then

$$\lim_{n\to\infty}\int_{\gamma} f_n(z)\mathrm{d}z = \int_{\gamma} f(z)\mathrm{d}z \tag{2.24}$$

Because by continuity of every f_n and Lemma 2.3.3 the maximum $M(f - f_n)$ tends to zero as $n \to \infty$.

Definition 2.3.2. Let $f : \Omega \to \mathbb{C}$, then if $F : \Omega \to \mathbb{C}$ is holomorphic and $F' = f$ in Ω, F is called the *primitive* (anti-derivative) of f in Ω.

It is not assumed that f itself is holomorphic, even though it follows from the fact that by F being holomorphic, it is also infinitely differentiable.

Theorem 2.3.4. *(Fundamental Theorem of Complex Calculus) Let F be a primitive of a continuous function f on Ω. Let $a, b \in \Omega$ be endpoints of a piecewise-smooth path γ in Ω (i.e.: $\gamma(0) = a$ and $\gamma(1) = b$). Then*

$$\int_{\gamma} f(z)dz = F(b) - F(a) \tag{2.25}$$

The result follows from (2.18) and the real-valued analog of Theorem 2.3.4. Obviously, (2.25) is zero if γ is a loop. Also if $f \equiv 0$, then $F \equiv C$, $C \in \mathbb{C}$ just like for real-valued functions.

Lemma 2.3.5. *Let $f : \Omega \to \mathbb{C}$ be continuous on Ω. Let $\Delta \subseteq \Omega$ be a triangle in Ω. If*

$$\int_{\Delta} f(z)dz = 0$$

then f has a primitive on Ω.

Proof. Let Δ have vertices z_0, a and z. The path integral along the edge between a and z is $F(z) = \int_{[a,z]} f(w)\mathrm{d}w$. And by the assumption

$$0 = \int_{[z_0,a]} f(w)\mathrm{d}w + \int_{[a,z]} f(w)\mathrm{d}w + \int_{[z,z_0]} f(w)\mathrm{d}w = -F(z_0) + F(z) + \int_{[z,z_0]} f(w)\mathrm{d}w$$

such that for $z \neq z_0$

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z,z_0]} f(w)\mathrm{d}w$$

And by Lemma 2.3.3

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \leq \frac{|z - z_0|}{|z - z_0|}\max_{w\in[z_0,z]} |f(w) - f(z_0)| = M(f - f(z_0))$$

Because f is continuous in Ω, the M term on the right-hand side tends to zero as $z \to z_0$, thus f is differentiable at z_0 and $F'(z_0) = f(z_0)$. □

The following result seems rather surprising, especially considering that it was discovered relatively late[6] and was not known to the founding fathers of the theory.

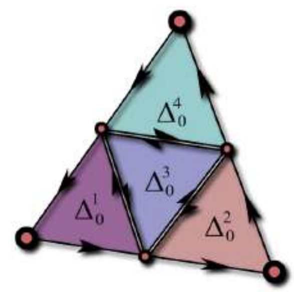


Figure 2.8: Triangle subdivision in Lemma 2.3.6

Lemma 2.3.6. *(Goursat) If f is differentiable on Ω and $\Delta \subseteq \Omega$ is any triangle in Ω, then*

$$\oint_{\partial\Delta} f(z)dz = 0 \tag{2.26}$$

Proof. Set $\Delta_0 = \Delta$ and δ_0 be its standard parametrization. Divide Δ_0 into triangles Δ_1^j, $j \in \{1,2,3,4\}$ by midpoints on the edges (see Fig.2.8). If δ_1^j is the standard parametrization of $\partial\Delta_1^j$ we have

$$\oint_{\partial\Delta_0} f(z)\mathrm{d}z = \oint_{\delta_0} f(z)\mathrm{d}z = \oint_{\delta_1^1} f(z)\mathrm{d}z + \oint_{\delta_1^2} f(z)\mathrm{d}z + \oint_{\delta_1^3} f(z)\mathrm{d}z + \oint_{\delta_1^4} f(z)\mathrm{d}z$$

which follows from Theorem 2.3.4 (Newton-Leibniz) and the fact that coincident edges of triangles Δ_1^j have opposing orientations and thus are negatives of each other.

Consequently, there exists $j \in \{1,2,3,4\}$ such that

$$\left|\oint_{\delta_0} f(z)\mathrm{d}z\right| \leq 4\left|\oint_{\delta_1^j} f(z)\mathrm{d}z\right|$$

(triangle Δ_1^j with the largest integral contribution). Define $\Delta_1 = \Delta_1^j$ and denote δ_1 the standard parametrization of its boundary. Applying the procedure again, one of the four sub-triangles Δ_2^j satisfies $\left|\oint_{\delta_0} f(z)\mathrm{d}z\right| \leq 4^2\left|\oint_{\delta_2^j} f(z)\mathrm{d}z\right|$ Proceeding recursively we obtain a sequence $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset ... \subset \Delta_k \subset ...$ of nested triangles, such that the integrals of f along the standard parameterizations δ_k of their boundaries satisfy $\left|\oint_{\delta_0} f(z)\mathrm{d}z\right| \leq 4^k\left|\oint_{\delta_k} f(z)\mathrm{d}z\right|$. Then $\bigcap_{k=0}^{\infty} \Delta_k = z_0 \in \Delta$.

Because $\Delta \subseteq \Omega$, f is differentiable at z_0, that is: $f(z) = f(z_0) + f'(z_0)(z - z_0) + r(z)$ where $r(z) = \mathcal{O}(z - z_0)$. And consequently, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|r(z)| < \varepsilon|z - z_0|$ whenever $|z - z_0| < \delta$.

Let d denote the circumference of Δ. Then all $z \in \Delta_k$ satisfy (by triangle inequality): $|z - z_0| \leq d/2^k$ so that for sufficiently large k: $|r(z)| < \varepsilon d/2^k$ if $z \in \Delta_k$.

Since the affine function $z \mapsto f(z_0) + f'(z_0)(z - z_0)$ has a primitive, the integral over that part of f along δ_k vanishes. Hence by $|r(z)| < \varepsilon d/2^k$ and the standard integral estimate (Lemma 2.3.3):

$$\left|\oint_{\delta_0} f(z)\mathrm{d}z\right| \leq 4^k\left|\oint_{\delta_k} r(z)\mathrm{d}z\right| \leq 4^k L(\delta_k)\varepsilon\frac{d}{2^k} = 4^k\varepsilon\frac{d}{2^k}\frac{d}{2^k} = \varepsilon d^2$$

where ε can be chosen arbitrarily small, so the integral on the left-hand side must be zero. ☐

Combining Lemma 2.3.5 and (*Goursat*) Lemma 2.3.6 we get the following important corollary:

Corollary 2.3.2. *If f is differentiable in Ω, then it has a primitive in Ω.*

The gap between holomorphic and analytic functions can be closed. So far, we know that any analytic function is differentiable (Theorem 2.2.1). The converse is rather tricky to prove.

A direct attempt might perhaps aim at proving that a holomorphic function f is infinitely differentiable, and then trying to show that it can be represented locally as power series. Instead of doing so, take a step back and show that the primitive F of a holomorphic function f is analytic. Once it is shown that F is analytic, we can conclude that all derivatives of F (and specifically f) are analytic too.

[6] By Edouard Goursat in 1884

Lemma 2.3.7. *Let $f : \mathbb{D} \to \mathbb{C}$ be differentiable on a disk $\mathbb{D}$ with radius R centered in a. Fix $0 < r < R$ and let $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$. If F is a primitive of f, then for any $z_0 \in \mathbb{D}$ and a closed path γ in $\mathbb{D}$*

$$F(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{F(z)}{z - z_0} dz \tag{2.27}$$

Proof. Let $z_0 \in \mathbb{D}$ be fixed. For h in $\mathbb{D}_0 = \{z \in \mathbb{C} | z + z_0 \in \mathbb{D}\}$ define $\varphi(h) = F(z_0 + h) - F(z_0) - f(z_0)h - \frac{1}{2}f'(z_0)h^2$. φ is differentiable in $\mathbb{D}_0$ and its derivative is $\varphi'(h) = f(z_0 + h) - f(z_0) - f'(z_0)h$.

Since f is differentiable at z_0, the right hand side of $\varphi'(h)$ is of order $\mathcal{O}(h)$ as $h \to 0$, that is: for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varphi'(h)| \le \varepsilon |h|$ whenever $|h| < \delta$.

φ' is continuous, whence map $[0, 1] \to \mathbb{C} : t \mapsto \varphi(th)$ is continuously differentiable (with respect to t), so that by the fundamental theorem (2.3.4)

$$\varphi(h) = \int_0^1 (\varphi(th))' \mathrm{d}t = \int_0^1 \varphi'(th) h \mathrm{d}t$$

Using the standard estimate for integrals in combination with $|\varphi'(h)| \le \varepsilon |h|$ we can conclude that $|\varphi(h)| \le h^2 \varepsilon$ for all h such that $|h| < \delta$.

Since ε can be chosen arbitrarily small

$$\lim_{h \to 0} \frac{\varphi(h)}{h^2} = 0 \tag{2.28}$$

Let $G : \mathbb{D} \to \mathbb{C}$ such that

$$G(z) = \begin{cases} \frac{F(z) - F(z_0)}{z - z_0} & \text{if } \ z \in \mathbb{D} \backslash \{z_0\} \\ f(z_0) & \text{if } \ z = z_0 \end{cases}$$

G is differentiable in $\mathbb{D} \backslash \{z_0\}$. In order to prove that G is also differentiable at z_0 consider

$$\frac{G(z) - G(z_0)}{z - z_0} = \frac{1}{(z - z_0)^2} \big[F(z) - F(z_0) - (z - z_0) f(z_0) \big] = \frac{1}{2} f'(z_0) + \frac{\varphi(z - z_0)}{(z - z_0)^2}$$

and for $z \to z_0$: $G(z_0) = \frac{1}{2} f'(z_0)$.

Because G is differentiable in $\mathbb{D}$, it has a primitive in $\mathbb{D}$, so by (Goursat) Lemma 2.3.6:

$$0 = \oint_\gamma G(z) \mathrm{d}z = \oint_\gamma \frac{F(z) - F(z_0)}{z - z_0} \mathrm{d}z = \oint_\gamma \frac{F(z)}{z - z_0} \mathrm{d}z - F(z_0) \oint_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{2.29}$$

The second integral on the right-hand side is $2\pi i F(z_0)$ by (A.2) in Lemma A.0.6. □

Definition 2.3.3. Let γ be a closed piecewise smooth path in $\mathbb{C}$ and assume $\varphi : [\gamma] \to \mathbb{C}$ is continuous, then $f : \mathbb{C} \backslash [\gamma] \to \mathbb{C}$ such that

$$f(z) = \frac{1}{2\pi i} \oint \frac{\varphi(w)}{w - z} \mathrm{d}w \ , \ z \in \mathbb{C} \backslash [\gamma] \tag{2.30}$$

is called the *Cauchy integral* with *density* φ along γ

Now comes a crucial result that bridges the gap between holomorphic and analytic functions.

Theorem 2.3.8. *Let γ be a piecewise-smooth closed path in $\mathbb{C}$ and assume $\varphi : [\gamma] \to \mathbb{C}$ is continuous. Then f defined by the Cauchy integral (2.30) is analytic on $\Omega = \mathbb{C} \backslash [\gamma]$ and tends to zero as $z \to \infty$. For a unit disk $\mathbb{D}_0 \subseteq \Omega$ with center z_0, the Taylor series $f(z) = \sum a_k (z - z_0)^k$ of f at z_0 converges in $\mathbb{D}_0$ and*

$$a_k = \frac{1}{2\pi i} \oint_\gamma \frac{\varphi(z)}{(z - z_0)^{k+1}} dz \tag{2.31}$$

Proof. Fix $z \in \mathbb{D}_0$. Because $[\gamma]$ is compact there exists $q < 1$ such that $|z - z_0|/|w - z_0| \leq q < 1$ for all $w \in [\gamma]$. Consequently

$$\frac{\varphi(w)}{w - z} = \frac{\varphi(w)}{(w - z_0) - (z - z_0)} = \frac{\varphi(w)}{w - z_0}\frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{\varphi(w)}{w - z_0}\sum_{k=0}^{\infty}\left(\frac{z - z_0}{w - z_0}\right)^k$$

Since $w \mapsto \varphi(w)/(w - z_0)$ is continuous and bounded on $[\gamma]$, the series converges uniformly with respect to $w \in [\gamma]$.

Changing the order of summation

$$2\pi i f(z) = \oint_\gamma \frac{\varphi(w)}{w - z}\mathrm{d}w = \sum_{k=0}^{\infty}\left(\oint_\gamma \frac{\varphi(w)}{(w - z_0)^{k+1}}\mathrm{d}w\right)(z - z_0)^k \tag{2.32}$$

for all $z \in \mathbb{D}_0$. And the standard integral estimate (Lemma 2.3.3) yields that $f(z) \to 0$ for $z \to \infty$. $\square$

Theorem 2.3.9. *Any $f : \Omega \to \mathbb{C}$ differentiable on domain Ω is analytic on Ω.*

Proof. Since both differentiability and analyticity are local properties, it is assumed that $\Omega = \mathbb{D}$. The Corrolary of Lemma 2.3.6 guarantees that f has a primitive F in $\mathbb{D}$. By Lemma 2.3.7 F can be written as a Cauchy integral, by Theorem 2.3.8 F is analytic, and consequently by $F' = f$, we get the analyticity of f. $\square$

Theorem 2.3.10. *Let f be analytic on a disk $\mathbb{D}(z_0)$. Then the Taylor series of f converges in the entire disk.*

Proof. Denote R the radius of $\mathbb{D}(z_0)$ and fix $0 < r < R$. The path $\gamma : I \to \mathbb{D}(z_0)$ such that $\gamma_r(t) = z_0 + r\mathrm{e}^{it}$ lies in $\mathbb{D}$ and $|f|$ is bounded on its trace $[\gamma_r]$ by some constant M. Applying the standard integral estimate (Lemma 2.3.3) to the Taylor coefficient formula (2.31 and Theorem 2.3.1): $|a_k| \leq M/r^k$ for $k \in \mathbb{N}$. The Taylor series converges for all z such that $|z - z_0| < r$ and because r can be chosen arbitrarily close to R, it converges everywhere in $\mathbb{D}$. $\square$

Theorem 2.3.11. *(Morera) Let $f : \Omega \to \mathbb{C}$ be continuous. If $\oint_\Delta f(z)dz = 0$ where $\Delta \subseteq \Omega$ is a triangle, then f is analytic in Ω.*

Proof. By Lemma 2.3.5 any f with a vanishing integral over a triangle has a primitive F. Then if f has a primitive the derivative $F' = f$ is analytic. $\square$

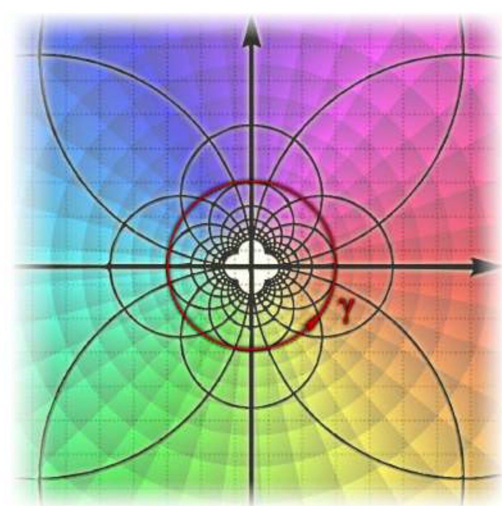


Figure 2.9: An enhanced phase portrait of the inversion map $z \mapsto 1/z$ with the image (black) of square grid (dashed), and the unit circle γ (red).

Theorem 2.3.12. *(Liouville) Any bounded entire function is constant.*

Proof. If f is analytic in all of $\mathbb{C}$ and $|f| \leq M$, $M \geq 0$, the estimate $|a_k| \leq M/R^k$ for the Taylor coefficients applies for every $R > 0$, so $a_k = 0$ for $k \in \mathbb{N}$. $\square$

By establishing that differentiability implies analyticity one of the main goals was achieved. The existence of a primitive, however, is still a problem for general domains.

It should be noted that the result in Theorem 2.3.10 is not a tautology. Being analytic in a disk means that f can be locally represented by power series, but there is no obvious reason why the series should converge globally in the entire (general) domain of f. One can take a look at the section about *analytic continuation* in Appendix B to see not only how to extend analytic functions from a disk onto its entire domain, but also that for each function there is a unique such extension.

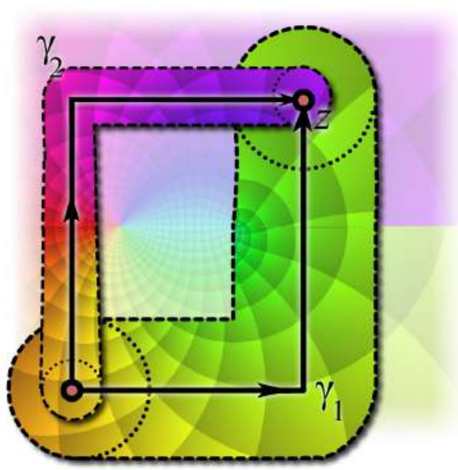


Figure 2.10: Analytic continuation to z of primitives of the inversion map $z \mapsto 1/z$ along two different paths γ_1 and γ_2.

Consider the *inversion function* $f : \mathbb{C}\backslash\{0\} \to \mathbb{C} : z \mapsto 1/z$. This function maps every point, except zero, on the interior of a unit circle onto its exterior, and vice versa . On the unit circle it acts as identity (see Fig. 2.9). If f has a primitive in $\mathbb{C}\backslash\{0\}$ then the integral of f over any loop in $\mathbb{C}\backslash\{0\}$ should vanish. However taking $\gamma = \mathbb{S}^1 : [0, 2\pi] \to \mathbb{C} : t \mapsto \mathrm{e}^{it}$ (a unit circle) we get:

$$\int_{\mathbb{S}^1} \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{i\mathrm{e}^{it}}{\mathrm{e}^{it}}\, \mathrm{d}t = 2\pi i \neq 0$$

$z \mapsto z^{-1}$ is a special case among power functions of type $z \mapsto z^n$, $n \in \mathbb{N}$. The primitive for all $n \neq -1$ is $F : z \mapsto z^{n+1}/(n+1)$. To accomplish the same even in cases like the inversion map, a more advanced technique must be used. The following construction is often referred to as finding the *primitive along a path*, and relies on the methods devised in Appendix B on analytic continuation, more specifically relying on unifying patches of primitives of f defined on disks, i.e.: constructing a *chain of function elements* (see Def. B.0.8).

Definition 2.3.4. A *primitive along a path* γ is a chain of function elements (F_0, D_0) ⚭ ... ⚭ (F_n, D_n) covering γ and satisfying $F_k' = f$ on D_k for $k = 0, 1, ..., n$.

Regardless of whether the path γ is *regular* or not (see Def. A.0.8), the following theorem shows that this procedure always works:

Theorem 2.3.13. *Let $f : \Omega \to \mathbb{C}$ be analytic on its domain and let (F_0, D_0) be a primitive of f in $D_0 \subseteq \Omega$. Then (F_0, D_0) has an unrestricted analytic continuation in Ω, and all the resulting function elements are primitives of f on their domains.*

Proof. According to the definition of an unrestricted analytic continuation (see Def.B.0.10) a function element (F_0, D_0) has an analytic continuation F_γ along any path in Ω such that $\gamma(0) = z_0 \in D_0 \subseteq \Omega$. By the Path Covering Lemma (Lemma A.0.1), there exists a chain of disks $D_0, D_1, ..., D_n$ covering γ. Because f is holomorphic in D_k, $k = 0, 1, ..., n$ it has a primitive on D_k. So find $(F_0, D_0), ..., (F_n, D_n)$ such that every F_k is a primitive of f on D_k. $F_k - F_{k-1}$ is a constant on $D_{k-1} \cap D_k$, so there exist constants $C_0 = 0$ and $C_1, C_2, ..., C_n \in \mathbb{C}$ such that $F_{k-1} + C_{k-1} = F_k + C_k$ on the disk intersection. Consequently the resulting chain of function elements: (F_0, D_0) ⚭ $(F_1 + C_1, D_0)$ ⚭ ... ⚭ $(F_n + C_n, D_n)$ is an analytic continuation of (F_0, D_0) along γ composed of primitives of f in the covering disks D_k. □

Theorem 2.3.14. *If f is analytic on a simply-connected domain Ω, then it has a primitive on Ω.*

Proof. Fix a disk $D_0 \subseteq \Omega$ centered at $z_0 \in \Omega$. Then f has a primitive F_0 on D_0. Considering Theorem 2.3.13 and the assertion II. of the Monodromy Principle (Theorem B.0.14), the analytic continuation of

(F_0, D_0) along all paths γ with $\gamma(0) = z_0$ and (variable) $\gamma(1) = z \in \Omega$ is a function of z. Since all function elements involved in the analytic continuation are primitives of f in their respective domains, $F : z \mapsto F(\gamma, z)$ is a primitive of f on Ω. $\square$

In general, the situation is more complicated. The idea of constructing a primitive from function elements (F_k, D_k) may encounter an obstacle, if Ω is not simple-connected. If the assumptions of the Monodromy Principle II. (Theorem B.0.14) are not satisfied, analytic continuation along different paths with the same endpoints may yield as far as uncountably many different values $F(\gamma, z)$ (Theorem B.0.16).

Theorem 2.3.15. *(Extended Fundamental Theorem): Let $\gamma : [\alpha, \beta] \to \Omega$ be a piecewise-smooth path. If f is holomorphic on Ω and F_γ is a primitive of f along γ then*

$$\int_\gamma f(z)\ dz = F_\gamma(\beta) - F_\gamma(\alpha) \tag{2.33}$$

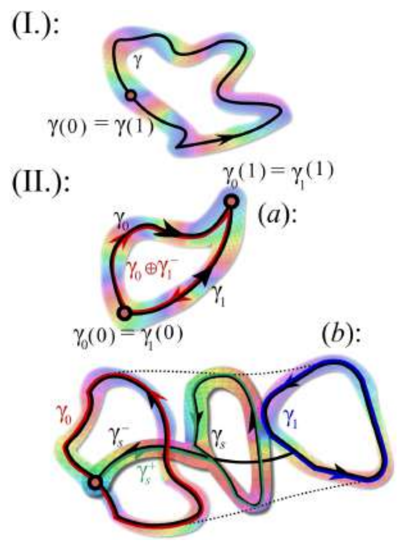
Proof. Choose a chain of function elements (F_k, D_k) covering γ. Then by $[t_{k-1}, t_k] \subseteq [\alpha, \beta]$ one can restrict γ onto $[t_{k-1}, t_k]$ and get γ_k which maps this interval into D_k and consequently write $\gamma = \gamma_1 \oplus \gamma_2 \oplus ... \oplus \gamma_n$. Since by the definition of path integral (Def. 2.18), the resulting integral is the sum of integrals along individual concatenated curve segments, each of which is defined on a disk D_k where f has a primitive. Using the Fundamental Theorem (Theorem 2.3.4) for γ_k one obtains only the difference of primitives at endpoints. $\square$

Now comes an important identity, widespread throughout most topics in complex analysis. This version is more general, making use of the notions of *homotopy* (Def.A.0.12), *free homotopy*, and *null-homotopy* (Def.A.0.15) of paths in $\mathbb{C}$.

Theorem 2.3.16. *(Cauchy Integral Theorem) Let f be analytic on $\Omega \subseteq \mathbb{C}$. Then*

(I.) *if a closed path (loop) γ is null-homotopic in Ω then*
$$\oint_\gamma f(z)dz = 0 \tag{2.34}$$

(II.) *if γ_0 and γ_1 are paths homotopic in Ω then*
$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz \tag{2.35}$$



Proof. (I.): Any primitive F_γ along a path $\gamma : [0,1] \to \Omega$ is obtained via an analytic continuation of a function element (F_0, D_0) centered at an initial point $z_0 = \gamma(0)$. If γ is null-homotopic in Ω then from the Monodromy Principle II. (Theorem B.0.14) we get $F_\gamma(0) = F_\gamma(1)$ and then the Extended Fundamental Theorem (Theorem 2.3.15) yields (2.34).

(II.): Paths γ_0 and γ_1 can either be homotopic with fixed endpoints (a) or freely homotopic (b).

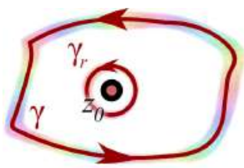
(a): If γ_0 and γ_1 are homotopic with fixed endpoints, then let $\gamma = \gamma_0 \oplus \gamma_1^-$. γ is, of course, a closed path, and thus null-homotopic in Ω (Lemma A.0.4). So according to part (I.) of this theorem

$$\int_{\gamma_1} f(z)\mathrm{d}z - \int_{\gamma_2} f(z)\mathrm{d}z = \int_\gamma f(z)\mathrm{d}z$$

(b): Now assume that γ_0 and γ_1 are closed loops and freely homotopic in Ω. Then there exists a homotopy $h : [0,1] \times [0,1] \to \Omega : (s,t) \mapsto h(s,t) = \gamma_s(t)$ such that h continuously connects γ_0 with γ_1. Define $\gamma_s^+(t) = h(st, 0) = h(st, 1)$, where $s \in [0,1]$ and $\gamma_s^- = -\gamma_s^+$ be the reverse of such path. Then $\gamma_s^* = \gamma_s^+ \oplus \gamma_s \oplus \gamma_s^-$ is a family of closed paths with fixed endpoints $\gamma_s^*(0) = \gamma_s^*(1) = \gamma_0(0) = \gamma_0(1)$ in Ω. Naturally γ_s^* is generated by a homotopy from $\gamma_0^* = \gamma_0$ to path $\gamma_1^* = \gamma_1^+ \oplus \gamma_1 \oplus \gamma_1^-$, with fixed endpoints. Applying the result of case (a) for paths homotopic with fixed endpoints, we get

$$\oint_{\gamma_0} f(z)\mathrm{d}z = \oint_{\gamma_0^*} f(z)\mathrm{d}z = \oint_{\gamma_1^*} f(z)\mathrm{d}z = \int_{\gamma_1^+} f(z)\mathrm{d}z + \oint_{\gamma_1} f(z)\mathrm{d}z + \int_{\gamma_1^-} f(z)\mathrm{d}z = \oint_{\gamma_1} f(z)\mathrm{d}z$$

where the path integrals along mutually negative paths γ_1^+ and γ_1^- cancel out (see identity (2.19)). □



A striking result follows when applying the Cauchy Integral Theorem to the formula for Taylor coefficients (2.31). One can then directly compute the value of an analytic function f at a point. But what's more, all the derivatives of f can be computed by integration:

Theorem 2.3.17. *(Cauchy Integral Formula I.) Let f be analytic in its simply-connected domain Ω and let $z_0 \in \Omega$. If γ is a closed path in $\Omega \backslash \{z_0\}$ with wind$(\gamma, z_0) = 1$, then*

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0} dz \quad \textit{and also} \quad f^{(k)}(z_0) = \frac{k!}{2\pi i}\oint_\gamma \frac{f(z)}{(z - z_0)^{k+1}} dz \ , k \in \mathbb{N} \tag{2.36}$$

Proof. The Taylor coefficients a_k of f are given by

$$a_k = \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(z)}{(z - z_0)^{k+1}}\mathrm{d}z \tag{2.37}$$

where γ_r is a positively-oriented circle inside the disk of convergence of this series. Since the integrand of (2.37) is analytic in $\Omega \backslash \{z_0\}$ and $[\gamma] \subset \Omega \backslash \{z_0\}$ with wind$(\gamma, z_0) = 1$, γ is freely homotopic to γ_r in the punctured domain $\Omega \backslash \{z_0\}$. Thus replacing the circle γ_r with γ does not change the value of the integral. Then by the definition of Taylor coefficients $a_k = f^{(k)}(z_0)/k!$ we get the result. □

The theorem can be generalized to arbitrary (even multiply-connected) domains Ω, allowing integration along "paths" composed of several discontinuous pieces.

Definition 2.3.5. A *chain* $\Gamma \subset \Omega$ is a finite collection of paths $\gamma_1, ..., \gamma_n$. If all paths in Γ are closed loops, then Γ is called a *cycle*. If $z \notin [\Gamma]$ then wind$(\Gamma, z) = \sum_{k=1}^n$ wind(γ_k, z) and the same holds for integrals $\int_\Gamma f(z)\mathrm{d}z = \sum_{k=1}^n \int_{\gamma_k} f(z)\mathrm{d}z$. A cycle $\Gamma \subset \Omega$ is called *null-homologous* if wind$(\Gamma, z) = 0$ for any $z \in \mathbb{C}\backslash\Omega$.

Clearly a cycle Γ composed of paths null-homotopic in Ω is null-homologous, but the converse is by no means true (see Fig.2.11). Denote $E = \mathrm{Ext}(\Omega) = \mathbb{C}\backslash\Omega$ the exterior. Then the winding number of any cycle Γ about any point in E is zero only when Ω is simply-connected.

A null-homologous cycle can be thought of as a collection of paths that wind and unwind the same amount of times around each hole of a multiply-connected domain. Γ can also completely ignore some holes, all that matters is that it has a zero winding number around all the points in them.

Theorem 2.3.18. *(Cauchy Integral Formula II.): Let Γ be a null-homologous cycle in $\Omega \subseteq \mathbb{C}$. If f is analytic on Ω, then:*

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0} dz = \textit{wind}(\Gamma, z_0)\ f(z_0) \qquad z_0 \in \Omega\backslash[\Gamma] \tag{2.38}$$

Proof. Denote $\Omega_0 = \Omega \backslash [\Gamma]$. In order to verify (2.38) consider

$$g_0 : \Omega_0 \to \mathbb{C} : z \mapsto \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} \mathrm{d}w - \mathrm{wind}(\Gamma, z)\ f(z) \tag{2.39}$$

The goal is to show that g_0 can be analytically extended to an entire function g vanishing at infinity, so that by Liouville's theorem (Theorem 2.3.12) $g \equiv 0$ on $\mathbb{C}$.

so let $g_0(z) = g_1(z) + g_2(z)$. By the definition of a cycle, g_1 is a sum of Cauchy integrals along the components of Γ, so it can be extended to an analytic function in $\Omega \backslash [\Gamma]$.

To show that the second summand $g_2(z)$ defines an analytic function in $\Omega_0 = \backslash [\Gamma]$, write

$$g_2(z) = \begin{cases} -\mathrm{wind}(\Gamma, z)\ f(z) & \text{if } z \in \Omega_0 \\ 0 & \text{if } z \in E = \mathbb{C} \backslash \Omega \end{cases}$$

A corollary of Morera's Theorem (Theorem 2.3.11) is that if a function is continuous on a domain Ω and analytic on $\Omega \backslash S$ where S is a finite union of line segments, then it is also analytic on all of Ω. Using this fact, and the fact that the components of Γ can be homotopically transformed without a change in the winding number into a polygonal approximation $\widehat{\Gamma}$ (which can be verified using part (5) of Theorem A.0.7). The need for polygonal approximation $\widehat{\Gamma}$ also arises from the fact that Γ might not, in general, be piecewise smooth, so the Standard Integral Estimate would not hold. It remains to be shown that $g = g_1 + g_2$ is, in fact, continuous.

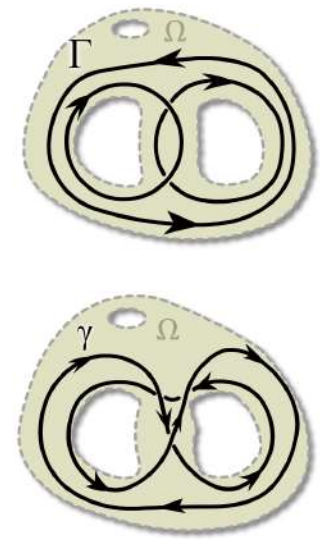


Figure 2.11: Example of a null-homologous cycle Γ and a null-homologous path γ which is not null-homotopic on a multiply-connected domain Ω.

For $z \in \Omega_0$ use the formula in (Lemma A.2) for the winding number and write

$$g_0(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(z)}{w - z} \mathrm{d}w \quad z \in \Omega_0$$

and then define a function

$$h : \Omega \times \Omega \to \mathbb{C} : (w, z) \mapsto \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ 0 & \text{if } w = z \end{cases}$$

Now it needs to be shown that h is continuous on $\Omega \times \Omega$. It is trivial for $w \neq z$, but it needs to be shown for all $(z_1, z_1) \in \Omega \times \Omega$. Fix $z_1 \in \Omega$ and denote $D_1 \subseteq \Omega$ a disk centered at z_1. Take a line segment $[z, w] \subset D_1$, and from the Fundamental Theorem (2.3.4) it follows that

$$h(w, z) = \frac{f(w) - f(z)}{w - z} = \frac{1}{w - z} \int_{[z,w]} f'(\zeta) \mathrm{d}\zeta = \int_0^1 f'(z + t(w - z)) \mathrm{d}t$$

and from the Standard Integral Estimate (Lemma 2.3.3):

$$\left| h(w, z) - h(z_1, z_1) \right| \leq \max_{t \in [0,1]} \left| f'(z + t(w - z)) - f'(z_1) \right| \tag{2.40}$$

Therefore since f' is continuous, the right-hand side of (2.40) tends to zero as $(w, z) \to (z_1, z_1)$, which proves the continuity of h on $\Omega \times \Omega$.

Now g_0 needs to be extended to all of Ω. Set $g_0(z) = \frac{1}{2\pi i}\int_\Gamma h(w,z)\mathrm{d}w$. In order to prove that g_0 is continuous on Ω pick $z_1 \in \Omega$ and a closed disk $K \subseteq \Omega$ centered in z_1. Then for all $z \in K$:

$$\big|g_0(z) - g_0(z_1)\big| = \frac{1}{2\pi i}\left|\int_\Gamma h(w,z) - h(w,z_1)\mathrm{d}w\right| \le \frac{L(\Gamma)}{2\pi}\max_{w\in[\Gamma]}\big|h(w,z) - h(w,z_1)\big| \tag{2.41}$$

Since h is uniformly continuous on $[\Gamma] \times K$, the right-hand side of (2.41) tends to zero as $z \to z_1$, which proves the continuity of g_0 and hence of g on Ω. So g has an analytic extension from $\mathbb{C}\backslash[\Gamma]$ to the entire complex plane.

Finally, the standard estimate for the Cauchy integral defining g_1 and the fact that $g_2 = 0$ on $\mathbb{C}\backslash\Omega = E$ yield that $g(z) \to 0$ at infinity and by Liouville's theorem $g \equiv 0$ on all of $\mathbb{C}$. $\square$

Corollary 2.3.3. *Let f be analytic on an open set Ω, and Γ be a null-homologous cycle in Ω, then*

$$\mathit{wind}(\Gamma, z_0)\ f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{k+1}}dz \quad k \in \mathbb{N}_0\ ,\ z_0 \in \Omega \tag{2.42}$$

Corollary 2.3.4. *(Cauchy Integral Theorem for Null-Homologous Cycles): Let f be analytic on an open set Ω, and Γ be a null-homologous cycle in Ω, then*

$$\int_\Gamma f(z)dz = 0 \tag{2.43}$$

2.4 Residues

Where Taylor series expansion fails to properly assess the behavior of a function f, another more general tool needs to be used. Generalizing the concept of power series while including also negative powers of $z - z_0$ leads to a new type of (doubly infinite) function series:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k(z-z_0)^k = \sum_{k=0}^{\infty} c_k(z-z_0)^k + \sum_{k=1}^{\infty} c_{-k}(z-z_0)^{-k}\ . \tag{2.44}$$

Expansion of this form is called *Laurent series* with center z_0. The first sum with coefficients c_k is said to be the *regular part*, and the second sum with c_{-k} as coefficients is referred to as the *main* (*principal* or *singular*) part of the Laurent series.

Because it is equivalent to the power series (2.10), the regular part of a Laurent series converges absolutely in a disk. Similarly, the singular part converges absolutely on the exterior of a closed disk. Both domains of convergence may be empty or cover the entire complex plane. If the regular part converges for all $|z - z_0| < R_1$ and the singular part for $|z - z_0| > R_0$ with $R_0 < R_1$ then the Laurent series converges on a *ring domain*:

$$\Omega = \big\{z \in \mathbb{C} \mid 0 \le R_0 < |z - z_0| < R_1 \le \infty\ \big\} \tag{2.45}$$

Its sum is a function analytic in Ω. Conversely, any analytic function on a ring (punctured disk) domain has a Laurent series expansion.

Theorem 2.4.1. *Any function $f : \Omega \to \mathbb{C}$ analytic on a ring domain Ω can be uniquely represented by a Laurent series expansion which converges absolutely in Ω. The Laurent coefficients are given by:*

$$c_k = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}dz\ , \quad k \in \mathbb{Z} \tag{2.46}$$

Proof. Fix r_0 and r_1 so that $R_0 < r_0 < r_1 < R_1$ and denote γ_0 and γ_1 standard parameterizations of circles with radii r_0 and r_1 respectively. Cycle $\Gamma = \gamma_1 + \gamma_0^-$ is null-homologous in Ω. Since $\mathrm{wind}(\Gamma, z) = 1$ for all $z \in \{z \in \mathbb{C} \mid r_0 < |z - z_0| < r_1\}$, the homology version of the Cauchy Integral Formula (Theorem 2.34) gives

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\mathrm{d}w = \frac{1}{2\pi i}\left[\oint_{\gamma_1}\frac{f(w)}{w-z}\mathrm{d}w - \oint_{\gamma_0}\frac{f(w)}{w-z}\mathrm{d}w\right]$$

Similarly to the expansion by geometric series in expression (2.32), one can now write both terms with $1/(w-z)$ as convergent series:

$$f(z) = \frac{1}{2\pi i}\left[\oint_{\gamma_1}\frac{f(w)}{w-z_0}\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k\mathrm{d}w - \oint_{\gamma_0}\frac{f(w)}{z-z_0}\sum_{k=1}^{\infty}\left(\frac{w-z_0}{z-z_0}\right)^k\mathrm{d}w\right] \tag{2.47}$$

and factor the infinite sum to get

$$f(z) = \frac{1}{2\pi i}\sum_{k=-\infty}^{\infty}\left(\oint_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\mathrm{d}w\right)(z-z_0)^k = \sum_{k\in\mathbb{Z}} c_k(z-z_0)^k\ . \qquad \square$$

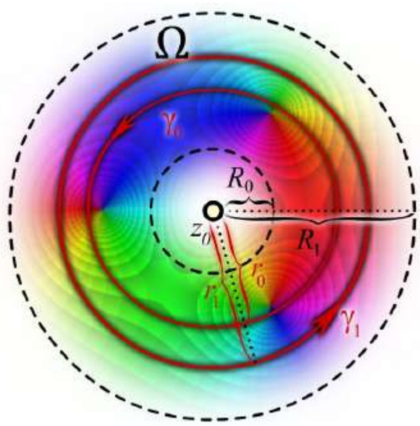


Figure 2.12: A ring domain of convergence of Laurent series.

We can fuse both integrals in (2.47) because any closed loop γ in Ω is homotopic to both γ_0 and γ_1.

Due to the fact that r_0 and r_1 can be chosen arbitrarily close to R_0 and R_1 respectively, the Laurent expansion holds for all $z \in \Omega$.

To show that Laurent series (2.44) is uniformly convergent on compact subsets of Ω, fix radius r such that $R_0 < r < R_1$ and let γ be a positively oriented parametrization of a circle $\mathbb{S}_r(z_0)$. If $M(f) = \max_{z\in[\gamma]}|f(z)|$ then by Theorem 2.3.1 extended to $k \in \mathbb{Z}$, and the Standard Integral Estimate (Lemma 2.3.3):

$$|c_k| = \frac{1}{2\pi i}\left|\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\mathrm{d}z\right| = \frac{1}{2\pi r^k}\left|\int_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\mathrm{e}^{-ikt}\mathrm{d}t\right| \leq$$

$$\leq \frac{1}{2\pi r^k}L(\gamma)M(f) = \frac{M(f)}{r^{k-1}} \tag{2.48}$$

So by Weierstrass M-Test (Theorem B.0.2), the regular part of Laurent series converges absolutely for all $z \in \mathbb{C}$ with $|z - z_0| < r$ while the singular part converges absolutely for $|z - z_0| > r$. Since r can be chosen arbitrarily from $]R_0, R_1[$, both series converge absolutely in Ω. Moreover, inequality: $|c_k| \leq M(f)/r^{k-1}$ guarantees uniform convergence in any ring $\{z \in \mathbb{C}\ |R_0 < r_0 \leq |z - z_0| \leq r_1 < R_1\}$ and hence on any compact subset of Ω. $\square$

Remark. Surprising as it may be, Laurent coefficients expressed as

$$c_k = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\mathrm{d}z = \frac{1}{2\pi r^k}\int_0^{2\pi} f(z_0 + r\mathrm{e}^{it})\mathrm{e}^{-ikt}\mathrm{d}t\ , \qquad k \in \mathbb{Z} \tag{2.49}$$

might stand out in the eye of an experienced reader as nothing less than Fourier coefficients defined on a circle $\mathbb{S}_r(z_0)$ with standard parametrization: $t \in [0, 2\pi]$.

For example, consider $f : \mathbb{C}\backslash\{0\} : z \mapsto z^2 + \frac{1}{z}$. Without any computation the Laurent coefficients of f at $z_0 = 0$ are $c_{-1}, c_2 = 1$. The outer radius of convergence is infinite and the inner is zero, with the inner disk containing only point 0.

A more complicated example is $f : \mathbb{C}\backslash\mathcal{Z}_{\sin} \to \mathbb{C} : z \mapsto \frac{1}{\sin z}$ where $\mathcal{Z}_{\sin} = \{m\pi \mid m \in \mathbb{Z}\}$ is the set of all zeros of $\sin z$ (Fig. 2.15). Let $z_0 = 0$, then $R_1 = \pi$ will be the distance to the closest zero of sin-function, hence the radius of the outer disk of convergence. Laurent series will, in fact, converge in a punctured disk $\dot{D} = \{z \in \mathbb{C} \mid 0 < |z| < \pi\}$. Because f has a pole at 0, we can write $f(z) = z^{-k}\tilde{f}(z)$ where $\tilde{f}$ is holomorphic on $\mathbb{C}$ with $\tilde{f}(0) \neq 0$. In particular: $\lim\limits_{z\to 0} z^k f(z) = \tilde{f}(0)$ exists and is non-zero. Using L'Hôpital's rule one can verify that this happens only when $k = 1$ and the limit in that case equals 1. So we have found the singular part of the Laurent series containing only one term: $1/z$. The remaining coefficients can be computed by (2.49), integrating over a unit circle $\mathbb{S}^1$. The resulting Laurent series is:

$$f(z) = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \frac{127z^7}{604800} + \frac{73z^9}{3421440} + \mathcal{O}(z^{11})$$

Point $z_0 = 0$ is an *isolated singularity* (see Def.B.0.7), meaning that a sufficiently small disk around z_0 contains only one singularity, that is z_0. Evidently, singularities of functions like $z \mapsto \sqrt{z}$ or $z \mapsto \log z$, or even pathological functions like $z \mapsto \mathrm{e}^{-1/z^2}$ or $z \mapsto \sin 1/z$, have quite different properties. Singularities of the square root and log functions have been described in Appendix B using *Riemann surfaces*. Singularities at which a Laurent series expansion exists can be divided into three categories:

Definition 2.4.1. An isolated singularity z_0 of an analytic function f with Laurent series expansion (2.44) is called:
(1.) a *removable singularity*, if $c_k = 0$ for all $k < 0$,
(2.) a *pole* of *order* m, if $c_{-m} \neq 0$ and $c_k = 0$ for all $k < -m < 0$,
(3.) an *essential singularity*, if $c_k \neq 0$ for infinitely many $k < 0$.

Functions analytic on $\mathbb{C}\backslash S$, where S is the set of singularities, which only have isolated singularities are called *meromorphic*.

It can be shown that an analytic function with Laurent series is bounded in a neighborhood of every removable singularity (see Fig.2.13 (a)), and for any pole $\lim\limits_{z\to z_0} f(z) = \infty$.

Around an essential singularity, on the other hand, the function shows "wild" behavior. For $z \to z_0$ the function assumes infinities in all possible directions, and the limit does not exist (see Fig.2.13 (b)). Taking a look at the phase portrait in Fig.2.5, one notices infinitely many isochromatic lines (curves in $\mathbb{C}$ where $\operatorname{Arg} f(z) = const.$) intersecting the essential singularity.

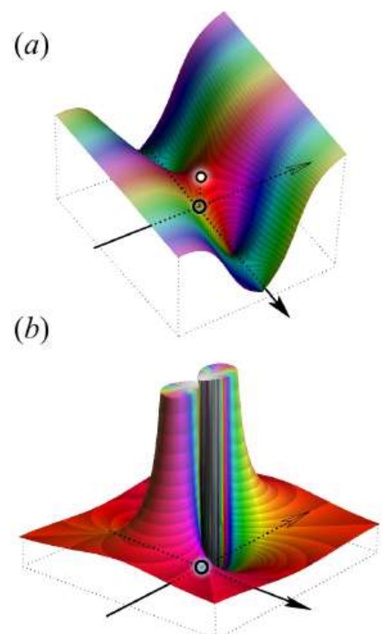


Figure 2.13: Analytic landscapes: (a) a removable singularity of $z \mapsto \frac{\sin z}{z}$ at $z_0 = 0$, and (b) an essential singularity of $z \mapsto \exp(-1/z^2)$ at 0 as well.

The Uniqueness Principle (Theorem B.0.6) suggests that an analytic function is completely determined by its Laurent series at an isolated singularity z_0. A considerable amount of information is contained in a single coefficient:

Definition 2.4.2. Let $z_0 \in \mathbb{C}$ be an isolated singularity of f. Coefficient c_{-1} of Laurent series (2.44) is called the *residue* of f at z_0:
$\operatorname{Res}(f, z_0) = c_{-1}$.

When f is analytic in a punctured disk $\dot{D} = D\backslash\{z_0\}$, then using formula (2.46) for Laurent series coefficients we get

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i}\oint_\gamma f(z)\mathrm{d}z \tag{2.50}$$

where γ is a closed loop with $\text{wind}(\gamma, z_0) = 1$ in $\dot{D}$. Hence, the name "residue" which comes from "what is left" after integrating along a path that winds exactly once around an isolated singularity z_0. Recall the example of the inversion map $z \mapsto 1/z$. Integrating along a unit circle we get $2\pi i$ which when substituted into (2.50) gives 1. The residue at a removable singularity, of course, vanishes.

If z_0 is a *simple pole* (a pole with order 1) the residue can be easily computed by

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) \tag{2.51}$$

If f has a pole of order n at z_0, then function g such that $g(z) = (z - z_0)^n f(z)$ has a removable singularity at z_0, and its $(n-1)$-th derivative satisfies $g^{(n-1)}(z_0) = (n-1)!\ c_{-1}$, which yields:

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \tag{2.52}$$

Now we extend the generalized Cauchy Integral Theorem (Corollary 2.3.4) for analytic functions with isolated singularities:

Theorem 2.4.2. *(Residue Theorem): Let $\Omega \subseteq \mathbb{C}$ be an open set, and $S = \{z_1, z_2, ..., z_n\} \subset \Omega$, and assume $f : \Omega \backslash S \to \mathbb{C}$ is analytic. Then for any cycle in $\Omega \backslash S$ which is null-homologous in Ω:*

$$\frac{1}{2\pi i} \int_\Gamma f(z) dz = \sum_{k=1}^{n} \mathit{wind}(\Gamma, z_k)\ \mathit{Res}(f, z_k) \tag{2.53}$$

Proof.
The cycle Γ is null-homologous in Ω, but not in $\Omega \backslash S$, so we complement Γ by adding paths encircling the singularities. The new cycle $\widetilde{\Gamma}$ is null-homologous in $\Omega \backslash S$. Since all $z_k \in \Omega$, find pairwise disjoint closed disks with centers in the singularities and with sufficiently small radii r_k. For $k = 1, ..., n$ define paths $\gamma_k : [0, 1] \to \Omega : t \mapsto r_k \mathrm{e}^{-2n_k it}$ where $n_k = \text{wind}(\Gamma, z_k)$. Then $\widetilde{\Gamma} = \Gamma + \gamma_1 + ... + \gamma_n$ is a null-homologous cycle in $\Omega \backslash S$. Then for all $z \in \mathbb{C} \backslash \Omega$: $\text{wind}(\gamma_k, z) = 0$ because all γ_k are null-homotopic to individual points z_k in $\mathbb{C} \backslash \{z\}$. Thus $\text{wind}(\widetilde{\Gamma}, z) = \text{wind}(\Gamma, z)$. If $z = z_j \in S$ then

$$\text{wind}(\gamma_k, z_j) = \begin{cases} 0 & \text{for} \quad k \neq j \\ -n_j & \text{for} \quad k = j \end{cases}$$

so $\text{wind}(\widetilde{\Gamma}, z_j) = \text{wind}(\Gamma, z_j) + \text{wind}(\gamma_j, z_j) = n_j - n_j = 0$. Then using the integral definition of the residue (2.50), we get that the integral of f along γ_j is just $n_j \text{Res}(f, z_j)$. The result then follows from the Cauchy Integral Theorem. □

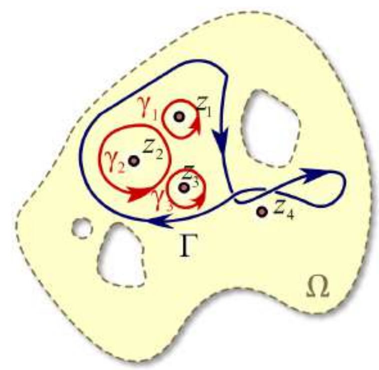


Figure 2.14: Complementing a cycle Γ null-homologous in Ω to a cycle $\widetilde{\Gamma}$ null-homologous in $\Omega \backslash \{z_1, z_2, z_3, z_4\}$.

The Residue Theorem is not only useful in complex analysis. It has applications in real analysis as well. In particular, it is a powerful tool for evaluating integrals of rational trigonometric functions (see Wegert [2], p.187, Theorem 4.5.4).

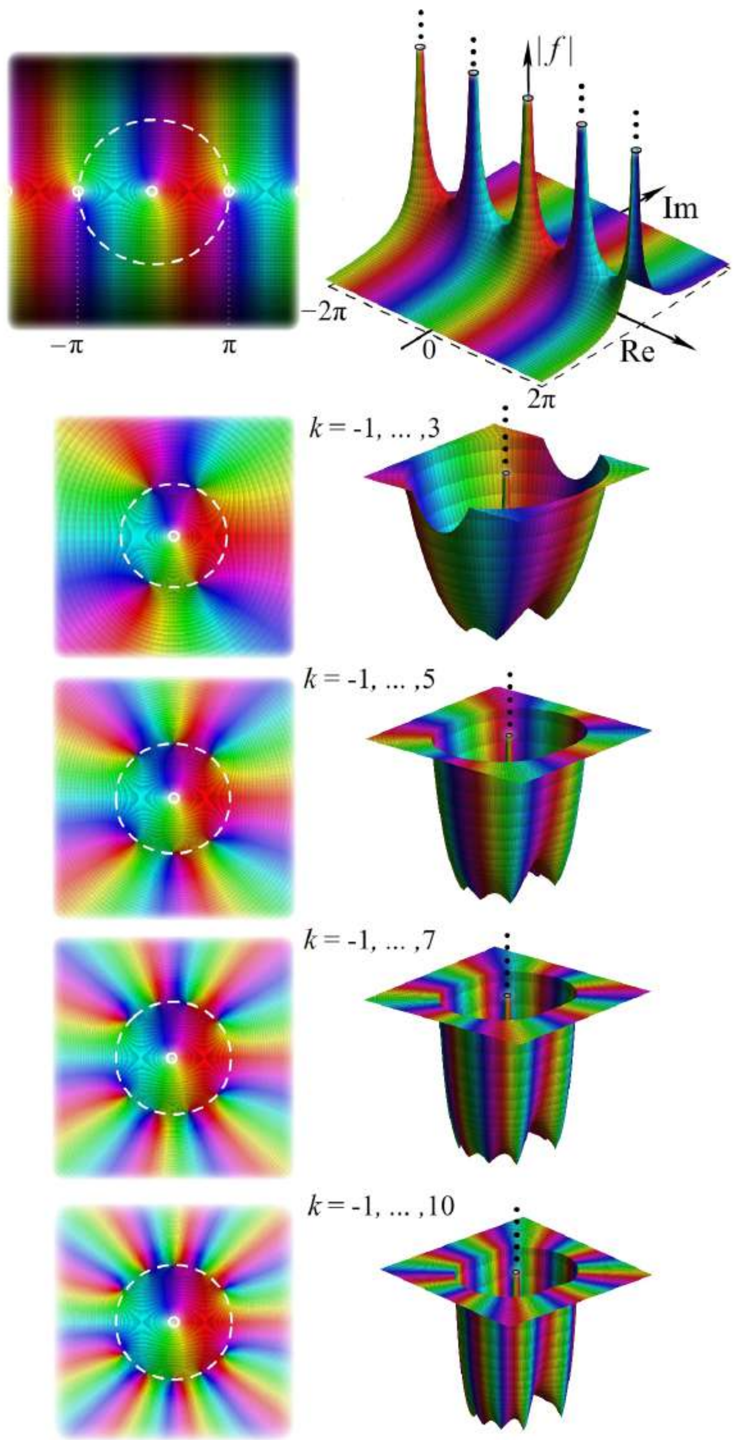


Figure 2.15: An enhanced phase portrait with domain coloring (moduli are tinted black when close to zero, and white when approaching infinity) (top left) and an *analytic landscape* depicting modulus in the z-coordinate (top right). All of which corresponding to a function $f : z \mapsto \frac{1}{\sin z}$ with singularities $\{z_m = m\pi, m \in \mathbb{Z}\}$. In the images below one can see partial sums of a Laurent series expansion around $z_0 = 0$ with outer radius of convergence $R_1 = \pi$ and inner radius $R_0 = 0$.

Chapter 3

Harmonic Functions and Complex Potential Flow

3.1 Conjugate Harmonic Functions

One of many practical applications of complex analysis (especially in physics and engineering) is its wide overlap with the theory of partial differential equations. In general, it is not easy to solve *boundary value problems* on various domains. Specifically, equations of second and higher order usually do not have a solution that can be explicitly written. Many such tasks require the use of numerical methods that may (or may not) properly converge. A fortunate exception is a set of problems governed by the *Laplace equation*: $\Delta u = 0$.

Looking back to the properties of analytic functions on connected subsets of $\mathbb{C}$, one immediately realizes that properties like the Uniqueness Principle (Theorem B.0.6) may turn out to be essential for finding solutions to boundary value problems.

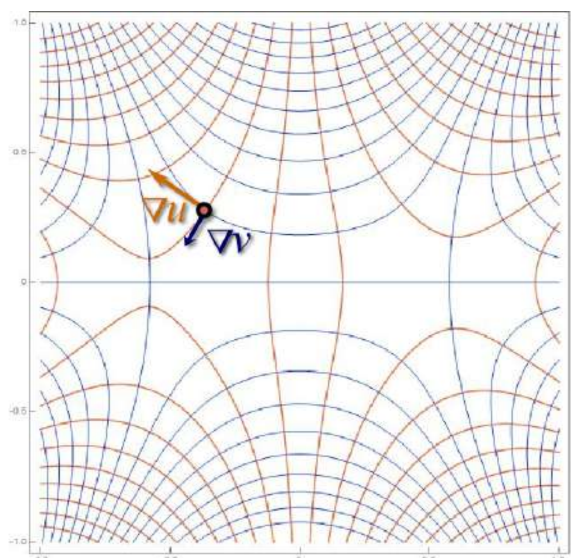


Figure 3.1: Level contours of the real and the imaginary parts of $f : z \mapsto z^3 - z + 1$.

Definition 3.1.1. A function $u : \Omega \to \mathbb{R}$ is called *harmonic* in $\Omega \subseteq \mathbb{R}^2$ if $u \in C^2(\Omega)$ (it is twice continuously differentiable) and satisfies the *Laplace equation*:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3.1}$$

Once again, the reach of properties like differentiability and analyticity, intimately related as they are, extends to other areas, more specifically to boundary value problems on subsets of $\mathbb{R}^2$ through Cauchy-Riemann equations (2.2). The astoundingly simple relationship is described by the following theorem:

Theorem 3.1.1. *Let $f : \Omega \to \mathbb{C}$ be analytic in an open set $\Omega \subseteq \mathbb{C}$, then $u = Re(f)$ and $v = Im(f)$ are harmonic functions on Ω.*

Proof. The analyticity of f guarantees the existence of continuous derivatives of all orders, therefore it only needs to be shown that u and v satisfy the Laplace equation (3.1). Using the Cauchy-Riemann equations (2.2) on u we get:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

And the same holds for v. □

By Theorem 3.1.1, any analytic function produces two real harmonic functions coupled by the Cauchy-Riemann equations. If u and v are such harmonic functions, then they are called *conjugate harmonic*. More specifically, v is conjugate harmonic to u and u is conjugate harmonic to $-v$.

Given that u and v are coupled by Cauchy-Riemann equations we find out that the sets of contour lines: $\{(x,y) \in \mathbb{R}^2 \mid u(x,y) = \text{const.}\}$ and $\{(x,y) \in \mathbb{R}^2 \mid v(x,y) = \text{const.}\}$ have an interesting property. From vector calculus we know that the gradients ∇u and ∇v are essentially vector fields in Ω that point in the direction of the "steepest ascent" of the function. For conjugate harmonic functions, these gradients are perpendicular to each other everywhere, and the following holds:

Theorem 3.1.2. *If u and v are conjugate harmonic and have non-vanishing gradients, then their contour lines are mutually orthogonal.*

Proof. Let $\nabla u = \left(\frac{\partial u}{\partial x}\ ,\ \frac{\partial u}{\partial y}\right)$ and $\nabla v = \left(\frac{\partial v}{\partial x}\ ,\ \frac{\partial v}{\partial y}\right)$. Then

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0\ .$$

And since gradients are orthogonal to the tangents of the contour lines, the contour lines are mutually orthogonal. □

Take, for example, $f : z \mapsto z^3 - z + 1\ ,\ z \in \mathbb{C}$, with $u : (x,y) \mapsto 1 - x + x^3 - 3xy^2$ and $v : (x,y) \mapsto 3x^2y - y - y^3$, $(x,y) \in \mathbb{R}^2$. Then $(\nabla u \cdot \nabla v)(x,y) = \left(3x^2 - 1 - 3y^2\ ,\ -6xy\right) \cdot \left(6xy\ ,\ 3x^2 - 1 - 3y^2\right) = 0$. Taking a look at Fig. 3.1, examining the contours of u and v we notice that the gradient ∇u , for instance, forms a direction field that (after integration) gives rise to a family of (integral) curves that are identical to the contour lines of v.

The converse of Theorem 3.1.2, however, is not true. Despite $u : (x,y) \mapsto x$ and $v : (x,y) \mapsto 2y$ satisfying the gradient orthogonality: $(\nabla u \cdot \nabla v)(x,y) = (1,0) \cdot (0,2) = 0$, these functions are not conjugate harmonic. It suffices to see that they do not satisfy the Cauchy-Riemann equations.

On the other hand, the idea that gradient orthogonality may imply that u and v are conjugate harmonic, should not be completely dispensed with. The gradients ∇u and ∇v may be orthogonal, but their lengths may vary in such way that they do not satisfy the Cauchy-Riemann equations. After only a minor adjustment, the following holds:

Theorem 3.1.3. *Let u and v be real-valued harmonic functions with non-vanishing gradients in Ω. If the contour lines of u and v are mutually orthogonal, then there exists $c \in \mathbb{R}$ such that cv is conjugate harmonic to u, that is: $u + icv$ is analytic in the complex domain Ω.*

Proof. By assumption, u and v are from $C^2(\Omega)$ and $\nabla u \cdot \nabla v = 0$, so ∇u (for example) is obtained by the rotation (with angle $\pi/2$) and scaling of ∇v, that is: there exists $c : \Omega \to \mathbb{R}\backslash\{0\}$ such that

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = -ci\left(\frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y}\right)$$

Separating the real and imaginary parts we get:

$$\frac{\partial u}{\partial x} = c\frac{\partial v}{\partial y}\ ,\ \frac{\partial u}{\partial y} = -c\frac{\partial v}{\partial x} \tag{3.2}$$

When differentiating the first equation in (3.2) with respect to x and the second equation with respect to y, and adding up the results, we get

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial c}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial c}{\partial y}\frac{\partial v}{\partial x} = \nabla c \times \nabla v \tag{3.3}$$

which means that the gradients ∇c and ∇v are parallel in Ω. Using the same process (assuming $c \neq 0$) on equations (3.2) for v (using its harmonicity):

$$\frac{\partial}{\partial y}\left(\frac{1}{c}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 v}{\partial y^2}\ ,\ \frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 v}{\partial x^2} \tag{3.4}$$

and subtracting the equations

$$0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{1}{c}\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{1}{c}\frac{\partial u}{\partial y}\right) =$$

$$= -\frac{1}{c^2}\frac{\partial c}{\partial y}\frac{\partial u}{\partial x} + \frac{1}{c}\frac{\partial^2 u}{\partial y \partial x} + \frac{1}{c^2}\frac{\partial c}{\partial x}\frac{\partial u}{\partial y} - \frac{1}{c}\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{c^2}\left(\frac{\partial c}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial c}{\partial y}\frac{\partial u}{\partial x}\right) = \frac{1}{c^2}\nabla c \times \nabla u \tag{3.5}$$

Since gradients ∇c and $\nabla(1/c) = (1/c^2)\nabla c$ are parallel and $\mathbb{R}^2 = \mathrm{span}\{\nabla u, \nabla v\}$, ∇c is vanishing on Ω, and thus c is a constant. □

The following result answers the question: which harmonic functions have conjugates?

Theorem 3.1.4. *Any harmonic function $u : \Omega \to \mathbb{R}$ on a simply connected domain Ω has a harmonic conjugate v on Ω. Moreover, v is uniquely determined up to an additive constant.*

Proof. First we need to show the existence and uniqueness of a harmonic conjugate v on a disk $D_0 \subseteq \Omega$. Using the harmonicity of v and the Cauchy-Riemann equations (2.2) define:

$$v(z) = \int_\gamma \frac{\partial u}{\partial x}(z)\mathrm{d}y - \int_\gamma \frac{\partial u}{\partial y}(z)\mathrm{d}x \tag{3.6}$$

where γ is a path from $z_0 = x_0 + iy_0$ to any $z = x + iy \in \Omega$. Then $f_0 = u + iv$ is analytic on D_0. Since the Cauchy-Riemann equations determine the gradient of the conjugate harmonic function v, any two such functions differ at most by a constant.

If D_0 and D_1 are two overlapping disks in Ω, the corresponding analytic functions f_0 and f_1 constructed earlier by (3.6) can be given in such way that $f_0 = f_1$ on $D_0 \cap D_1$. f_0 then automatically has an unrestricted analytic continuation in Ω and by the Monodromy Principle II (Theorem B.0.14) it generates an analytic function f in Ω. Naturally, the imaginary part of f is a harmonic conjugate of u. □

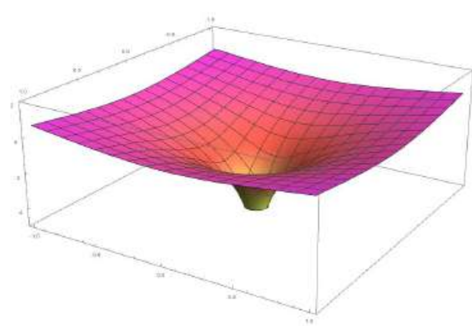


Figure 3.2: Logarithmic potential $u : (x, y) \mapsto \log|x + iy| = \log(x^2 + y^2)$.

Despite the fact that most harmonic functions have conjugates, it is not always the case. Since Theorem 3.1.4 implies that the existence of a harmonic conjugate ultimately depends on the topology (simple-connectedness) of the domain, one can easily find exceptions.

The *logarithmic potential*: $u : \mathbb{R}^2\backslash\{0\} \to \mathbb{R} : (x, y) \mapsto \log|x + iy|$ is an example of a harmonic function that *does not have a global harmonic conjugate.* The function is harmonic in $\mathbb{C}\backslash\{0\}$ since it is the real part of any branch of $z \mapsto \log z$. Assuming that a harmonic conjugate v exist on the domain, it can be normalized so that $v(1) = 0$. Then $u + iv$ is analytic on $\mathbb{C}\backslash\{0\}$ and coincides with the principal branch of the Log function in a neighborhood of $z_0 = 1$. However, such function is discontinuous at $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ (to see why, take a look at the definition of a global logarithm in Appendix B).

3.2 Complex Potential and Ideal Flow

The Laplace equation (3.1) (with its many variants) occurs in a wide range of engineering applications, from plane electrostatics to hydrodynamics. In the physics of each model, vector quantities like force or velocity usually have scalar counterparts that represent the *potential.* In these specific formulations, the vector fields are *conservative*, meaning that their integrals are path-independent. Reflecting the real world, potential field formulations of given models assume that scalar quantities like energy are always balanced out (conserved) by some form of potential, so that the overall sum of energies is constant at all times. So for example, a (charged) particle in a force field like the electrostatic field gains kinetic energy as it plummets down the potential scalar field. It is the simplest (and still the most effective) formulation of action at a distance in such natural phenomena.

The electric field $\boldsymbol{E} = (E_x, E_y)$ in a region containing no charges is *irrotational*: $\nabla \times \boldsymbol{E} = \mathbf{0}$ and has zero divergence $\nabla \cdot \boldsymbol{E} = 0$. In two dimensions these conditions correspond to

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \ , \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

which are essentially the Cauchy-Riemann equations of an analytic function $f = E_x - iE_y$. This formulation using the *complex electric field* is, however, seldom used to model problems in plane electrostatics. Instead, a real-valued scalar field ϕ, that is: the *electrostatic potential*, is used and then the electric (vector) field is $\boldsymbol{E} = -\nabla\phi$ (the field points in the direction of the *steepest descent*). The local existence of ϕ is guaranteed by $\nabla \times \boldsymbol{E} = \mathbf{0}$ while $\nabla \cdot \boldsymbol{E} = 0$ implies that the potential is harmonic: $\Delta\phi = \nabla \cdot \nabla\phi = -\nabla \cdot \boldsymbol{E} = 0$. The contour lines of ϕ are called *equi-potential lines*, and since the negative gradient of ϕ is normal to them, the contours of the harmonic conjugate ψ to ϕ (called the *field lines*) are orthogonal to the equi-potential lines at every point. The analytic function $F = \phi + i\psi$ is said to be a *complex potential* of $\boldsymbol{E}$ since

$$E_x = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} \ , \quad E_y = -\frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

the complex electric field $f = E_x - iE_y$ is then the negative derivative of the complex potential: $f = -F'$.

Taking the *inverse square law*[1] into account, it is, in fact, the logarithmic potential which is used to describe the electrostatic field around a charged particle. Up to physical constants, the potential of a positive unit charge (at the origin) is described by a potential $\phi : z \mapsto -\log|z|$. Its potential lines are concentric circles. It is the real part of any branch of $-\log(z) = -\log|z| - i\mathrm{Arg}(z)$, and hence $\psi : z \mapsto -\mathrm{Arg}(z)$ is a (local) harmonic conjugate of ϕ. The field lines $\psi =$ const. are radial rays emerging from the origin. Unlike the complex potential $F : z \mapsto -\log z$, the complex electric field $f : z \mapsto -F'(z) = 1/z$ is well defined in all of $\mathbb{C}\backslash\{0\}$. Let $g = \exp(\phi + i\psi)$. Since $\phi = c$ and $\log|g(z)| = c$ are equivalent, the modulus contour lines in the enhanced phase portrait of g coincide with the potential lines, and because the isochromatic lines (where $\mathrm{Arg}(z) =$ const.) are orthogonal to the lines with constant modulus, they are the field lines of f (see Figure 3.3 (a)).

[1] the field intensity decreases with the inverse square of the distance from the origin.

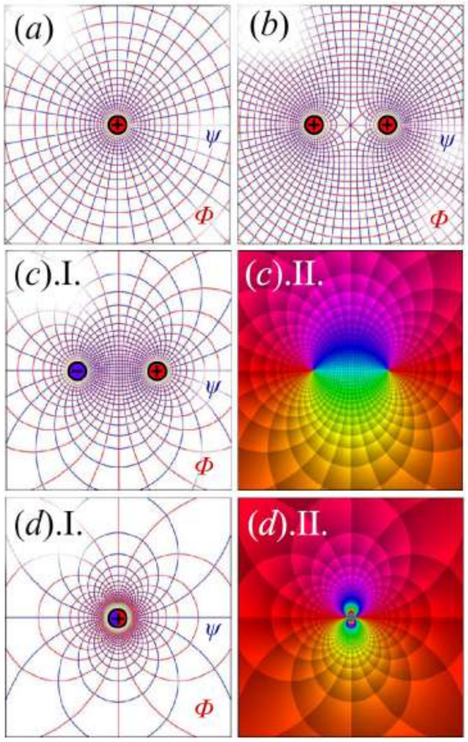


Figure 3.3: Equipotential lines (red) and field lines (blue) of different configurations of point charges, with enhanced phase portraits.

When multiple charges are interacting, the resulting potential is the sum of their individual potentials: $\phi(z) = \sum_{i=0}^{n} q_i \log|z - z_i|$, where q_i are charges of the individual particles. The potential lines are then the modulus contour lines of a rational function $g : \mathbb{C}\backslash\{z_{m+1}, ..., z_n\} \to \mathbb{C}$ such that:

$$g(z) = \exp(\phi(z) + i\psi(z)) = \frac{(z - z_1)...(z - z_m)}{(z - z_{m+1})...(z - z_n)}$$

Denote $d \in \mathbb{R}^+$ the distances of two opposite point charges from the origin (as in Fig. 3.3 (c)), that is $z_1 = -d$ and $z_2 = d$. When these charges approach each other, the potential ϕ needs to be scaled up by the reciprocal distance $1/(2d)$ in order to obtain a limit:

$$\phi(z) = \lim_{d \to 0} \frac{1}{2d} \log\left|\frac{z + d}{z - d}\right|$$

Instead of determining this limit, one can find the corresponding limit of the complex electric field:

$$f(z) = \lim_{d \to 0} \frac{1}{2d}\left(\frac{1}{z - d} - \frac{1}{z + d}\right) = \lim_{d \to 0} \frac{1}{(z - d)(z + d)} = \frac{1}{z^2}$$

and from $F' = -f$ get the complex potential $F : z \mapsto \phi(z) + i\psi(z) = 1/z$. The contour lines of ϕ and ψ then correspond to the modulus and isochromatic lines of $g : z \mapsto \exp(1/z)$ (see Fig.3.3 (d)). This particular configuration is called an *electric dipole*.

A similar formulation applies when describing *incompressible* and *irrotational planar flow*, that is if $\boldsymbol{v} : \Omega \mapsto \mathbb{R}^2$ is the *velocity vector field* describing the instantaneous velocity (of any particle) at $\boldsymbol{x} \in \Omega$ (also called the *Eulerian description of flow*), then $\nabla \cdot \boldsymbol{v} = 0$ (incompressibility) and $\nabla \times \boldsymbol{v} = \mathbf{0}$ (irrotationality). The velocity vector field $\boldsymbol{v}$ can then be written as the gradient of some scalar function $\phi : \Omega \to \mathbb{R}$ referred to as *potential*. The incompressibility condition then gives $\nabla \cdot \boldsymbol{v} = \nabla \cdot \nabla\phi = \nabla^2\phi = \Delta\phi = 0$. So the flow potential is the solution of the Laplace equation.

In general, fluid flow problems are difficult to solve, especially for fluids with non-zero viscosity. The whole problem is then governed by a system of tensor partial differential equations, also known as the *Navier-Stokes equations*. In practice, these equations are usually solved using numerical methods, and because some types of viscous flow might involve turbulence[2], one might even encounter difficulties with numerical solutions when dealing with such situations. In order to make use of the harmonic function theory, we will use the following simplifications of the flow:

(1.) zero viscosity
(2.) incompressibility: $\nabla \cdot \boldsymbol{v} = 0$.
(3.) irrotationality: $\nabla \times \boldsymbol{v} = \mathbf{0}$.
(4.) steady-state: that is, quantities like $\boldsymbol{v}$, ρ (density) and the potential ϕ are constant with respect to time.

[2]the viscous forces that arise in regions with larger velocity gradient $\nabla\boldsymbol{v}$ cause the formation of vortices which "destabilize" the flow. The point at which this happens is usually characterized by the dimensionless *Reynolds number*: $\mathrm{Re} > 4000$.

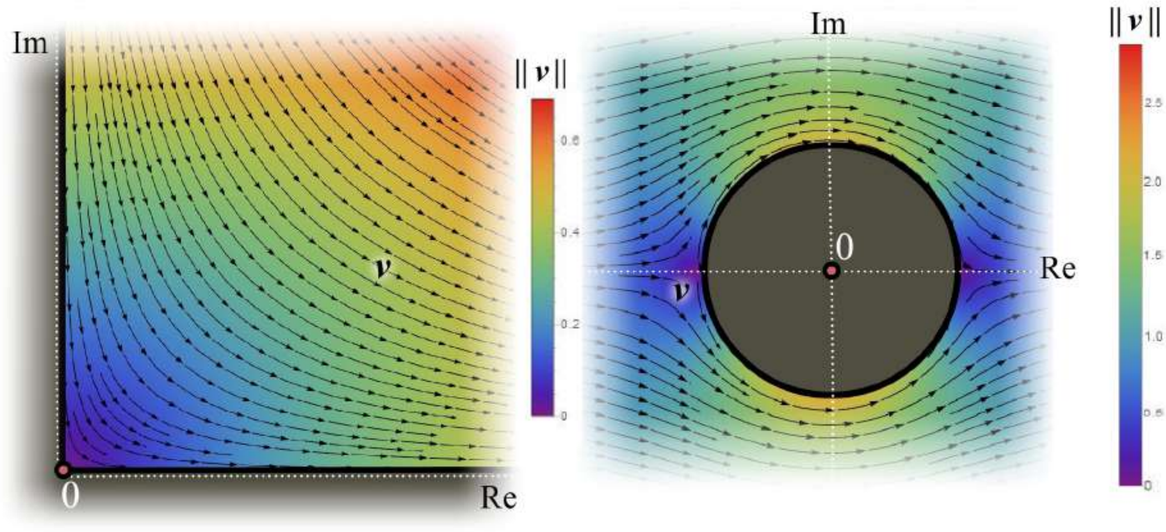


Figure 3.4: complex flow inside a corner (left) and around a disk (right) with velocity magnitude $||\boldsymbol{v}||$ depicted using color.

Neglecting the effects of viscosity, the Navier-Stokes equations reduce to the *Euler equation*:

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \boldsymbol{g} - \nabla p \ , \ \text{ where } \ \frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} \text{ (a material derivative)} \tag{3.7}$$

and where $p : \Omega \to \mathbb{R}$ is the pressure scalar field and $\boldsymbol{g}$ the vector field of gravitational acceleration. The Euler equation (3.7) describes the conservation of momentum in the system. Using identity $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \frac{1}{2}\nabla(\boldsymbol{v} \cdot \boldsymbol{v}) - \boldsymbol{v} \cdot \nabla \boldsymbol{v}$ write

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\nabla(\boldsymbol{v} \cdot \boldsymbol{v}) - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})\right) = \rho \boldsymbol{g} - \nabla p$$

and assuming irrotationality $\nabla \times \boldsymbol{v} = \boldsymbol{0}$ as well as $\boldsymbol{v} = \nabla \phi$:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{g} \cdot \boldsymbol{x}\right)$$

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{g} \cdot \boldsymbol{x}\right) = 0$$

and for steady state $\partial\phi/\partial t = 0$ we get

$$\frac{p}{\rho} + \frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v} - \boldsymbol{g} \cdot \boldsymbol{x} = \text{const.} \tag{3.8}$$

which is the well known *Bernoulli equation* for the conservation of energy. Notice that regions with lower dynamic pressure are also regions with higher flow velocity, because the term $p/\rho - \boldsymbol{g} \cdot \boldsymbol{x}$ has to be balanced out by the *kinetic energy per unit of density* term $\frac{1}{2}\boldsymbol{v} \cdot \boldsymbol{v}$ (see Fig.3.4).

Assuming (1.) - (4.) the flow is then called an *ideal flow*, and the assumptions are a preliminary to describing the system via complex analysis. For instance, the irrotationality and incompressibility conditions are linked by the Cauchy-Riemann equations on the complex domain Ω:

$$\frac{\partial v_y}{\partial x} = \frac{\partial v_x}{\partial y} \ , \quad \frac{\partial v_x}{\partial x} = -\frac{\partial v_y}{\partial y} \quad , \text{ where } \quad \boldsymbol{v} = (v_x, v_y) \tag{3.9}$$

corresponding to the *complex velocity* $f = v_x - iv_y$. A (local) primitive $F = \phi + i\psi$ of f is called a *complex velocity potential.* The derivative (gradient) of F gives

$$v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \ , \ v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad , \quad \boldsymbol{v} = \nabla \phi \tag{3.10}$$

As the real and imaginary parts of the analytic function F, functions ϕ and ψ are conjugate harmonic, and hence the gradient $\nabla\phi$ is tangent to the contour lines of ψ at every point in Ω. Hence a set of points where ψ = const. corresponds to the *stream lines* of the flow, and ϕ is called the *stream function*. So essentially the complex velocity conjugate gives rise to a system of ordinary differential equations:

$$\begin{array}{l} \frac{\mathrm{d}x}{\mathrm{d}t} = v_x(x,y) \\ \\ \frac{\mathrm{d}y}{\mathrm{d}t} = v_y(x,y) \end{array} \quad \Rightarrow \quad \frac{\mathrm{d}z}{\mathrm{d}t} = \overline{f(z)}$$

Solving the *initial value problem* of this system for some $z(t_0) = z_0$ gives the integral curve $z : [t_0, \infty[\to \mathbb{C} : t \mapsto z(t)$ which is the *trajectory* of a particle that happened to be at point z_0 when $t = t_0$. In Fig. 3.5 we can see the contour lines of the potential ϕ and the stream function ψ (on the left) of an ideal planar flow around the unit disk $\mathbb{D}_1$. The details of the expressions involved will be shown later on as we reveal more about *conformal mappings* and the *Riemann Mapping Theorem.*

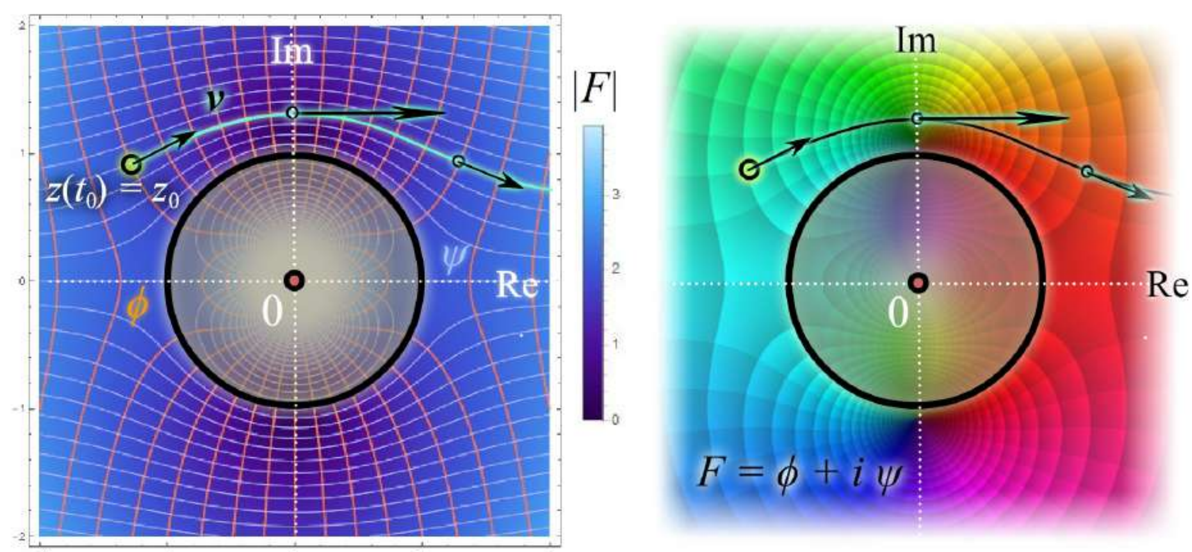


Figure 3.5: Flow around a disk modeled by a complex potential $F = \phi + i\psi$. One can see contour lines of the potential ϕ (orange) and of the stream function ψ (light blue), on the left. Taking $z(t_0) = z_0$ and integrating $\frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} - i\frac{\partial \psi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$ with respect to t one can then find the trajectory coincident with a stream line $\psi(z_0)$.

Let $F : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ such that

$$F(z) = v_\infty\left(z + \frac{1}{z}\right) \tag{3.11}$$

where v_∞ is the complex constant that corresponds to the (*asymptotic*) value of the complex potential at infinity. F is then the primitive of complex velocity $f = v_x - iv_y$ with:

$$f(z) = v_\infty\left(1 - \frac{1}{z^2}\right) \tag{3.12}$$

When integrating the velocity field (see Fig.3.4 on the right) with respect to time t from a given initial point z_0 we get the trajectory $z(t)$ of a particle which is coincident with a contour line of the stream function ψ at this point. Notice that the complex potential $F = \phi + i\psi$ has zeros at i and $-i$ and the instantaneous velocity of a particle increases as it approaches the zeros, which is what follows from the physical interpretation of the Bernoulli equation (3.8).

The complex potential F has a simple pole at $z = 0$, and the complex velocity f has a pole of order 2 at this point. This means that inside the unit disk $\mathbb{D}_1$, the velocity field will circulate with increasing magnitude closer to the origin, but since all of this is happening inside the disk, such behavior has no real physical interpretation and thus it is ignored. The boundary of a solid object has a *zero flux condition* which states that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ where $\boldsymbol{n}$ is the (outward) normal to the unit disk $\mathbb{D}_1$.

Points -1 and 1 are called the *stagnation points* of the flow described by f, these are the zeros of expression (3.12). Here, the velocity vector field $\boldsymbol{v}$ vanishes, and the fluid is stationary.

Another example is the flow inside a corner with internal angle α. In Fig.3.4 on the left we can see the resulting complex velocity field for $\alpha = \pi/2$. Generally (see Fig.3.7), the situation can be described using power functions $f : z \mapsto z^a$ where $a \in \mathbb{R}$. For $a \notin \mathbb{Z}$, however, (when $\alpha > \pi$) the stagnation point at $z = 0$ becomes an algebraic branch point. This means that the analytic continuation along any loop about 0 might be non-trivial (see Monodromy Principle II., Theorem B.0.14). The resulting branches $f_k = z \mapsto \exp\big(a\big(\log(z) + 2(k-1)\pi\big)\big)$ of the global analytic function can be combined to solve the given boundary value problem, which holds as long as one knows what results (of the velocity profile) to expect, of course.

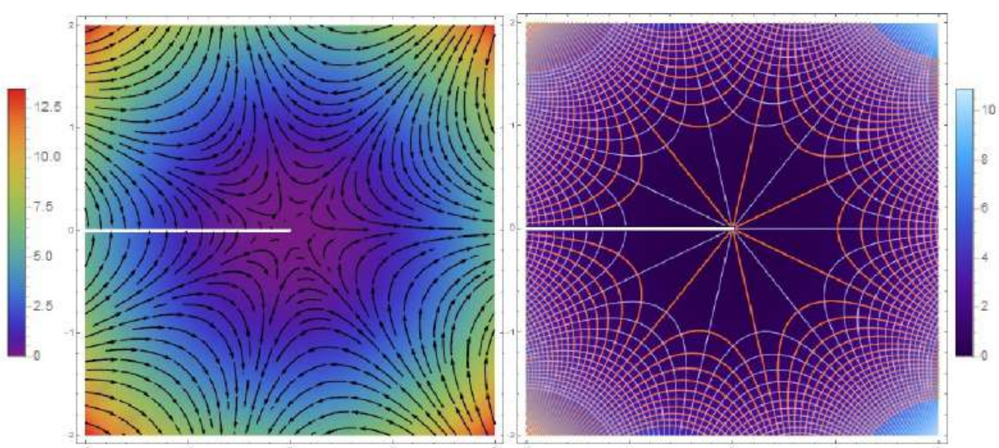


Figure 3.6: Complex velocity field (right) and complex potential (left) of k = 1st branch of a power function $f_k : z \mapsto \exp\big(2/3\big(\log(z) + 2(k-1)\pi\big)\big)$.

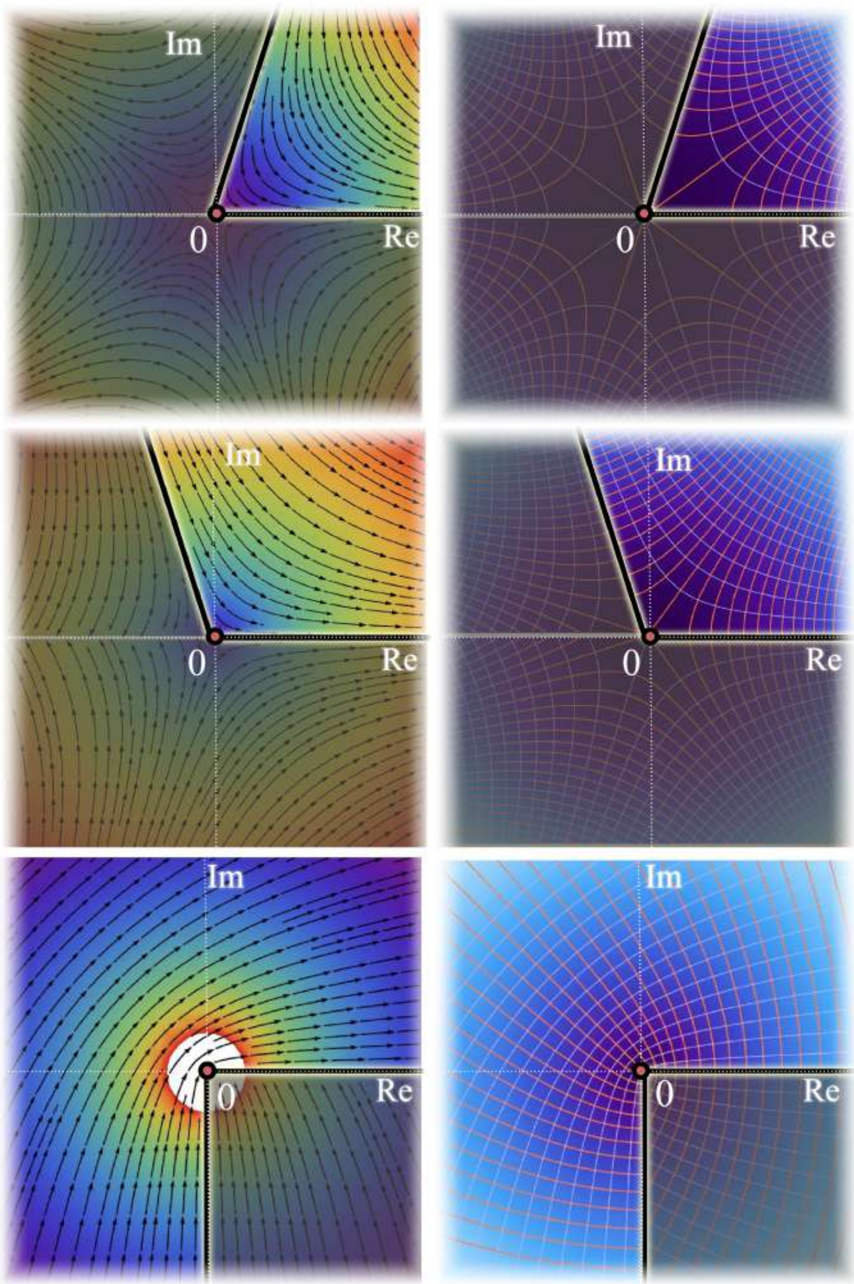


Figure 3.7: Potential flow around corners of various angles α. The complex velocity can be expressed as a global analytic function with branches $f_k : z \mapsto \exp\big(a\big(\log(z) + 2(k-1)\pi\big)\big)$, where $k \in \mathbb{Z}$ and $a \in \mathbb{R}$. Notice how for angles $\alpha > \pi/2$ the analytic continuation along a closed loop about $z = 0$ is non-trivial, and produces multiple valued functions. Combining branches f_1 and f_2 (in the bottom image with $a = -1/3$) gives a proper solution to the problem for $\alpha = 3\pi/2$. The white region in the last velocity stream function is due to range clipping of density plot in Mathematica.

Chapter 4

Conformal Mappings and the Riemann Mapping theorem

4.1 Transformations of Planar Domains

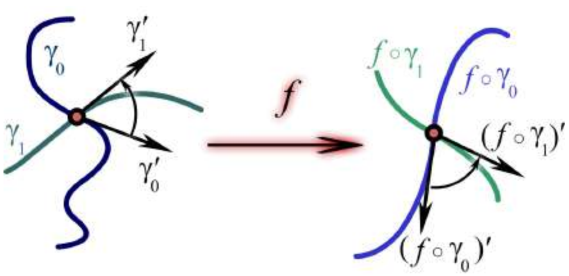


Figure 4.1: A conformal map preserving angles between paths.

In the previous chapter we stated the necessary conditions for flow described using the methods of complex analysis. The incompressibility, irrotationality, zero viscosity, and the steady-state condition reduce the complex potential and the complex velocity to locally *angle-preserving maps*. Indeed, one can easily see that zero curl $\nabla \times \boldsymbol{v} = \boldsymbol{0}$, combined with zero divergence $\nabla \cdot \boldsymbol{v} = 0$ imply the transformation $\boldsymbol{v}$ of a particular reference region Ω_0 does not locally shear two arbitrary adjacent vectors inside the region. Even though it might deform the entire set, it will balance out the twisting by a change in magnitude. This means that the velocity field $\boldsymbol{v}$ will always be tangent to the boundary or it will be vanishing (at the stagnation points).

Definition 4.1.1. Let $\gamma_0, \gamma_1 : [0,1] \to \Omega$ be two continuously differentiable paths in region Ω intersecting at some $z = \gamma_0(t) = \gamma_1(s)$, $t, s \in [0,1]$. A map $f : \Omega \to \mathbb{C}$ is called *conformal* if it preserves the angle $\mathrm{Arg}(\gamma_0'(t)) - \mathrm{Arg}(\gamma_1'(s))$ between γ_0 and γ_1 at z, that is: $\mathrm{Arg}((f \circ \gamma_0)'(t)) - \mathrm{Arg}((f \circ \gamma_1)'(s)) = \mathrm{Arg}(\gamma_0'(t)) - \mathrm{Arg}(\gamma_1'(s))$.

Recall that we stated the Cauchy-Riemann equations (2.2) as the necessary condition for complex differentiability. The fact that a complex-valued function f satisfies the Cauchy-Riemann equations at a given point, however, is not a *sufficient condition* for it to be complex-differentiable. Take for example $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = \begin{cases} 0 & \text{if } \mathrm{Re}(z) = 0 \text{ or } \mathrm{Im}(z) = 0 \text{ or } z = 0 \\ 1 & \text{otherwise} \end{cases}$$

Then f certainly satisfies the Cauchy-Riemann equations at $z = 0$, but it is not holomorphic at this point because it is not even continuous there. Hence we define a stronger condition:

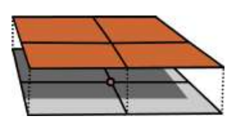


Figure 4.2: Real part of a function that satisfies the Cauchy-Riemann equation, but is not holomorphic.

Definition 4.1.2. $f : \Omega \to \mathbb{C}$ is said to be $\mathbb{R}$-*differentiable* at $z_0 \in \Omega$ if there exist $A, B \in \mathbb{C}$ such that $f(z) = f(z_0) + A(x - x_0) + B(y - y_0) + r(z)$, where $z = x + iy$, $z_0 = x_0 + iy_0$ and the remainder r satisfies $r(z) = \mathcal{O}(|z - z_0|)$, that is $r(z)/|z - z_0| \to 0$ as $z \to z_0$.

If f satisfies these conditions, it is also said to be *Fréchet differentiable* as a function of x and y. It is easy to see that if f is holomorphic (complex-differentiable) at z_0, then

$$f(z) = f(z_0) + \frac{\partial f}{\partial x}(z_0)(x - x_0) + \frac{\partial f}{\partial y}(z_0)(y - y_0) + \mathcal{O}(|z - z_0|) \tag{4.1}$$

and in the previous example we see that the ratio $r(z)/|z - z_0|$ does not tend to zero as $z \to z_0$.

(4.1) might suggest that f can locally act as a linear map (represented by a suitable matrix). The question that follows is: under what conditions is this map angle-preserving?

Lemma 4.1.1. *Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation represented by a matrix $\mathbf{A}$. Then A is conformal if and only if $\mathbf{A}$ is orthogonal with positive determinant.*

Proof. If A is conformal then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$:

$$\frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}||\ ||\mathbf{y}||} = \frac{\langle \mathbf{Ax}, \mathbf{Ay} \rangle}{||\mathbf{Ax}||\ ||\mathbf{Ay}||} \tag{4.2}$$

where $\langle .\ ,\ . \rangle$ denotes scalar (inner) product and $||.||$ the Euclidean norm. Now take the standard orthonormal basis $\{\mathbf{e}_i\}$. If A is angle-preserving, then for $i \neq j$

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0 \implies \langle \mathbf{Ae}_i, \mathbf{Ae}_j \rangle = 0$$

then

$$0 = \langle \mathbf{Ae}_i, \mathbf{Ae}_j \rangle = (\mathbf{Ae}_i)^\top \mathbf{Ae}_j = \mathbf{e}_i^\top \mathbf{A}^\top \mathbf{Ae}_j$$

which holds if and only if $\mathbf{A}^\top \mathbf{A} = \mathbf{I}$, so $\mathbf{A}$ must be orthogonal.

Since $\mathbf{A}$ is orthogonal: $\mathbf{A}^\top \mathbf{A} = \mathbf{A}\mathbf{A}^\top = \mathbf{I}$, then $1 = \det \mathbf{I} = \det(\mathbf{A}^\top \mathbf{A}) = \det(\mathbf{A}^\top)\det(\mathbf{A})$, and because for any matrix $\det(\mathbf{A}^\top) = \det(\mathbf{A})$ we just get: $(\det \mathbf{A})^2 = 1$, thus $\det(\mathbf{A}) = \pm 1$ for any orthogonal matrix. Moreover, if A not only preserves angles, but also their orientation, then $\det(\mathbf{A}) > 0$ (in which case $\det(\mathbf{A}) = 1$). ☐

From the proof of Lemma 4.1.1 we see that local orthogonality and positive determinant of a matrix ensure not only that the orthonormal basis preserves its orthonormality, but also its orientation under A. Transformations like reflection (with $\det(\mathbf{A}) = -1$) are orthogonal, but do not preserve angles with their orientation.

Now we use this information to see under what conditions mappings on the complex plane preserve angles:

Theorem 4.1.2. *Let $f : \Omega \to \mathbb{C}$ be an $\mathbb{R}$-differentiable map on $\Omega \subseteq \mathbb{C}$. Then f is conformal at $z \in \Omega$ if and only if it is holomorphic at z and $f'(z) \neq 0$. f is conformal on the entire domain Ω if and only if it is analytic and $f'(z) \neq 0$ for all $z \in \Omega$.*

Proof. Let $\gamma' = \xi + i\eta$ be a tangent vector of a differentiable path γ at some point $z = \gamma(t) \in \Omega$. Then by the chain rule: $(f \circ \gamma)'(t) = \left(\frac{\partial u}{\partial x}\xi + \frac{\partial u}{\partial y}\eta\right) + i\left(\frac{\partial v}{\partial x}\xi + \frac{\partial v}{\partial y}\eta\right)$, which can also be expressed as a linear map:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mathbf{J}_f \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{4.3}$$

where $\mathbf{J}_f$ is the Jacobian of f. Consequently, by Lemma 4.1.1: f is conformal if and only if $\mathbf{J}_f$ is orthogonal and $\det(\mathbf{J}_f) > 0$. This is, of course, true only when the Cauchy-Riemann equations (2.2) hold, so we can write

$$\det(\mathbf{J}_f) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \geq 0$$

and the determinant is positive if and only if $f'(z) \neq 0$. The (global) result then follows for all $z \in \Omega$. $\square$

Note that $\det(\mathbf{J}_f)$ represents the local scaling factor by which f expands or contracts the area measure of the region.

Among the simplest conformal maps $z \mapsto z^2$ stretches and twists the square grid as can be seen in Fig.2.2 and Fig.2.4. As one might expect, points on the imaginary axis will get mapped onto the negative real axis and the points on the negative real axis find themselves on the positive real axis. The conformality of the square map can be seen in the image of the square grid where mutually orthogonal lines get mapped onto curves that are locally orthogonal.

For an example of a map which is not conformal see Fig.4.3.

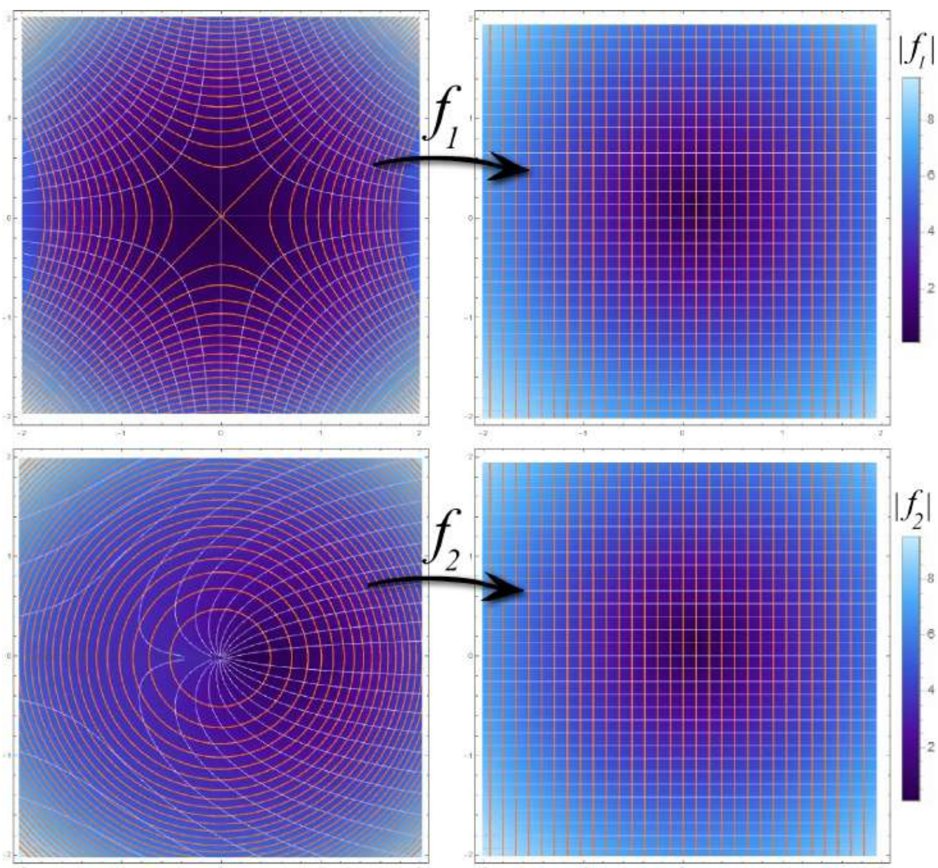


Figure 4.3: Transformation (*pull-back*) of the complex plane under $f_1 : z \mapsto z^2$ and a non-conformal map $f_2 : z \mapsto z\overline{z} + i(\mathrm{Arg}(z) + \mathrm{Im}(z))$.

So far, the definitions of conformality did not require the mapping to be injective. If $f : \Omega \to G$, where $G = f[\Omega]$ is injective, by the Open Mapping Principle (Theorem B.0.10) f must be a bijection.

Definition 4.1.3. A conformal injection $f : \Omega \to \mathbb{C}$ is called a *univalent conformal map*, and if there exists an analytic bijection $f : \Omega \to G$, where $G = f[\Omega]$, then domains Ω and G are said to be *conformally equivalent*. A *conformal automorphism* of Ω is a conformal bijection of Ω onto itself.

The simplest bijective maps on the complex plane are $z \mapsto az + b$, where $a \neq 0$ and $b \in \mathbb{C}$, all of which are compositions of *translation* (by a complex constant b), *rotation*, and *scaling* (by a non-zero complex constant $a = |a|e^{i\theta}$). Clearly any such affine function is a conformal automorphism of $\mathbb{C}$. Conversely if f is an automorphism of $\mathbb{C}$ then it has exactly one zero $z_0 \in \mathbb{C}$, so by Theorem B.0.4: $f(z) = (z - z_0)g(z)$ where g is analytic and $g(z_0) \neq 0$. Consequently there exists an open neighborhood $\mathcal{U}$ of z_0 where $1/g$ is bounded. By the Open Mapping Principle (Theorem B.0.10) $f[\mathcal{U}]$ covers a neighborhood $\mathcal{V}$ of 0 and since $f[\mathcal{U}] \cap f[\mathbb{C}\backslash\mathcal{U}] = \varnothing$, $1/f$ is bounded on $\mathbb{C}\backslash\mathcal{U}$. This means that $1/g$ is also bounded on $\mathbb{C}\backslash\mathcal{U}$ and thus on all of $\mathbb{C}$. By Liouville's theorem (Theorem 2.3.12) g must be constant. Naturally, if the "rotostretch" coefficient a of the *affine map* is not zero, g has to be a non-zero constant.

Let $f : z \mapsto \frac{az+b}{cz+d}$, where $ad - bc \neq 0$. These types of maps are called the *Möbius transformations*, and they are indeed conformal automorphisms of the *extended complex plane* $\widehat{\mathbb{C}} = \mathrm{cl}(\mathbb{C})$. An example of an enhanced phase portrait of such transformation can be seen on Fig.3.3 (c).II. where $z \mapsto (z-1)/(z+1)$ (also called the *Cayley map*).

To make matters more convenient, every Möbius transformation has an inverse $f^{-1} : z \mapsto \frac{dz-b}{a-cz}$ on the extended complex plane, so it maps $\widehat{\mathbb{C}}$ onto itself. Conversely if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a bijection, then by the Maximum Modulus Principle (Theorem B.0.7) it has at most one zero and one simple pole. If it neither has a zero nor a pole, it is constant, and if it only has a zero it is a affine map from the previous example.

A special type of Möbius transformations are *Blaschke factors*: $z \mapsto c\frac{z\ -\ z_0}{1\ -\ \overline{z}_0 z}$, where $|c| = 1$ (a unimodular constant) and $|z_0| < 1$. Not only do the Blaschke factors map $\widehat{\mathbb{C}}$ bijectively onto itself, but are also conformal

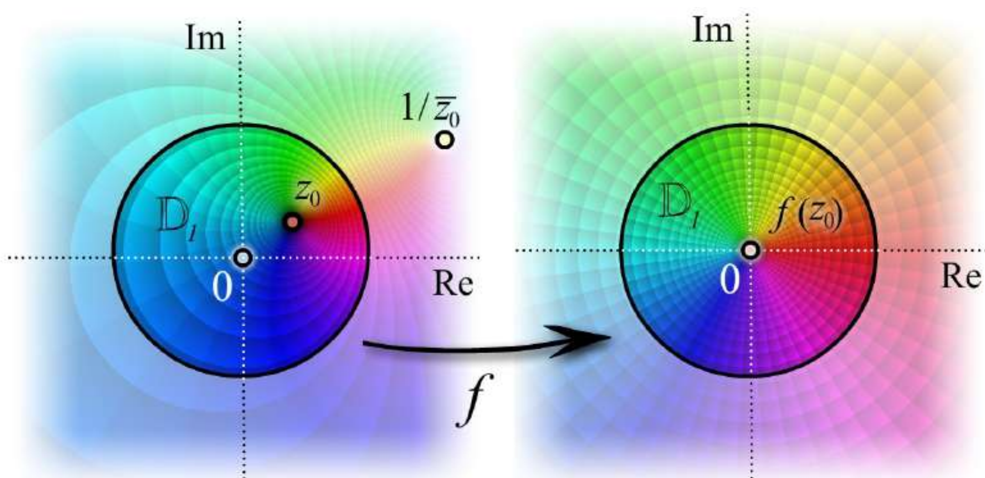


Figure 4.4: A conformal automorphism of the unit disk $\mathbb{D}_1$ by a Blaschke factor $f : z \mapsto c\frac{z-z_0}{1-\overline{z}_0 z}$ where $c = 1$ and $z_0 = 0.45 + 0.25i$.

automorphisms of the unit disk $\mathbb{D}$.

The inversion of a Blaschke factor is easily found as $f^{-1} : z \mapsto \overline{c}\frac{z + z_0}{1 + \overline{c}z_0}$. We see that for $z \in \mathbb{S}^1$ we get

$$f(z) = f(\mathrm{e}^{i\theta}) = c\frac{\mathrm{e}^{i\theta} - z_0}{1 - \overline{z}_0\mathrm{e}^{i\theta}} = c\ \mathrm{e}^{i\theta}\frac{1 - z_0\mathrm{e}^{-i\theta}}{\overline{1 - z_0\mathrm{e}^{-i\theta}}}$$

and $|f(z)| = 1$. Since f is analytic and non-constant in a neighborhood of $\mathbb{D}$, by the Maximum Modulus Principle (Theorem B.0.7): $|f(z)| < 1$ for all $z \in \mathbb{D}$. This means that f bijectively maps the unit disk onto itself because f is non-constant and has no maximum inside $\mathbb{D}$, and if it has any minimum in $\mathbb{D}$, it is the zero $z = z_0$.

Conversely any analytic bijection $g : \mathbb{D} \to \mathbb{D}$ is a Blaschke factor. To show this, let $z_0 = g^{-1}(0)$ and $f(z) = c\frac{z - z_0}{1 - \overline{z}_0 z}$ with $z_0 = 0$. Then $h = f \circ g^{-1}$ and $h^{-1} = g \circ f^{-1}$ are automorphisms of $\mathbb{D}$ which satisfy: $h(0) = h^{-1}(0) = 0$. By the Schwarz Lemma (Lemma B.0.11): $|h(z)| \leq |z| = |h^{-1}(h(z))| \leq |h(z)|$ which means that $|h(z)| = |z|$ for all $z \in \mathbb{D}$, and also $h(z) = cz$, where $|c| = 1$. Then $(f \circ g^{-1})(z) = h(z) = cz$ for all $z \in \mathbb{D}$ implies that $f(z) = cg(z)$ so g is indeed a Blaschke factor.

If $f = \prod_{k=1}^{n} f_k$ where f_k is a Blaschke factor, then f is called a *Blaschke product of order n*.

From the examples given in this section we can conclude that there are only three types of conformal automorphisms of domains $\mathbb{C}$, $\widehat{\mathbb{C}}$, and the unit disk $\mathbb{D}$:

Proposition 4.1.3. *If $f : \Omega \to \Omega$ is a conformal automorphism, then*

(I.) If $\Omega = \mathbb{C}$, then $f : z \mapsto az + b$, with $a \neq 0$ is a linear transformation

(II.) If $\Omega = \widehat{\mathbb{C}}$ (the extended complex plane), then $f : z \mapsto \frac{az + b}{cz + d}$ where $ad - bc \neq 0$ is a Möbius transformation.

(III.) If $\Omega = \mathbb{D}$ then $z \mapsto c\frac{z - z_0}{1 - \overline{z}_0 z}$, where $|c| = 1$ and $|z_0| < 1$ is a Blaschke factor.

Theorem 4.1.4. *Let $z_0 \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Then there exists exactly one conformal automorphism f of $\mathbb{D}$ such that*

$$f(z_0) = 0 \quad , \qquad \text{and } Arg(f'(z_0)) = \alpha \tag{4.4}$$

namely, the Blaschke factor $f : z \mapsto e^{i\alpha}\frac{z - z_0}{1 - \overline{z}_0 z}$, $|z_0| < 1$.

Proof. According to part (III.) of Proposition 4.1.3, any conformal automorphism with $f(z_0) = 0$ has the form of a Blaschke factor. Since $f'(z) = c/(1 - |z_0|^2)$ and $|c| = 1$, the condition $\mathrm{Arg}(f'(z_0)) = \alpha$ holds if and only if $c = \mathrm{e}^{i\alpha}$. □

We will proceed to show one more result, crucial for determining conformality of analytic functions. But first define:

Definition 4.1.4. An analytic function $f : \Omega \to \mathbb{C}$ is *locally injective* if for every $z \in \Omega$ there exists a neighborhood U of z such that $f : U \to \mathbb{C}$ is injective.

Proposition 4.1.5. *An analytic function $f : \Omega \to \mathbb{C}$ has a vanishing derivative at $z_0 \in \Omega$ if and only if it fails to be locally injective.*

Proof. Without loss of generality assume $z_0 = 0$ and $f(z_0) = 0$. Then by the Local Normal Form (Theorem B.0.4) there exists an analytic function $h : \Omega \to \mathbb{C}$ such that $f(z) = z^n h(z)$ with $n \in \mathbb{N}$ and any $z \in \Omega$. So there exists an open neighborhood U of 0 contained in Ω, such that $h(z) \neq 0$ for all $z \in U$. This implies that there exists an analytic function $g : U \to \mathbb{C}$ such that $g^n(z) = h(z)$ for all $z \in U$ (an analytic branch of $w \mapsto w^{1/n}$, for example). Thus $f(z) = z^n g^n(z)$, $z \in U$.

Note that $z \mapsto zg(z)$ has a non-vanishing derivative at 0 (because $g(0) = h^{1/n}(0) \neq 0$) so by the inverse function theorem, for small enough U, $z \mapsto zg(z)$ has an inverse on U. Which means that by the Argument Principle (Theorem B.0.8) $f : z \mapsto z^n g^n(z)$ is an n-to-one function on $U\backslash\{0\}$.

If, in fact, $f'(0) = 0$, by considering the power series (Taylor series, for example) of f, we have $n \geq 2$. This means that if f has a vanishing derivative, the multiplicity $n \geq 2$ implies that it fails to be locally injective at z_0. $\square$

The *complex exponential* $z \mapsto \mathrm{e}^z$, for example is locally injective for all $z \neq 0$. It maps rectangular domains $\{z \in \mathbb{C} \mid a < \mathrm{Re}(z) < b \;,\; c < \mathrm{Im}(z) < d\}$ onto annular sections $\{z \in \mathbb{C} \backslash \{0\} \mid \mathrm{e}^a < |z| < \mathrm{e}^b \;,\; c < \mathrm{Arg}(z) < d\}$. For $d - c > 2\pi$, the complex exponential fails to be injective.

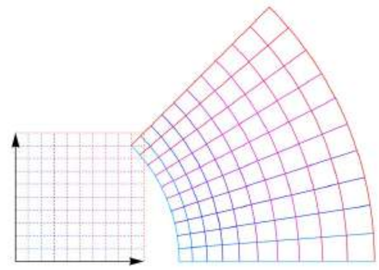


Figure 4.5: The image of $\{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < \pi/4 \;,\; 0 < \mathrm{Im}(z) < \pi/4\ \}$ by the complex exponential $z \mapsto \mathrm{e}^z$.

4.2 The Riemann Mapping Theorem

So far we have covered the elementary analytic function theory with a brief tour of harmonic functions and also specified conformal mappings of complex domains. Additional preliminaries, regarding notions like normal convergence and normal families of functions, for the proof of the *Riemann Mapping Theorem* can be found in Appendix C.

Now we proceed to show the core idea behind the methods of this work:

Theorem 4.2.1. *(The Riemann Mapping Theorem): Any simply-connected (proper) subset $\Omega \subset \mathbb{C}$ is conformally equivalent to the unit disk $\mathbb{D}$. That is: there exists a unique conformal analytic map $g : \Omega \to \mathbb{D}$ such that*

$$g(z_0) = 0 \;, \quad g'(z_0) > 0 \tag{4.5}$$

Proof.

1) First we want to verify that $\mathcal{F} = \big\{ f : \Omega \to \mathbb{D} \;\big|\, f \text{ is analytic, injective and } f(z_0) = 0 \big\} \neq \varnothing$ and is a normal family on Ω (see Def. C.0.3).

Choose a point $p \in \Omega \backslash \{z_0\}$ and show that $\mathcal{F}$ contains functions g, satisfying an *extremal condition*: $|g(p)| = \max \big\{ |f(p)| \;\big|\; f \in \mathcal{F} \big\}$. Afterwards we prove that any such extremal function g maps Ω conformally onto $\mathbb{D}$.

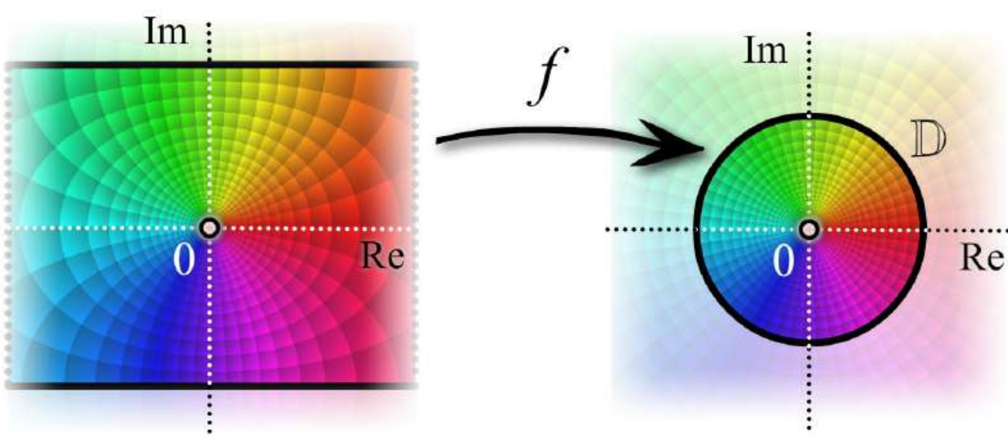


Figure 4.6: A conformal map of $\{z \in \mathbb{C} \mid -\pi/2 < \mathrm{Im}(z) < \pi/2\}$ onto a unit disk by $f : z \mapsto \frac{\mathrm{e}^z - 1}{\mathrm{e}^z + 1}$.

2) In order to find out whether $\mathcal{F} \neq \varnothing$, we distinguish three cases (with increasing complexity):

(Case 1): If Ω is bounded, there exists an affine map $f : \Omega \to D \subseteq \mathcal{D} : z \mapsto az + b$ which maps Ω onto a subset of the unit disk, such that $f(z_0) = 0$ and $f \in \mathcal{F}$.

(Case 2): If $a \in \mathbb{C}\backslash\Omega$, the map $z \mapsto 1/(z-a)$ maps Ω onto a simply connected domain, which reduces the situation to Case 1.

(Case 3): In general, let $a \in \mathbb{C}\backslash\Omega$. Then $z \mapsto z - a$ is non-vanishing inside Ω, and since Ω is simply-connected, there exists an analytic branch of the logarithm $z \mapsto \log(z-a)$ on Ω (see Lemma B.0.15). Define $h : z \mapsto \exp\left(\frac{1}{2}\log(z-a)\right)$, then h is an analytic branch of the square root of $z - a$, that is: $h^2(z) = z - a$ for all $z \in \Omega$. Clearly h is injective, so it maps Ω conformally onto a simply-connected domain $h[\Omega]$.

Moreover $h(z) = -h(w)$ with some $z, w \in \Omega$ implies: $z - a = h^2(z) = (-h(w))^2 = w - a$ from which $z = w$, $h(z) = -h(z)$, $h(z) = 0$, and $a = z \in \Omega$, which is a contradiction with the assumption $a \notin \Omega$. Consequently $h[\Omega] \cap -h[\Omega] = \varnothing$ and because $-h[\Omega] \neq \varnothing$ is also simply-connected, $\mathbb{C}\backslash h[\Omega] \neq \varnothing$, so the situation reduces to Case 2.

3) Since $\mathcal{F} \neq \varnothing$, let $s = \sup\left\{|f(p)| \;\middle|\; f \in \mathcal{F}\right\} > 0$ and we can choose a sequence $\{f_n\} \subseteq \mathcal{F}$ with $|f_n(p)| \to s$. Functions f_n are *uniformly bounded*, that is: they satisfy the conditions (C.2) of Theorem C.0.4. Thus by Montel's Theorem (Theorem C.0.6), a subsequence of $\{f_n\}$ converges normally to an analytic function g on Ω. This limit function satisfies $g(z_0) = 0$, $|g(p)| = s$ and $|g(z)| \leq 1$ for all $z \in \Omega$. Because $s \neq 0$, by the Maximum Modulus Principle (Theorem B.0.7), g is non-constant and $|g(z)| < 1$ for all $z \in \Omega$. Moreover, using Corollary C.0.1 of Hurwitz' Theorem (Theorem C.0.3) we can easily conclude that g is injective (univalent). Thus we have shown that $g \in \mathcal{F}$ is an extremal function.

4) In order to verify that $g : \Omega \to \mathbb{D}$ is conformal we prove that any extremal function in $\mathcal{F}$ is surjective. The following construction shows that if $\mathcal{D} = g[\Omega]$ is a proper subset of $\mathbb{D}$, then g cannot be extremal, that is: there exists $f \in \mathcal{F}$ such that $|f(p)| > |g(p)|$ (see Fig.4.7 for illustration).

So assume that there is a point $a \in \mathbb{D}\backslash\mathcal{D}$. Then the Möbius transformation

$$\varphi_a : z \mapsto \frac{z-a}{1-\bar{a}z}\ , \quad z \in \mathcal{D}$$

maps $\mathcal{D}$ conformally onto a simply-connected domain $\mathcal{D}_a \subset \mathbb{D}$. Now since $\varphi_a(a) = 0$ we can conclude that $0 \notin \mathcal{D}_a$, and as in Case 3, choose an analytic branch σ of the square root, conformally mapping $\mathcal{D}_a$ onto a simply-connected domain $\mathcal{D}_b \subset \mathbb{D}$. It should be noted that there are exactly two such branches σ and $-\sigma$ (see (B.12) in Appendix B). Now set $b = \sigma(a)$ and consider another Möbius transformation

$$\varphi_b : z \mapsto \frac{z-b}{1-\bar{b}z}\ , \quad z \in \mathcal{D}_b$$

which maps $\mathcal{D}_b$ conformally onto a simply-connected domain $\mathcal{D}_c \subset \mathbb{D}$. The composition $\psi_c = \varphi_b \circ \sigma \circ \varphi_a$ maps $\mathcal{D}$ conformaly onto $\mathcal{D}_c$ and satisfies $\psi_c(0) = 0$, so $f = \psi_c \circ g \in \mathcal{F}$.

5) Now we need to prove that $|f(p)| > |g(p)|$ which shows that g is not extremal. The crucial observation is that $\psi_c^{-1} : \mathcal{D}_c \to \mathcal{D}$ can be extended to an analytic function $\Psi_c : \mathbb{D} \to \mathbb{D}$ with $\Psi_c(0) = 0$. Indeed, in $\psi_c^{-1} = \varphi_a^{-1} \circ \sigma^{-1} \circ \varphi_b^{-1}$ maps φ_a^{-1}, σ^{-1}, and φ_b^{-1} are the restrictions of

$$\Phi_a : z \mapsto \frac{z+a}{1+\bar{a}z}\ , \quad S : z \mapsto z^2\ , \quad \Phi_b : z \mapsto \frac{z+b}{1+\bar{b}z}\ ,\ z \in \mathbb{D}$$

on $\mathcal{D}_a$, $\mathcal{D}_b$, and $\mathcal{D}_c$ respectively. Consequently ψ_c^{-1} is the restriction onto $\mathcal{D}_c$ of a Blaschke product of order two, namely

$$\Phi_a \circ S \circ \Phi_b = \Psi_c : \mathcal{D}_c \to \mathbb{D} : z \mapsto (\Phi_a \circ S \circ \Phi_b)(z) = z\frac{z-c}{1-\bar{c}z}$$

where $c = \varphi_b(-b) = -2b/(1+|b|^2) \in \mathbb{D}$. Because Ψ_c maps $\mathbb{D}$ onto $\mathbb{D}$, satisfies $\Phi_c(0) = 0$, and Ψ_c is not a rotation, the Schwarz Lemma (Lemma B.0.11) tells us that $|\Psi_c(z)| < |z|$ for all $z \in \mathbb{D}\backslash\{0\}$. If we insert $z = f(p)$ we get

$$|\Psi_c(f(p))| < |f(p)| \quad \Longrightarrow \quad |g(p)| < |f(p)|.$$

6) Finally we show the uniqueness of the normalized mapping. Assume that g_1 and g_2 are two such conformal mappings that satisfy

$$g_i(z_0) = 0\ , \quad g_i'(z_0) > 0\ , \quad i \in \{1, 2\}$$

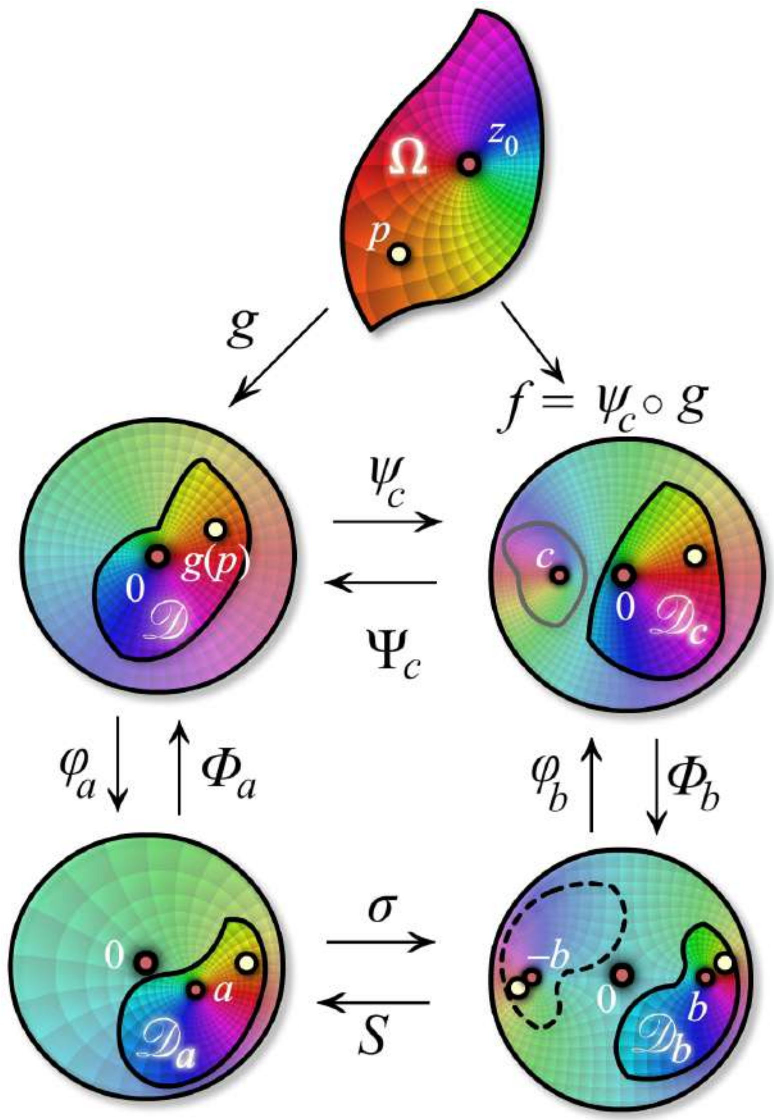


Figure 4.7: An illustration of the proof of Riemann Mapping theorem.

Then $h = g_1 \circ g_2^{-1}$ is a conformal automorphism of $\mathbb{D}$ with $h(0) = h^{-1}(0) = 0$ and $h'(0) > 0$. By Theorem 4.1.4, h is the identity map of $\mathbb{D}$, hence $g_1 = g_2$. $\square$

The methods of this proof were provided by a Greek mathematician Constantin Carathéodory (*1873 - †1950). The theorem itself is, of course, an extension of Riemann's original statement to domains with Jordan curves as boundaries. In some literature Theorem 4.2.1 is referred to as Carathéodory's theorem.

The consequences of this theorem are profound. Any domain which does not cover the entire complex plane can be conformally mapped onto a unit disk. And most boundary value problems on disk domains have been solved. It is only natural to devise a possible method of solving Dirichlet or Neumann problems on arbitrary simply-connected proper subsets of $\mathbb{C}$ by finding their unique conformal map onto the unit disk $\mathbb{D}$, utilize the existing solution for transformed boundary conditions, and transform the result with an inverse mapping g^{-1} back onto the original domain.

4.3 Dirichlet Problems on a Disk and the Poisson Integral

As the matter of fact, any harmonic function $u : \mathbb{D} \to \mathbb{R}$ can be expressed in terms of its values on the boundary. The underlying formula comes from Corollary 2.3.1, but the details of the proof rely on the theory of analytic functions and the existence of conformal automorphisms of the unit disk.

Consider a Möbius transformation

$$\varphi_z : w \mapsto \frac{z - w}{1 - \overline{z}w}\ , \quad w \in \mathbb{D} \tag{4.6}$$

Besides being an *involution* of the unit disk $\mathbb{D}$, that is: $\varphi_z \circ \varphi_z = \mathrm{id}_{\mathbb{D}}$, (4.6) has properties

$$\varphi_z'(w) = -\frac{1 - |z|^2}{(1 - \overline{z}w)^2}\ , \quad 1 - |\varphi_z(w)| = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2} \tag{4.7}$$

which can be easily verified. Also (4.6) is a Blaschke factor and thus an automorphism of the unit disk $\mathbb{D}$. Maps like these play a vital role in solving Dirichlet problems of the Laplace equation.

If u is harmonic, it is a component of an analytic function $f = u + iv$ on $\mathbb{D}$. Now consider the unit disk automorphism φ_z from (4.6) with $z = 0$. Then $\varphi_z(0) = 0$ and its inverse φ_z^{-1} exists and satisfies $\varphi_z^{-1}(0) = z$. So we can write $(f \circ \varphi_z^{-1})(0) = f(z)$. With the mean value approximation and the Cauchy Integral Formula (2.36) we get:

$$f(z) = (f \circ \varphi_z^{-1})(0) = \frac{1}{2\pi}\int_0^{2\pi} (f \circ \varphi_z^{-1})(\mathrm{e}^{i\theta})\mathrm{d}\theta = \frac{1}{2\pi i}\oint_{\partial\mathbb{D}=\mathbb{S}^1} \frac{(f \circ \varphi_z^{-1})(w)}{w}\mathrm{d}w\ , \quad w \in \mathbb{D}$$

and since φ_z is an automorphism of $\mathbb{D}$, we have $\varphi_z^{-1}[\mathbb{S}^1] = \mathbb{S}^1$, and

$$\frac{1}{2\pi i}\oint_{\partial\mathbb{D}=\mathbb{S}^1} \frac{(f \circ \varphi_z^{-1})(w)}{w}\mathrm{d}w = \frac{1}{2\pi i}\oint_{\varphi_z^{-1}[\mathbb{S}^1]=\mathbb{S}^1} \frac{(f \circ \varphi_z^{-1}) \circ \varphi_z(\zeta)}{\varphi_z(\zeta)}\varphi_z'(\zeta)\mathrm{d}\zeta = \frac{1}{2\pi i}\oint_{\mathbb{S}^1} \frac{f(\zeta)}{\varphi_z(\zeta)}\varphi_z'(\zeta)\mathrm{d}\zeta =$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\mathrm{e}^{i\theta})}{\varphi_z(\mathrm{e}^{i\theta})}\varphi_z'(\mathrm{e}^{i\theta})i\mathrm{e}^{i\theta}\mathrm{d}\theta \tag{4.8}$$

Now using the derivative formula in (4.7):

$$\frac{\varphi_z'(\mathrm{e}^{i\theta})i\mathrm{e}^{i\theta}}{\varphi_z(\mathrm{e}^{i\theta})} = \frac{1 - |z|^2}{(1 - \overline{z}\mathrm{e}^{i\theta})^2}\frac{1 - \overline{z}\mathrm{e}^{i\theta}}{\mathrm{e}^{i\theta} - z}i\mathrm{e}^{i\theta} = \frac{i(1 - |z|^2)}{(\mathrm{e}^{-i\theta} - \overline{z})(\mathrm{e}^{i\theta} - z)} = \frac{i(1 - |z|^2)}{(\overline{\mathrm{e}^{\theta} - z})(\mathrm{e}^{i\theta} - z)} = \frac{i(1 - |z|^2)}{|\mathrm{e}^{i\theta} - z|^2}$$

Substituting this back to (4.8) we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{i\phi})\frac{1-|z|^2}{|\mathrm{e}^{i\theta}-z|^2}\mathrm{d}\theta \tag{4.9}$$

(4.9) can also be written as

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta) f(\mathrm{e}^{i\theta})\mathrm{d}\theta \tag{4.10}$$

where function $P : (\theta, z) \mapsto \frac{1-|z|^2}{|\mathrm{e}^{i\theta}-z|^2}$ is called the *Poisson kernel*, and the expression (4.10) is called the *Poisson Integral Formula*. Taking $z = r\mathrm{e}^{i\psi}$ we get

$$P(z,\theta) = \frac{R^2-r^2}{|R\mathrm{e}^{i\theta}-r\mathrm{e}^{i\psi}|^2} = \frac{R^2-r^2}{[R(\cos\theta+i\sin\theta)-r(\cos\psi+i\sin\psi)][R(\cos\theta-i\sin\theta)-r(\cos\psi-i\sin\psi)]} =$$

$$= \frac{R^2-r^2}{[R(\cos\theta-r\cos\psi)+i(R\sin\theta-r\sin\psi)][R(\cos\theta-r\cos\psi)-i(R\sin\theta-r\sin\psi)]} =$$

$$= \frac{R^2-r^2}{[R(\cos\theta-r\cos\psi)]^2+[R(\sin\theta-r\sin\psi)]^2} = \frac{R^2-r^2}{R^2-2Rr\cos(\psi-\theta)+r^2}$$

where the formula on the right-hand side is often used for the Poisson kernel $P_{R,r}(\psi-\theta)$. If $f(z)=1$ for all $z \in \mathrm{cl}(\mathbb{D})$, then

$$\frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\mathrm{d}\theta = 1 \tag{4.11}$$

Theorem 4.3.1. *Let ϕ be continuous on $\mathbb{S}^1$ and define $u : \mathbb{D} \to \mathbb{R}$ by:*

$$u(z) = P[\phi](z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\phi(e^{i\theta})\,d\theta \tag{4.12}$$

and $u(z) = \phi(z)$ for $z \in \mathbb{S}^1$. Then u is continuous on $cl(\mathbb{D})$ and harmonic in $\mathbb{D}$.

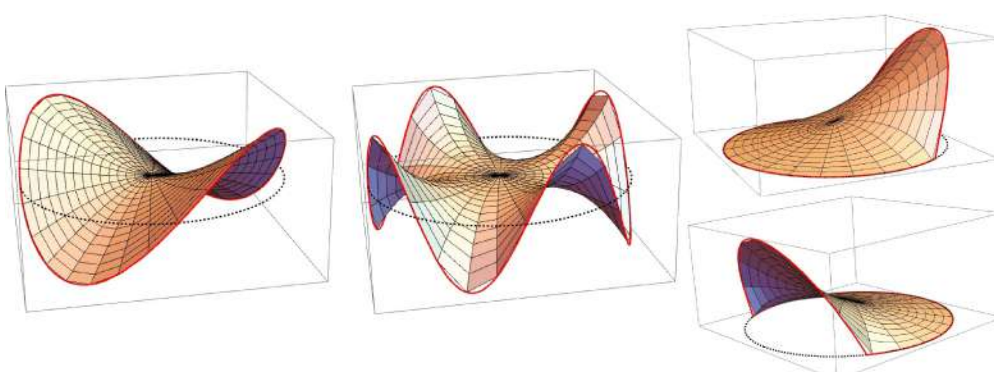


Figure 4.8: Numerically evaluated Poisson integrals for a Dirichlet problem on a unit disk $\mathbb{D}$ with boundary conditions $\phi_1(z) = 1/2\sin(2\mathrm{Arg}(z))$ (left), $\phi_2(z) = 1/2\sin(4\mathrm{Arg}(z))$ (middle), and $\phi_3(z) = 1-\mathrm{Arg}^2(z)$ when $1-\mathrm{Arg}^2(z) > 0$ and $\phi_3(z) = 0$ otherwise (right), for $z \in \mathbb{S}^1$.

Proof. The Poisson kernel can also be written as:

$$P(z,\theta) = \mathrm{Re}\left(\frac{\mathrm{e}^{i\theta}+z}{\mathrm{e}^{i\theta}-z}\right)$$

For a fixed $\mathrm{e}^{i\theta}$, P is a harmonic function of $z \in \mathbb{D}$ because P is the real part of an analytic map $w \mapsto \frac{w+z}{w-z}$, hence u is harmonic in $\mathbb{D}$.

To show that u is continuous on $\mathrm{cl}(\mathbb{D})$ fix $\mathrm{e}^{i\theta}$ on the unit circle and $\varepsilon > 0$. Choose $\delta > 0$ such that $|\phi(\mathrm{e}^{it}) - \phi(\mathrm{e}^{i\theta})| < \varepsilon$ whenever $|t-\theta| < \delta$. Using (4.11) and the Standard Integral Estimate (Lemma 2.3.3):

$$|u(z) - u(\mathrm{e}^{it})| = \frac{1}{2\pi}\left|\int_0^{2\pi} P(z,t)(\phi(\mathrm{e}^{it}) - \phi(\mathrm{e}^{i\theta}))\mathrm{d}t\right| \leq$$

$$\leq \frac{1}{2\pi}\int_{|t-\theta|\leq\delta} P(z,t)|\phi(\mathrm{e}^{it}) - \phi(\mathrm{e}^{i\theta})|\mathrm{d}t + \frac{1}{2\pi}\int_{|t-\theta|>\delta} P(z,t)|\phi(\mathrm{e}^{it}) - \phi(\mathrm{e}^{i\theta})|\mathrm{d}t \leq$$

$$\leq \varepsilon + \frac{1}{2\pi}\max_{\mathrm{e}^{it}\in\mathbb{S}^1}|\phi(\mathrm{e}^{it})|\int_{|t-\theta|>\delta} P(z,t)\mathrm{d}t$$

and the last term tends to 0 as $z \to \mathrm{e}^{i\theta}$, which means that u is indeed continuous on the unit circle $\mathbb{S}^1$. □

The solution (4.12) of a Dirichlet problem: $\Delta u = 0$, with $z \in \mathbb{D}$ and $u(z) = \phi(z)$ for all $z \in \partial\mathbb{D} = \mathbb{S}^1$, has to be unique due to the Uniqueness Principle of analytic functions (Theorem B.0.6).

Chapter 5

Ideal Flow Around Solid Objects

Equipped with analytic function theory and the Riemann Mapping Theorem, we proceed to utilize the properties of given constructs, more precisely to analyze how the complex velocity f and the complex potential F of an irrotational and incompressible steady-state flow (ideal flow) around different obstacles obtained from univalent transformations of the unit disk $\mathbb{D}$ onto the complex plane, change with respect to the domain geometry.

First, we demonstrate the effects of the *Joukovsky map* and its inverse on different circles, then model ideal flow around these solid objects without the effects of lift forces, then introduce the *Kutta-Joukovsky Theorem* and implement it for the examples.

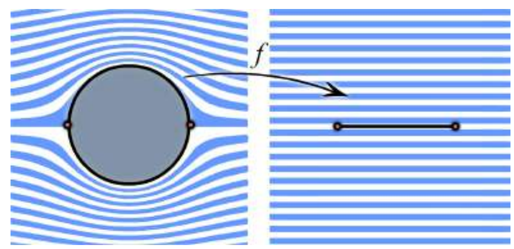


Figure 5.1: Transformation of the stream lines under the Joukovsky map.

5.1 Flow Around a Cylinder

We will examine flow around an infinitely long cylinder in the 3-dimensional case, so that the flow is isotropic with respect to the third dimension. Thanks to being isotropic with respect to the third dimension it can be formulated as a planar problem (for a disk centered at the origin), more precisely a problem in the complex plane. The complex velocity $f = v_x - iv_y$ satisfies boundary conditions

$$v_x = \frac{\partial \psi}{\partial y} = v_\infty,\ v_y = -\frac{\partial \psi}{\partial x} = 0 \quad \text{for} \quad |z| \to \infty \tag{5.1}$$

where v_∞ is the (asymptotic) flow velocity without the presence of an obstacle. On the boundary of the disk we assume zero-flux condition, which means that the boundary coincides with a streamline

$$\psi = \text{const. for } |z| = a \tag{5.2}$$

with a being the disk radius. The stream function ψ is, of course, harmonic: $\Delta\psi = 0$. Combining boundary conditions (5.1) and (5.2) gives a Dirichlet problem for ψ. Alternatively using the potential ϕ we obtaing a Neumann problem

$$\Delta\phi = 0\ ,\ v_x = \frac{\partial \phi}{\partial x} = v_\infty,\ v_y = \frac{\partial \phi}{\partial y} = 0 \quad \text{for } |z| \to \infty \quad \text{and } v_{\boldsymbol{n}} = \frac{\partial \phi}{\partial \boldsymbol{n}} = 0 \text{ for all } z \in \partial\mathbb{D}_a$$

where $\boldsymbol{n}$ is the outward normal to the disk boundary $\partial\mathbb{D}_a$. Since ψ and ϕ are conjugate harmonic, they are both solutions to the problem of complex potential $F = \phi + \psi i$ which reduces to finding the primitive of the

Laurent expansion of complex velocity

$$f(z) = \frac{\mathrm{d}F}{\mathrm{d}z} = C_0 + \frac{C_1}{z} + \frac{C_2}{z^2} + \frac{C_3}{z^3} + ...$$

which yields

$$F(z) = C_0 z + C_1 \log(z) - \frac{C_2}{z} - \frac{C_3}{2z^2} - ...$$

Now we can determine the complex constants $C_k = A_k + iB_k$ from the boundary conditions

$$\frac{\mathrm{d}F}{\mathrm{d}z} = v_\infty \text{ for } |z| \to \infty \quad \text{and} \quad \psi = \text{const. for } |z| = a$$

which immediately gives $C_0 = v_\infty$. For the remaining boundary condition write complex potential $F = \phi + i\psi$ in polar form:

$$F(re^{i\theta}) = v_\infty re^{i\theta} + (A_1 + iB_1)\log(re^{i\theta}) - \frac{A_2 + iB_2}{r}\mathrm{e}^{-i\theta} - \frac{A_3 + iB_3}{2r^2}\mathrm{e}^{-2i\theta} - ...$$

and separating the imaginary part for the stream function:

$$\psi(r,\theta) = v_\infty r\sin\theta + B_1\log r + A_1\theta + \frac{A_2}{r}\sin\theta - \frac{B_2}{r}\cos\theta + \frac{A_3}{2r^2}\sin 2\theta - \frac{B_3}{2r^2}\cos 2\theta + ... \tag{5.3}$$

Substituting $r = a$ into (5.3) then gives $A_1 = B_2 = A_3 = B_3 = ... = 0$ and $v_\infty a + A_2/a = 0$ because ψ needs to be constant, independent of θ, on the circle. Thus we get $A_2 = -v_\infty a^2$, and we have found all constants except B_1:

$$F(z) = v_\infty\left(z + \frac{a^2}{z}\right) + iB_1\log(z) \tag{5.4}$$

Omitting the $iB_1\log(z)$-term and setting $a = 1$ we get the complex potential (3.11) of ideal flow around the unit disk.

5.2 Joukovsky Profiles

Figure 5.2: Nikolay Yegorovich Joukovsky.

Consider complex potential (3.11). The formula was first used by a Russian mathematician and engineer Nikolay Yegorovich Joukovsky (*1847 - †1921) who published it in his 1910 paper. Historically, it was used especially for airfoil design. As we will see later, off-centered circles in $\mathbb{C}$ with certain radii get transformed by F into airfoil-shaped regions.

The versatility of the formula resides in the fact that it maps the unit circle onto a line segment $I_\pm = \{z \in \mathbb{C} \mid -2 < \mathrm{Re}(z) < 2\ , \mathrm{Im}(z) = 0\}$. The resulting complex potential then becomes an identity map with stream lines parallel to the real axis (see Fig.5.1). The inverse transformation

$$F^{-1} : z \mapsto \frac{1}{2}\left(z \pm \sqrt{z^2 - 4}\right) \tag{5.5}$$

has two algebraic branch points of order 2 at $z = 2$ and $z = -2$. The corresponding branches of F^{-1} map $\mathbb{C}\backslash I_\pm$ on the interior and the exterior of the unit circle respectively (see Fig.5.3).

Pulling back the parallel streamlines along a horizontal plate, that is: mapping them through F^{-1} gives a stream profile that satisfies the zero-flux condition, as well as conditions (1.) - (4.) in Section 3.2.

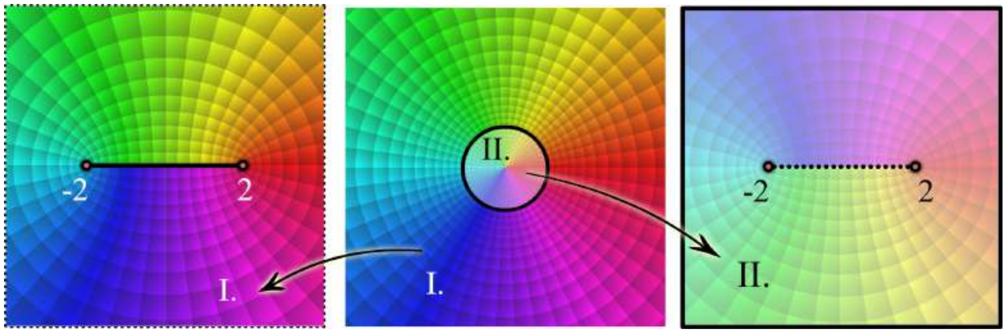


Figure 5.3: The effect of the inverse Joukovsky map F^{-1} on the complex plane, with 1 and -1 branches mapping the exterior and the interior of the unit disk (I.) and (II.) respectively

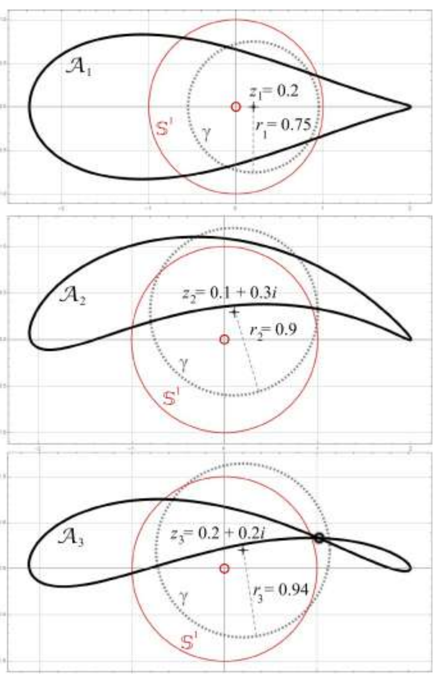


Figure 5.4: Airfoil contours $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ obtained from circles with radii $r_1 = 0.75$ and $r_3 = 0.94$ and centers $z_1 = 0.2$, $z_2 = 0.1 + 0.3i$, and $z_3 = 0.2 + 0.2i$.

Mapping arbitrary circles $\gamma : t \mapsto z_0 + re^{it}$, with $t \in [0, 2\pi]$ via $F : z \mapsto z + 1/z$ gives contours, some of which are Jordan curves that have been used in airfoil design. Taking a look at the images of different circles in Fig.5.4 provides a clue on how the position z_0 of the circle and its radius r determines the shape of contours $\mathcal{A} = F \circ \gamma$.

We notice that for $\mathrm{Im}(z_0) = 0$ the resulting contours are symmetrical with respect to the real axis, and for $z_0 = 0$ they become ellipses with varying lengths of semi-minor axes. Once the center z_0 moves away from the real line, more deformed asymmetrical contours are obtained. This is due to the inversion term $1/z$ in the Joukovsky map.

F is, of course, injective on every circle of radius $r \neq 1$. The circle of radius r and the circle of radius $1/r$ get mapped to the same ellipse because $F(z) = F(1/z)$ (which can be easily verified). This identity shows that F fails to be injective on the unit circle, in particular $F[\mathbb{S}^1] = I_\pm$ because if $|z| = 1$, $F(z) = z + \overline{z} = 2\mathrm{Re}(z) = 2\cos(\mathrm{Arg}(z))$. Thus for some r and z_0, as we can see in Fig.5.4, the contour $\mathcal{A} = F \circ \gamma$ is not a Jordan curve since it is not injective.

Moreover, if $\gamma(t) = \pm 1$ for some $t_0 \in [0, 2\pi]$, then $(F \circ \gamma)(t_0) = \mathcal{A}(t_0) = \pm 2$, and since $\mathcal{A}(t_0) \in \mathrm{cl}(I_\pm)$ curve $\mathcal{A}$ is not differentiable at t_0 and has a *cusp* at t_0 whenever the circle γ passes through points 1 or -1. In fact, we observe that whenever $\mathrm{Int}(\gamma)$ contains one of these points, the curve $\mathcal{A}$ intersects itself (see Fig.5.5). When both $1, -1 \in \mathrm{Int}(\gamma)$, then $\mathcal{A}$ becomes a Jordan curve again.

Taking the conjugate $\overline{\gamma}$ of the circle (with center $\overline{z}_0$) produces an airfoil $\overline{\mathcal{A}}$ symmetrical with respect to the real axis, and similarly for γ_- centered at $i\mathrm{Im}(z_0) - \mathrm{Re}(z_0)$ gives an image $\mathcal{A}^-$ symmetrical with respect to the imaginary axis.

Inversion of the circle γ, in general, does not produce a symmetrical image, since by the circle offset z_0 not lying on the real or the imaginary axis gives rise to an airfoil contour asymmetrically curved along the *camber line* which is the skeleton or the medial axis of a Jordan region.

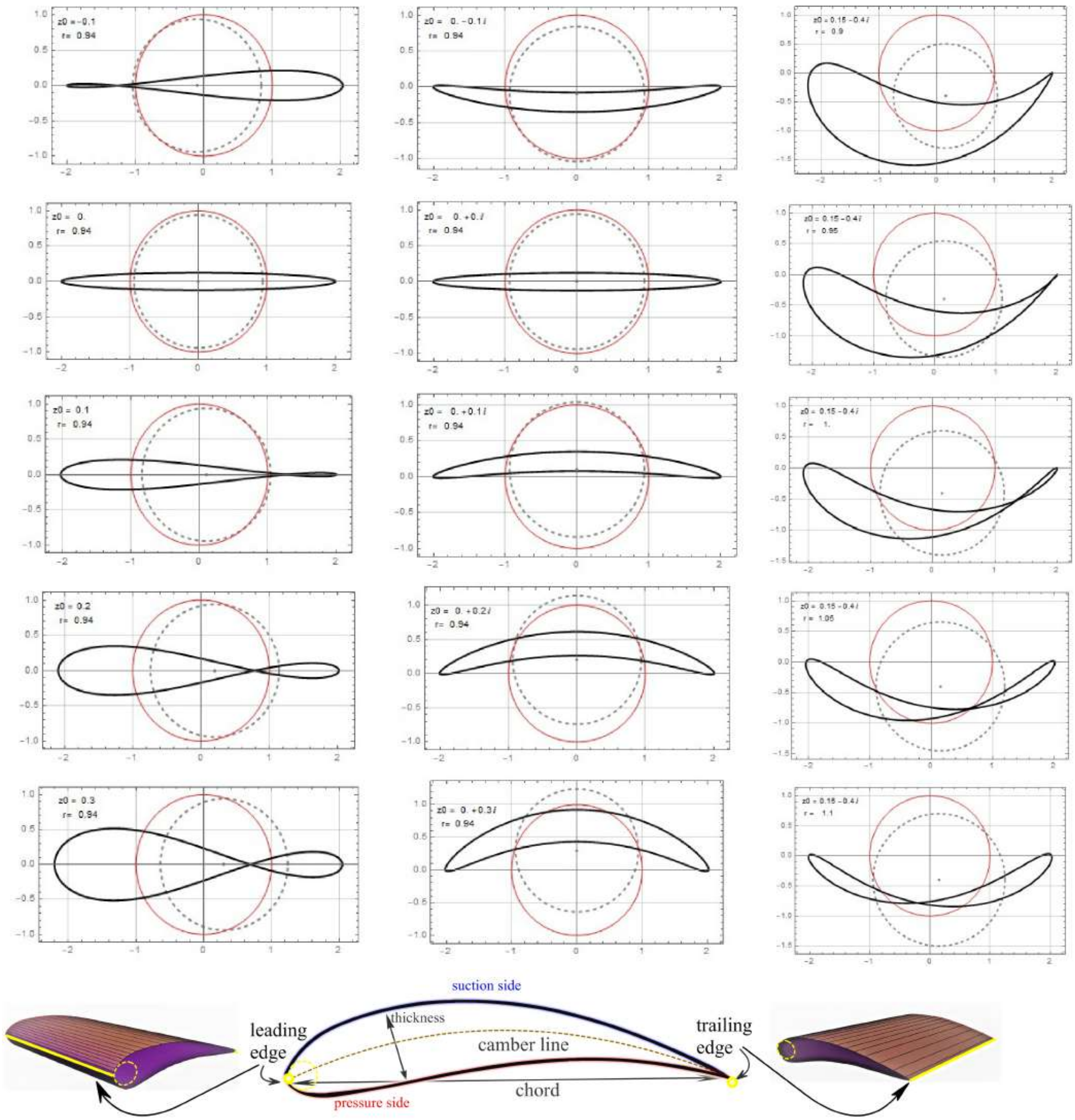


Figure 5.5: Changing parameters z_0 and r of circle γ (dashed, grey) under Joukovsky map with the unit circle (red), with common technical names for individual parts of an airfoil.

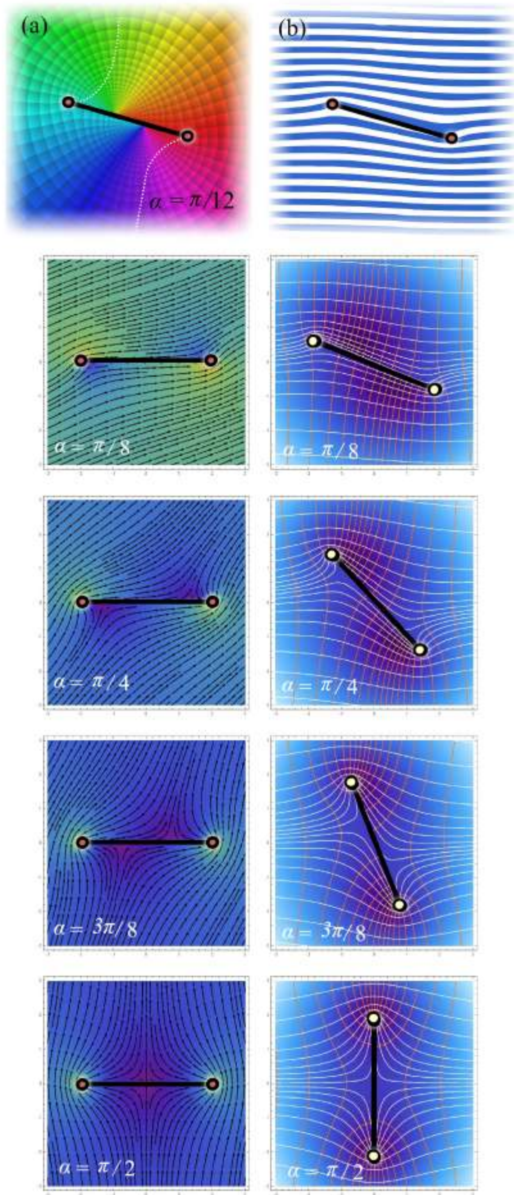


Figure 5.6: Enhanced phase portrait (a) and streamlines (b) of flow past a plate with tilt angle $\pi/12$, and velocity fields (left) with complex potentials (right) of flow past plates with various tilt angles.

A typical property of Joukovsky-type airfoils is that $r = \sqrt{(x_0-1)^2 + y_0^2}$, that is their *trailing edge* is cusped because the transformed circle γ passes through 1 (or -1 alternatively).

An important difference between the flow around a disk, and a general Joukovsky airfoil is that its orientation in space matters. Rotating a disk does not change the resulting complex potential, but changing the parameters of an airfoil $\mathcal{A}$ must result in noticeable changes that are essentially related to the angle at which the stream interacts with the solid body.

5.3 Flow Past a Tilted Plate

Clearly, rotating the unit disk around the origin has no effect on the resulting complex potenial. Yet, as we see in Fig. 5.1 the streamlines under the Joukovsky map are parallel to the line segment $I_\pm$ with a stagnation point at each end. What happens when the direction of these streamlines changes can be seen when applying transformation $z \mapsto \mathrm{e}^{-i\alpha}z$ where α is the *angle of attack* is simply that the complex potential becomes

$$\mathcal{Y}_\alpha : z \mapsto F_\alpha(\mathrm{e}^{-i\alpha}z) = \mathrm{e}^{-i\alpha}z + \frac{\mathrm{e}^{i\alpha}}{z}$$

and substituting back into the inverse Joukovsky map (5.5):

$$F_\alpha : z \mapsto \mathrm{e}^{i\alpha}\big(z\cos\alpha \pm i\sin\alpha\sqrt{z^2 - \mathrm{e}^{-2i\alpha}}\big) \tag{5.6}$$

Differentiating (5.6) with respect to z then gives the complex velocity:

$$f_\alpha : z \mapsto \mathrm{e}^{i\alpha}\big(\cos\alpha \pm i\sin\alpha\frac{z}{\sqrt{z^2 - \mathrm{e}^{-2i\alpha}}}\big) \tag{5.7}$$

The resulting complex potential has, of course, two branches each corresponding to the sign of the complex root in (5.5). Changing the sign in front of the $\sin\alpha$-term is equivalent to inverting the angle of attack from α to $-\alpha$, rotating the velocity profile by -2α (see Fig.5.6).

Solving the initial value problem for a sytem of ordinary differential equations with right-hand side $\overline{f_\alpha}$ essentially amounts to an unrestricted analytic continuation along a streamline of F_α. We can achieve the same effect just by fusing individual branches into a full profile of the complex potential and velocity (see Fig.5.6).

Flow past a plate with tilt angle $\pi/2$ has a complex potential $F_{\pi/2} : z \mapsto \pm i\sqrt{z^2-1}$ (see Fig.5.6 bottom images).

5.4 Flow Past Joukovsky Airfoils

In the previous section we implemented the angle of attack α by substituting $ze^{i\alpha}$ into the complex potential. One can then find a suitable solution for a special type of Joukovsky airfoil, called *Rankine oval*. Let $z \mapsto z + \frac{b^2}{z}$ be a Joukovsky-type transformation of the complex plane. Let $\zeta = \frac{1}{2}(z \pm \sqrt{z^2 - 4b^2})$ be the value under inverse transformation. Then the resulting complex potential is given by

$$F_\zeta : \zeta \mapsto v_\infty \left(\zeta + \frac{a^2}{\zeta} \right) \tag{5.8}$$

where a is the semi-major axis of the oval (in the calculations the asymptotic velocity $v_\infty = 1$). Parameters a and b determine the semi-axes of the ellipse, specifically when $b = 1$ the unit circle gets mapped onto a line segment.

Substituting $\zeta \to \zeta \mathrm{e}^{-i\alpha}$ (clockwise rotation of the oval) to produce the desired effect in formula (5.8) we get

$$F : z \mapsto v_\infty \left(\zeta(z) \mathrm{e}^{-i\alpha} + \frac{a^2}{\zeta(z)} \mathrm{e}^{i\alpha} \right) ,$$

$$\text{where} \quad \zeta(z) = \frac{1}{2}(z + \mathrm{sign}(\mathrm{Re}(z))\sqrt{z^2 - 4b^2}) \tag{5.9}$$

The resulting complex velocity field f can be obtained by differentiating 5.9 with respect to z. In Fig. 5.7 we see the velocity field as well as the complex potetnial with streamlines and equipotential lines for different angles of attack α. Obviously, for angles $\alpha \notin [0, \pi/2]$ the situation would be completely symmetrical to some angle in $[0, \pi/2]$.

We can then apply this principle to the general Joukovsky airfoil and implement the situation for airfoils generated from off-centered circles γ with center z_0. Let $z \mapsto az + z_0$, with $|a| = r$ be an affine map transforming a unit circle into γ. Let $a = re^{i\phi}$. Its inverse is then $z \mapsto \frac{z - z_0}{r} \mathrm{e}^{-i\phi}$, and since the argument ϕ will only be added to the existing angle of attack, without loss of generality, assume $\phi = 0$ and $a = r$. Now the resulting complex potential is

$$F : z \mapsto v_\infty \left(\frac{\zeta(z) - z_0}{r} \mathrm{e}^{-i\alpha} + \frac{a^2}{\zeta(z) - z_0} \mathrm{e}^{i\alpha} \right) ,$$

$$\text{where} \quad \zeta(z) = \frac{1}{2}(z + \mathrm{sign}(\mathrm{Re}(z))\sqrt{z^2 - 4b^2}) \tag{5.10}$$

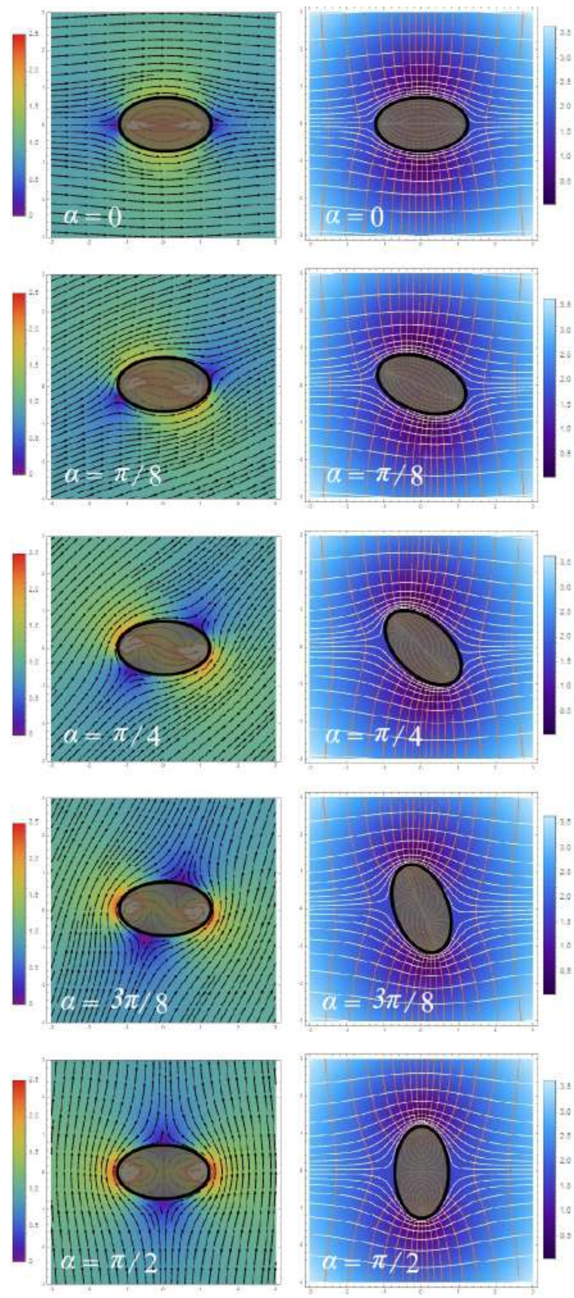


Figure 5.7: Complex velocities (left) with complex potentials (right) of a rotating Rankine Oval with $a = 1$ and $b = 0.5$.

Examples of the resulting flow can be seen in Fig.5.8.

The drawbacks of this approach are that for some parameter values, the resulting velocity profile might (appear to) not satisfy the zero-flux condition along the solid object because of conflicting analytic branches

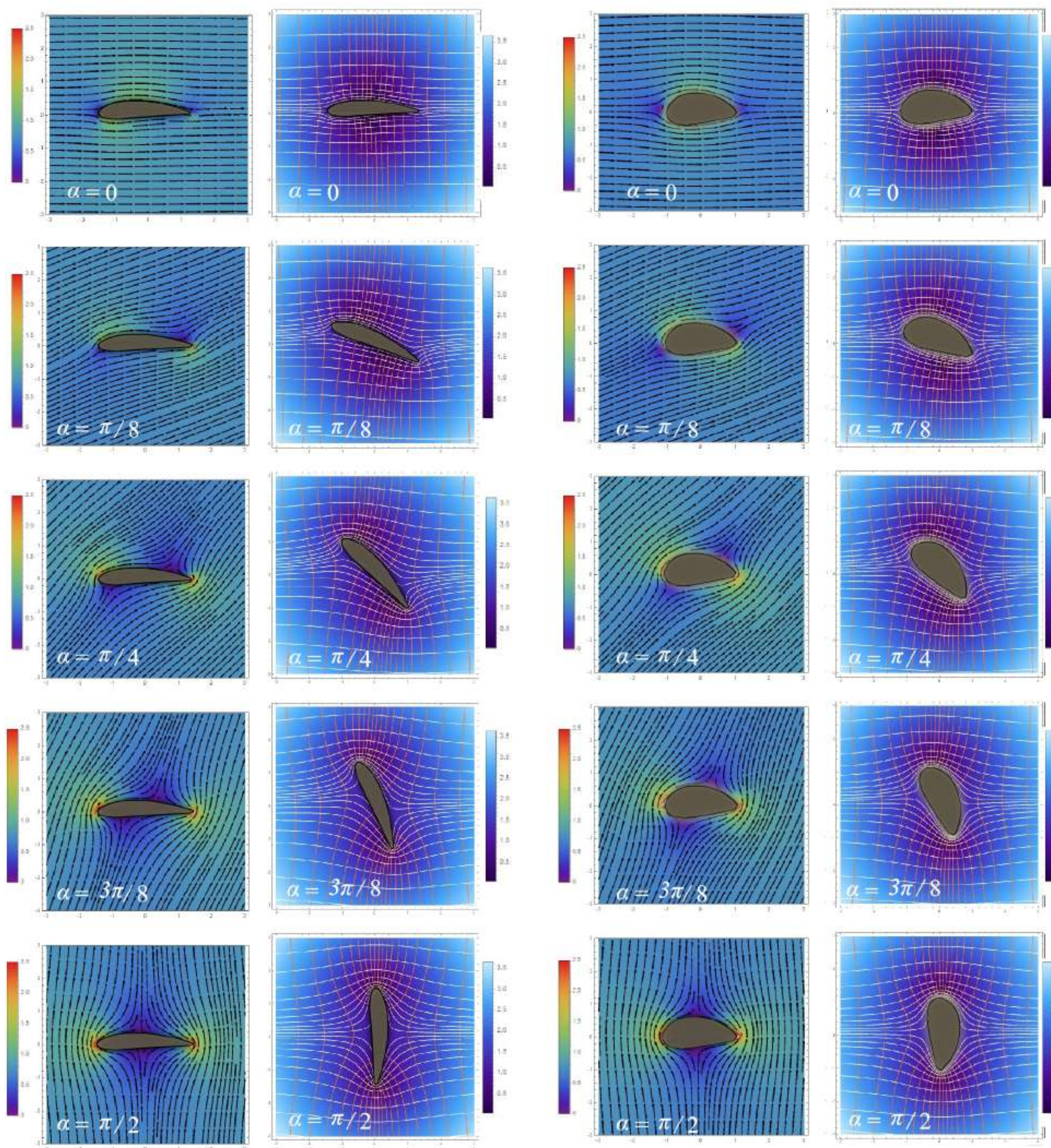


Figure 5.8: Flow past tilted Joukovsky airfoils that do not satisfy the Kutta Condition. In the left column we see the situation around an airfoil with parameters $z_0 = -0.1 + 0.1i$, $r = 0.8$, $a = 1$, and $b = 0.7$, while in the right column, parameter b changes to $b = 0.5$.

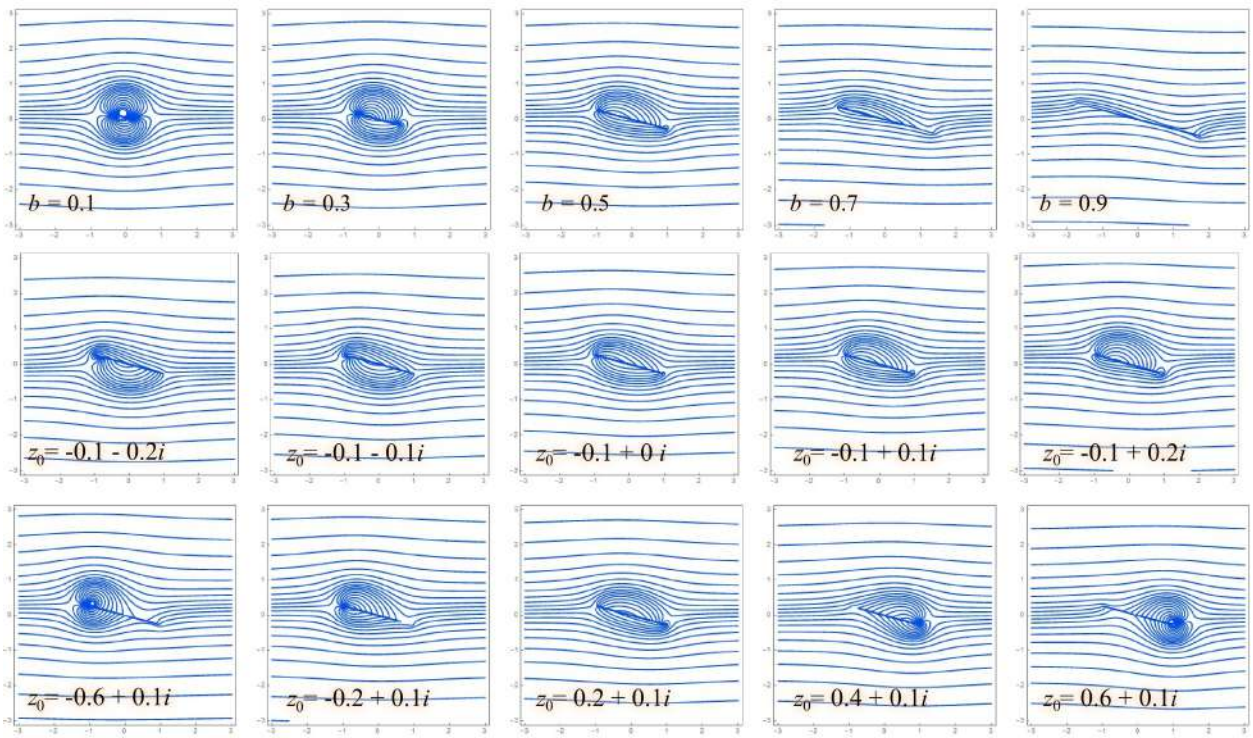


Figure 5.9: Streamlines of potential flow generated by formula (5.10) with changing individual parameters with $\alpha = \pi/12$ and $r = 0.8$. Default parameter values are $z_0 = -0.1 + 0.1i$ and $b = 0.5$

of the complex potential due to self-intersecting airfoil shapes. The streamlines in Fig.5.8 and in Fig.5.9 flow past an obstacle that is similar to a Joukovsky airfoil, but when we use the constants z_0 and r utilized in (5.10) the parametrization is not the same. Complex constant $z_0 = x_0 + iy_0$ produces a deformation of the Rankine oval similar to a Joukovsky airfoil with $z_0 = -x_0 + iy_0$, that is: the thicker area moves in the positive direction as x_0 increases, instead of negative as in Joukovsky profiles (see Fig.5.4).

Figure 5.10: Circulation of the velocity field along a closed loop.

Another, more fundamental, problem is that this type of flow has zero *circulation* along the boundary of the solid body. Airplanes with wings like the ones in Fig.5.8 would not fly because airfoils like these do not produce lift.

5.5 Circulation, Lift, and the Kutta Condition

When a smooth symmetric body (like the Rankin oval, for instance) moves with $\alpha = 0$, the motion of the fluid creates two stagnation points on the bod, one at the front (on the *leading edge*) and one in the back (at the *trailing edge*). Changing the angle of attack does not change the number of stagnation points of the flow. The position of these stagnation points changes with increasing α. The frontal stagnation point moves further back along the *pressure side* of the airfoil and the stagnation point in the back moves by the same amount along the *suction side* more towards the leading edge (see Fig.5.7). The frontal and the rear stagnation points would continue moving until $\alpha = \pi$ when the same situation would occur in reverse. For asymmetrical profiles the rate at which the stagnation points move changes only by a small amount (see Fig.5.8).

Let γ be a closed loop in $\Omega \subseteq \mathbb{C}$. Define the *circulation*[1] of the velocity field $\boldsymbol{v} : \Omega \to \mathbb{C}$ as

$$\Gamma = \oint_\gamma \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = \oint_\gamma (v_x \mathrm{d}x + v_y \mathrm{d}y) = \mathrm{Re}\left(\oint_\gamma \overline{f(z)} \mathrm{d}z \right) \tag{5.11}$$

where $\mathrm{d}\boldsymbol{s}$ is the tangent differential to path γ, and f is the corresponding complex velocity of the flow. The physical interpretation of (5.11) is the amount of flow along the contour, that is: flow in the direction tangent to the path.

Now we make use of the term $iB_1 \log(z)$ in (5.4). Since the velocity field is continuously differentiable the circulation about an arbitrary closed loop is equivalent to the circulation about a circle. Recall that the path integral of $z \mapsto 1/z$ along an arbitrary closed loop containing zero yields $2\pi i$ and since $z \mapsto \log(z)$ is also a (local) primitive of $z \mapsto 1/z$, we set the Laurent coefficient $B_1 = -\frac{\Gamma}{2\pi}$. The complex potential around a disk then becomes

$$F : z \mapsto v_\infty \left(z + \frac{a^2}{z} \right) - i \frac{\Gamma}{2\pi} \log(z) \tag{5.12}$$

Using this potential we can find the new location of stagnation points by solving for z in $\mathrm{d}F/\mathrm{d}z = 0$:

$$\frac{\mathrm{d}F}{\mathrm{d}z} = v_\infty \left(1 - \frac{a^2}{z^2} \right) - i \frac{\Gamma}{2\pi z} = 0$$

$$z_{1,2} = \frac{i\Gamma}{4\pi v_\infty} \pm \sqrt{-\frac{\Gamma^2}{16\pi^2 v_\infty^2} + a^2} \tag{5.13}$$

where we distinguish cases:

(1): $\Gamma < 4\pi a v_\infty$
(2): $\Gamma = 4\pi a v_\infty$
(3): $\Gamma > 4\pi a v_\infty$

In the first case we put $\Gamma/(4\pi a v_\infty) = \sin\theta$ and substitute into the solution (5.13):

$$z = a(\pm\cos\theta + i\sin\theta) = \begin{cases} a\mathrm{e}^{i\theta} \\ a\mathrm{e}^{i(\pi-\theta)} \end{cases}$$

which means that both stagnation points lie on the circle and their arguments are θ and $\pi - \theta$. More precisely, if $\Gamma > 0$ both critical points lie above the real axis, if $\Gamma < 0$ they lie below it.

For case (2) the discriminant in (5.13) vanishes and both stagnation points merge into $z = ia$ (if $\Gamma > 0$) or $z = -ia$ (if $\Gamma < 0$).

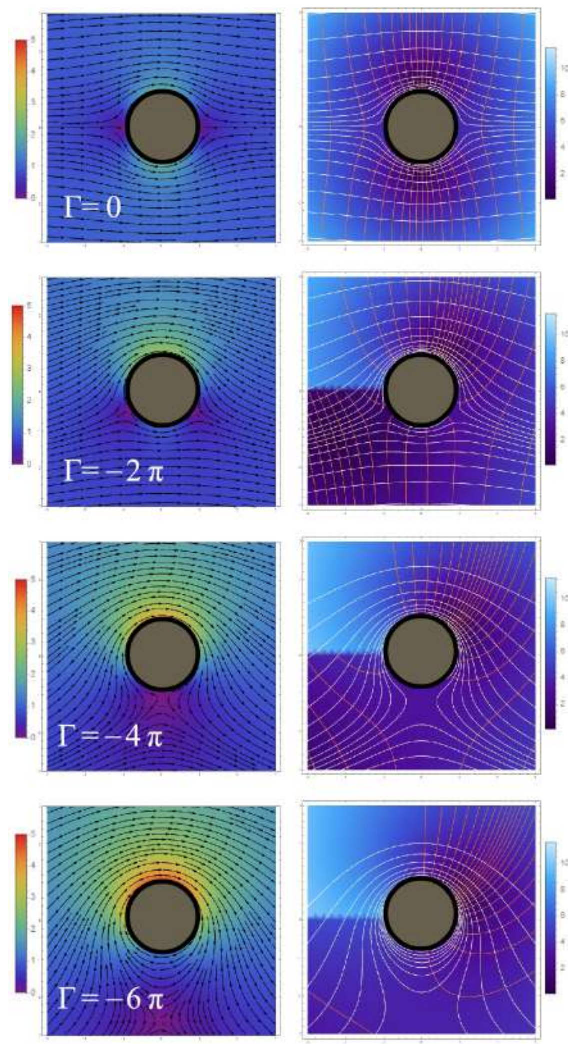


Figure 5.11: Flow around a disk with different values of circulation Γ.

[1] Γ is a real number, not to be confused with a cycle.

And finally for case (3) the discriminant in (5.13) becomes negative, hence we obtain two stagnation points on the imaginary axis:

$$z_1 = i\left(\frac{\Gamma}{4\pi v_\infty} + \sqrt{\left(\frac{\Gamma}{4\pi v_\infty}\right)^2 - a^2}\right)$$

$$\text{and} \quad z_2 = i\left(\frac{\Gamma}{4\pi v_\infty} - \sqrt{\left(\frac{\Gamma}{4\pi v_\infty}\right)^2 - a^2}\right)$$

where for $\Gamma > 0$: $|z_1| > a$ and $|z_2| < a$, and for $\Gamma < 0$ the other way around.

In Fig.5.11 we can see how the stagnation points merge into one point on the circle and then separate with one moving down the imaginary axis. In the density plots of the complex potential we notice that for $\Gamma \neq 0$ there is a discontinuity in the argument value at $\mathrm{Re}(z) < 0$ which corresponds to the branch cut for an analytic branch of the logarithm.

Going back to the example of Rankine oval and profiles generated by (5.10) we notice that the lack of circulation in these situations gives rise to a state of equilibrium. No resultant force acts on the solid, until the circulation term $-i\Gamma/2\pi \log(z)$ is added. Since the position of the rear stagnation point changes with rotating the airfoil (by α), by Bernoulli's principle, it would be more feasible to ensure that fluid with lower velocity stays coincident with the (lower) pressure side and the faster moving fluid occupies the (upper) suction side to generate a stable lift force. Interestingly enough, there is a principle for airfoil design that guarantees it.

A body with a *sharp trailing edge* moving through a fluid generates circulation (about itself) strong enough to *hold the rear stagnation point at the trailing edge*. This is known as the *Kutta condition*, and is achieved in the case of Joukovsky airfoils when the radius of transformed circle γ centered at $z_0 = x_0 + iy_0$ satisfies $r = \sqrt{(x_0 - 1)^2 + y_0}$ (if $v_\infty > 0$), which is when the Joukovsky transformation maps a circle intersecting $z = 1$ onto a cusped airfoil contour.

Since the radius (of curvature) of the sharp trailing edge is zero, vortex flow takes place at this location. In theory, the velocity around the trailing edge tends to infinity, and even though real fluids cannot move at infinite speed, they can move exceedingly fast which causes large velocity gradients leading to formation of vortices on the suction side of the airfoil. This type of vortex is called the *starting (trailing) vortex*. As the airfoil moves through air it carries this vortex along with it. The starting vortex is not bound to the top of the airfoil, but is left spinning in the air as the airplane flies further. Starting vortices have, in fact, been photographed by leading aerodynamicists in the 20th century (see Fig.5.12).

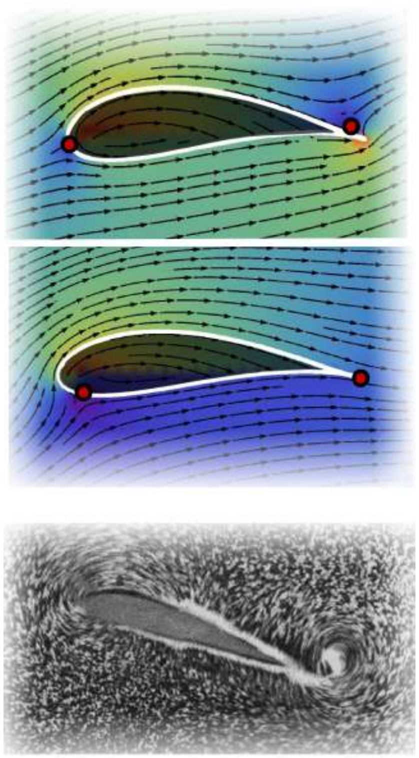


Figure 5.12: An airfoil which does not satisfy (top) and an airfoil which satisfies the Kutta Condition (middle), with stagnation points (red). (bottom) A trailing vortex in a water channel with aluminum particles, photographed by Ludwig Prandtl in 1934.

5.6 Flow Past Joukovsky Airfoils Revisited

We will now adjust formula (5.10) so that it satisfies the Kutta Condition as well as other attributes. Denote h the thickness of the airfoil and c the chord length (see Fig.5.4, (bottom)). Now let $L = c/4$ and since the real part of $z_0 = x_0 + iy_0$ determines the airfoil thickness let $x_0 = -h/5.2$. The imaginary part y_0 adjusts the curvature of the camber line. For $y_0 = 0$ the camber line coincides with the chord. Now using (5.12) with circulation Γ we complete the resulting complex potential:

$$F : z \mapsto v_\infty \left(\zeta(z) + \frac{a^2}{\zeta(z)} \right) - i\frac{\Gamma}{2\pi} \log \zeta(z)$$

$$\text{where} \quad \zeta(z) = \left(\frac{z + \operatorname{sign}(\operatorname{Re}(z))\sqrt{z^2 - 4L^2}}{2} - x_0 - iy_0 \right) \frac{\mathrm{e}^{-i\alpha}}{r}\,, \quad \text{and} \quad r = \sqrt{(x_0 - 1)^2 + y_0^2} \tag{5.14}$$

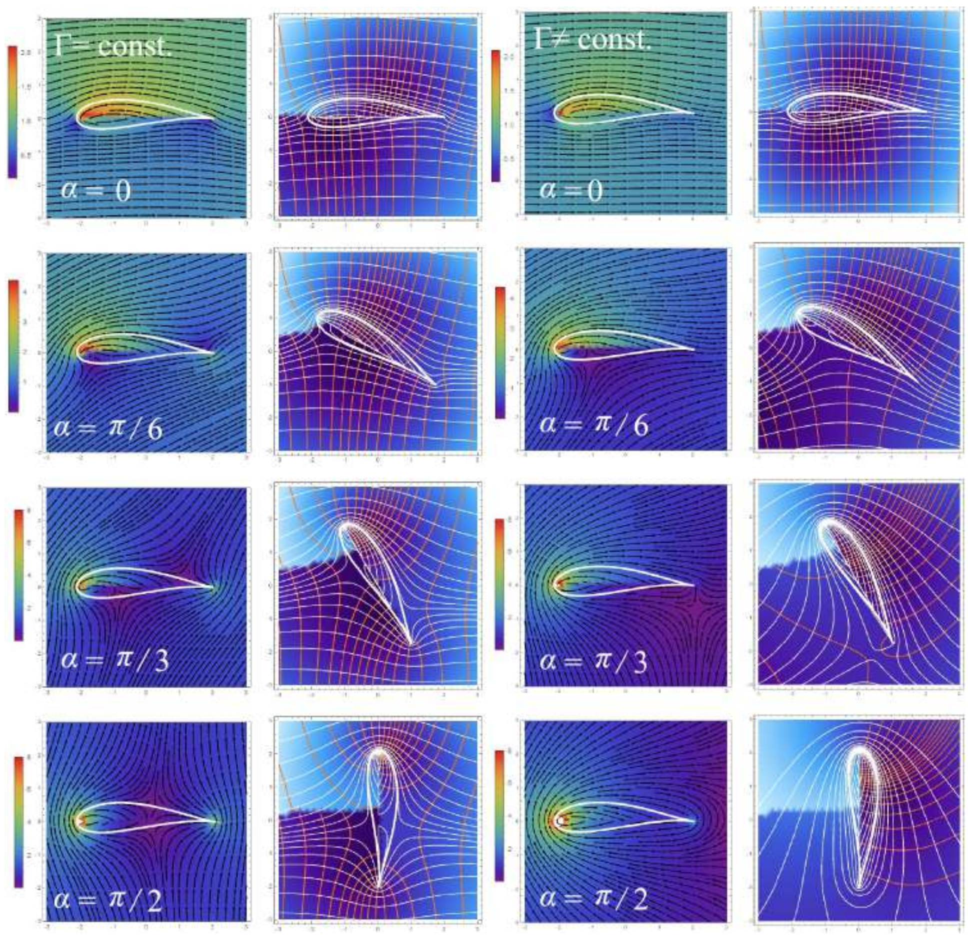


Figure 5.13: Upward pitching (changing α) of airfoils satisfying Kutta Condition, and with constant circulation $\Gamma = \pi v_\infty^2$ (left) and circulation determined by the pitch angle $\Gamma = -4\pi v_\infty^2 a(\alpha + \beta)$ (right). Other parameters are $c = 4$, $h = 1$, $y_0 = 0.1$.

The resulting airfoil shape can be parametrized by

$$\gamma : t \mapsto (z_0 + r\mathrm{e}^{it}) \ , \Longrightarrow \ \mathcal{A} : t \mapsto \left(\gamma(t) + \frac{a^2}{\gamma(t)}\right)\mathrm{e}^{-i\alpha} \tag{5.15}$$

Unlike for flow around a disk, the circulation depends on the angle of attack, otherwise pitching the airfoil "against" the flow (with $\alpha = \pi/2$) would cause no drag, which would conflict with reality. Drag force can be expressed via $F_D = \frac{1}{2}\rho v_\infty^2 C_D A$ where ρ is the fluid density, C_D the drag coefficient, and A the effective reference area. We will use a simplified version of this law to determine circulation: $\Gamma = -4\pi\rho v_\infty^2 a(\alpha + \beta)$, where $\beta = \arccos(L/a)$, and also putting $a = \sqrt{y_0^2 + L^2}$.

This means that the effect of circulation grows with y_0 (which causes the camber line curvature), and with the angle of attack α. Implementing all of these adjustments yields desired results which can be seen in Fig.5.13 compared with the same airfoil geometry with circulation independent of α.

Changing parameters h and y_0 yields expected results. Since y_0 is not bounded it can be set in such way that the airfoil shape (5.15) loses the cusp and becomes self-intersecting, even though Kutta Condition (for r) is satisfied. Changing the chord length c also changes parameter L which results in a more oval shape for $c < 4$ that, unfortunately, does not coincide with the parametrization (5.15) since for $c \neq 4$ the "cusp point" $z = 1$ is exceeded.

The resulting changes in the complex velocity field can be seen in Fig.5.14.

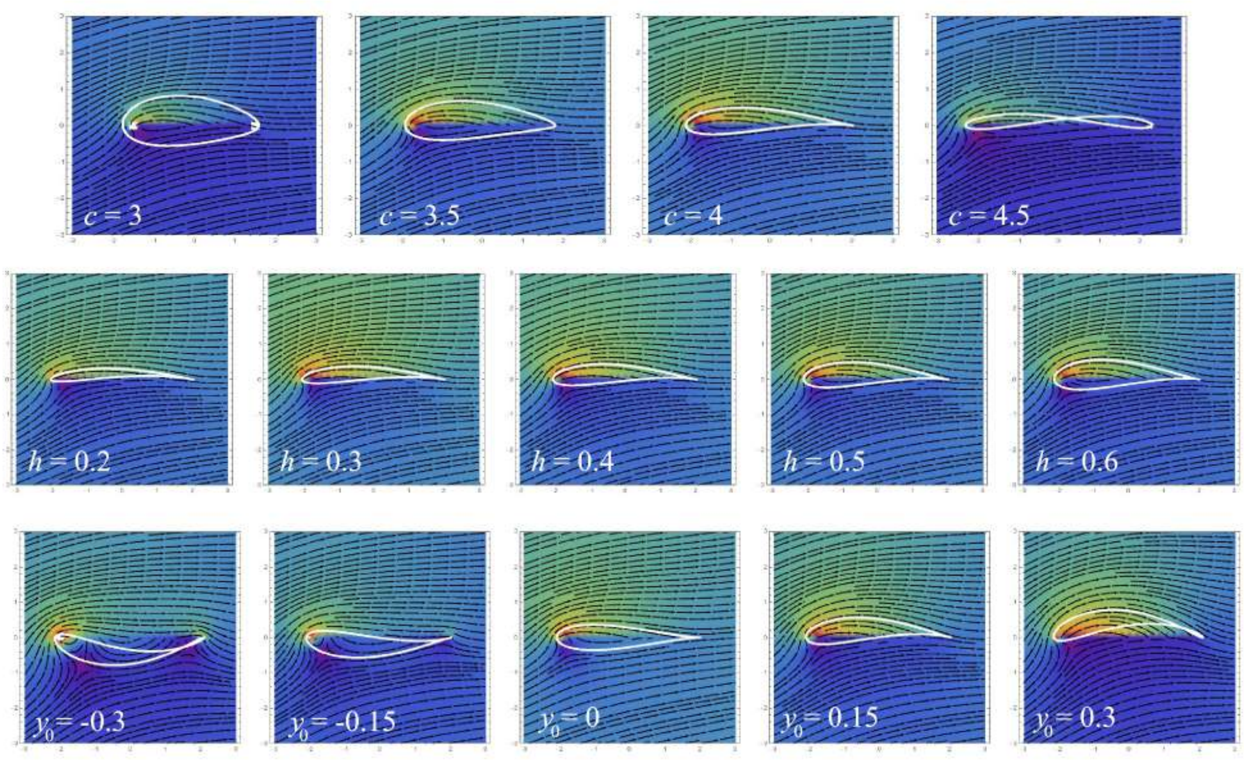


Figure 5.14: (top): changing chord lenght c with $h = 0.5$, $y_0 = 0.1$, and $\alpha = \pi/12$. (middle): changing airfoil thickness h with $y_0 = 0.1$, $x_0 = -h/3.2$, $c = 4$, and $\alpha = \pi/12$. (bottom): changing airfoil parameter y_0 with the remaining parameters from the previous row.

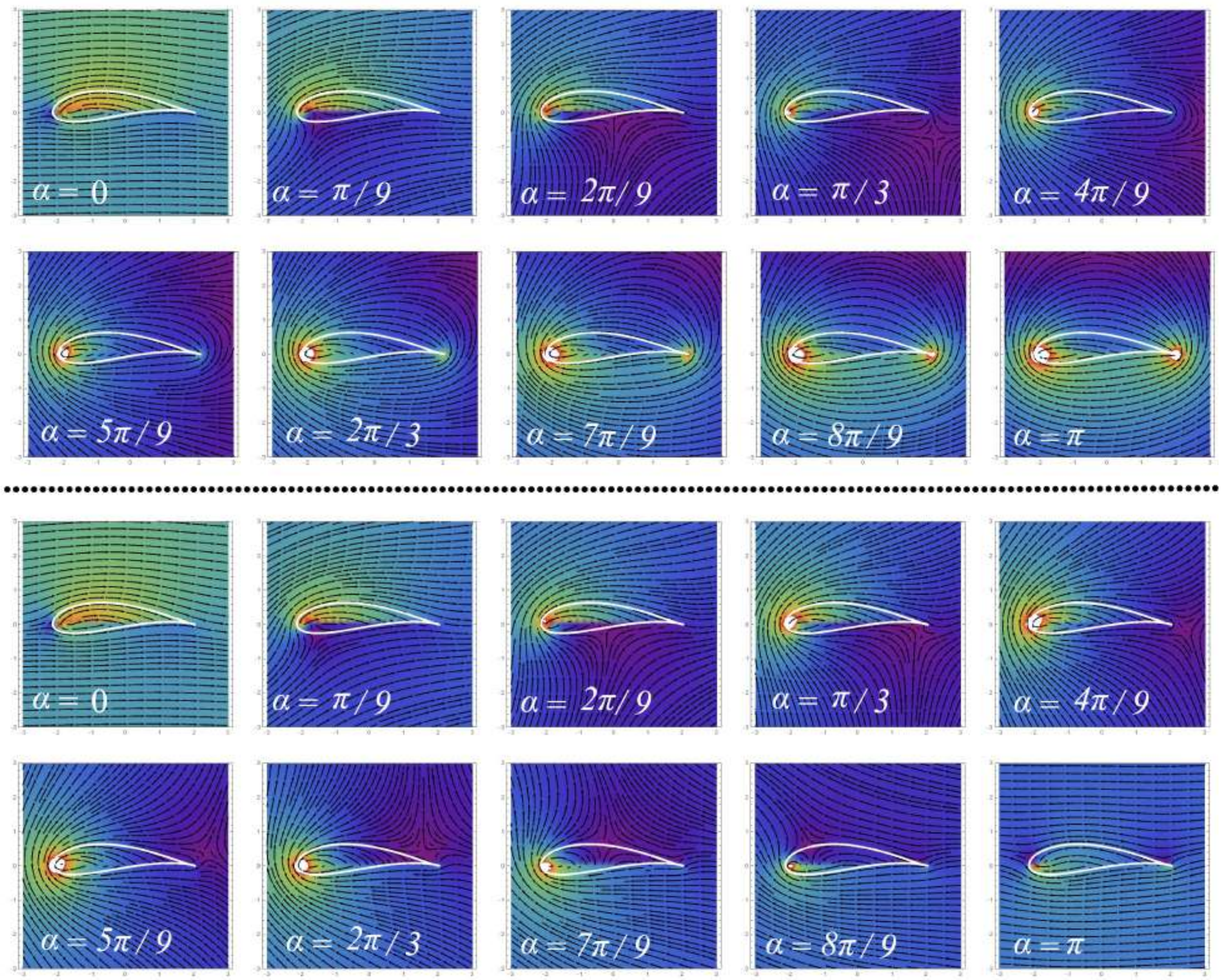


Figure 5.15: Inverting a Joukovsky airfoil with $\Gamma = -4\pi v_\infty^2 a(\alpha + \beta)$ (top) and with $\Gamma = -4\pi v_\infty^2 a(\sin\alpha + \beta)$

The approximation of Γ seems still far from reality if we want to model situations like full inversion of the airplane, that is: changing α from 0 to π. Using the simplified version $\Gamma = -4\pi v_\infty^2 a(\alpha + \beta)$ does not account for the real effect of rotation in the air since circulation Γ keeps accumulating. With only a minor adjustment $\Gamma = -4\pi v_\infty^2 a(\sin\alpha + \beta)$ accurately approximates reality (see Fig.5.15).

5.7 Lift and The Kutta-Joukovsky Theorem

The difference in dynamic pressure caused by the vast gradients of velocity generate a *lift* force $\boldsymbol{F}_l$, which will be measured per unit length of the airfoil (cylinder) with an arbitrary cross section. Let

$$\boldsymbol{F}_l = -\oint_{\mathcal{A}} p\boldsymbol{n}\,\mathrm{d}\boldsymbol{s}$$

where p is the total pressure acting on the boundary $\mathcal{A}$ and $\boldsymbol{n} = (n_x, n_y)$ is the outward normal to $\mathcal{A}$. The force can be expressed in the complex plane as

$$F_l = F_{l,x} + iF_{l,y} = -\oint_{\mathcal{A}} p(\sin\phi - i\cos\phi)\mathrm{d}s \ , \quad \text{where} \ \ \phi = \mathrm{Arg}(n_x + in_y)$$

with complex conjugate

$$\overline{F}_l = F_{l,x} - iF_{l,y} = -\oint_{\mathcal{A}} p(\sin\phi + i\cos\phi)\mathrm{d}s = -i\oint_{\mathcal{A}} p(\cos\phi - i\sin\phi)\mathrm{d}s = -i\oint_{\mathcal{A}} p\mathrm{e}^{-i\phi}\mathrm{d}s = -i\oint_{\mathcal{A}} p\ \mathrm{d}\overline{z}$$

Now using the pressure term from the Bernoulli equation (3.8) set $p = p_0 - \frac{1}{2}\rho||\boldsymbol{v}||^2$ with ambient fluid pressure p_0. Assuming incompressibility $\rho = \text{const.}$ we get

$$\overline{F}_l = -ip_0\oint_{\mathcal{A}} \mathrm{d}\overline{z} + i\frac{\rho}{2}\oint_{\mathcal{A}} ||\boldsymbol{v}||^2\mathrm{d}\overline{z}$$

where the first term is zero since a constant function is analytic in $\mathbb{C}$. Now let $v = \pm||\boldsymbol{v}||\mathrm{e}^{i\phi}$ be the corresponding complex velocity. Then $||\boldsymbol{v}||^2\mathrm{d}\overline{z} = v^2\mathrm{d}z$, and we obtain:

$$\overline{F}_l = i\frac{\rho}{2}\oint_{\mathcal{A}} v^2(z)\mathrm{d}z \tag{5.16}$$

Which is also known as the *Blasius-Chaplygin formula*. Since we already have complex velocity $v = F'$ (for a cylinder with disk cross section) where F is the corresponding complex potential (5.4), for example, we get

$$v^2(z) = \left(v_\infty - v_\infty\frac{a^2}{z^2} - i\frac{\Gamma}{2\pi z}\right)^2 = v_\infty^2 + \frac{a^2v_\infty^2}{z^4} + i\frac{a^2\Gamma v_\infty}{\pi z^3} - \frac{2a^2v_\infty^2}{z^2} - i\frac{\Gamma v_\infty}{\pi z}$$

Substituting this to (5.16) and using the Residue Theorem (Theorem 2.4.2) all Laurent coefficients c_{-1} are zero for each term except $-i\Gamma v_\infty/\pi z$, which gives

$$\overline{F}_l = i\frac{\rho}{2}\left(2\pi i\frac{v_\infty\Gamma}{\pi i}\right) = i\rho v_\infty\Gamma$$

so the resulting lift force is:

$$F_l = \rho\Gamma\overline{v}_\infty \tag{5.17}$$

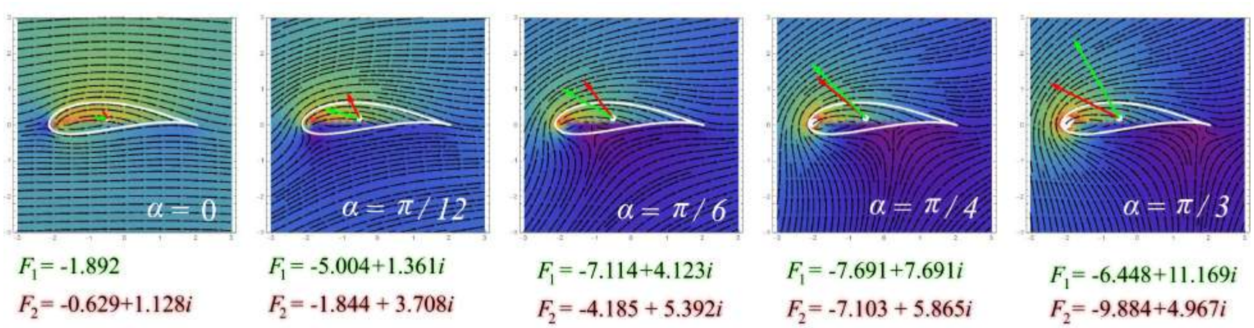


Figure 5.16: Lift forces F_1 (green) using (5.17) and F_2 (red) using (5.16) acting on an airfoil with different angles of attack (using scale factor 0.2 for the arrow lengths).

per unit length of the airfoil with cross-section $\mathcal{A}$. This relation is often referred to as the *Kutta-Joukovsky lift theorem* for a cylinder.

Formula (5.17) is, of course, an approximate computation of forces acting per unit length of a cylinder with circular cross section. For general airfoil contours we need to use the Blasius-Chaplygin formula (5.16) integrating the square of complex velocity over $\mathcal{A}$.

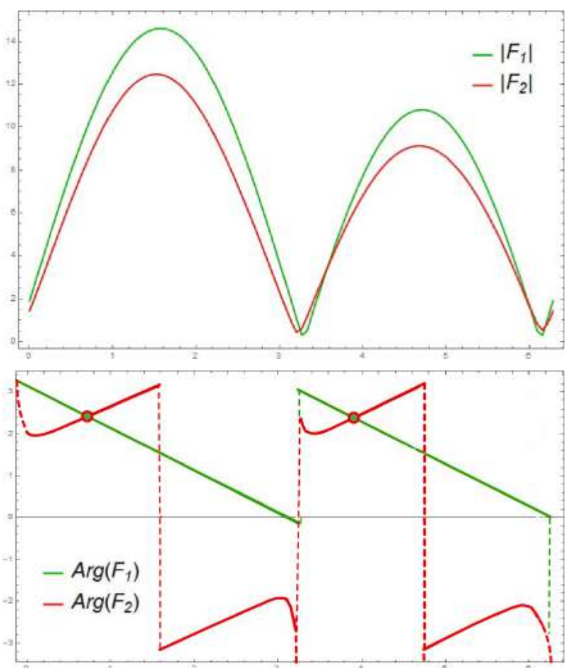


Figure 5.17: Changing magnitudes $|F_1|$ and $|F_2|$ (top) and arguments (bottom) of lift forces given by formulas (5.17) and (5.16), with respect to α.

One can see the resulting forces F_1 (using Kutta-Joukovsky formula) and F_2 (via Blasius-Chaplygin formula) in Fig.5.16, where they are compared in each image with different angles of attack using default parameters from the last result in Fig.5.15.

As we see in Fig.5.17, lift forces F_1 and F_2 do not differ in magnitude nearly as much as in arguments. The angles α for which $\mathrm{Arg}(F_1) \approx \mathrm{Arg}(F_2)$ are $\alpha \approx \pi/4$ as well as $\alpha \approx 5\pi/4$

Force F_2 generated by formula (5.16) appears to better describe the effects of lift in real flight mechanics. When the airplane speeds up (on the runway) there is a non-zero lift generated in the upward direction, causing a pitch of the aircraft, increasing the angle of attack, which leads to even greater lift. While (5.17) might be able to describe lift acting on a cylinder with circular cross section, it does not account for the cambered geometry of the Joukovsky type airfoil. In fact, during the take-off the *trailing flaps* are lowered to increase drag in the rear parts of the airfoils, and change the resulting direction of lift (forward), increasing the speed of the aircraft. *Slats* on the leading edges of wings, on the other hand, are used to increase drag on the in the front leading to downward pitch.

Fig. 5.18 shows a simulation of the take-off. Notice that as the aircraft speeds up on the runway (v_∞ increases) and changes its camber line curvature (by lowering trailing flaps) the resulting lift force causes upward pitch as well as more upward acceleration.

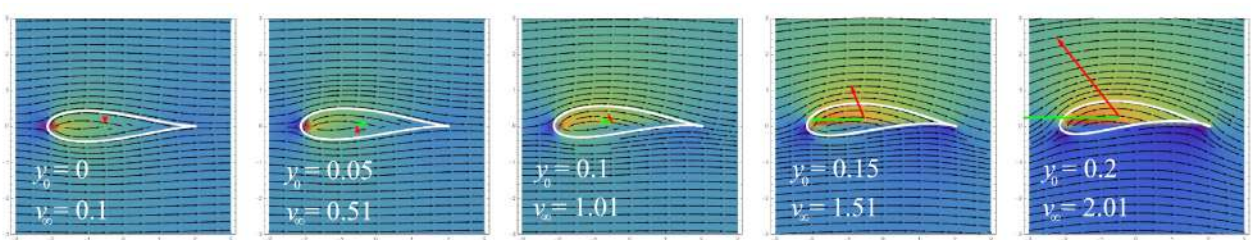


Figure 5.18: The effect on Kutta-Joukovsky lift F_1 and Blasius-Chaplygin lift F_2 while simulating aircraft take-off, that is: changing camber line curvature by y_0-parameter and increasing the asymptotic fluid velocity v_∞ (accelerating on the runway).

Chapter 6

Conclusion

With a solid background in complex function theory in chapters 2. - 4., multiple demonstrations of laminar (potential) flow past solid objects have been shown in Chapter 5. Being the basis for the transformations of the complex plane, the Joukovsky map not only appears as a solution to conjugate harmonic Dirichlet and Neumann problems on the exterior of a disk, but also provides a transformation of the system so that it approximates other geometries, like the tilted plate, Rankine Oval, and various Joukovsky type airfoils.

While the majority of literature presents the problem as a mere transformation of complex variables, some adjustments were required to approach realistic flow scenarios to produce flow past Joukovsky-like shapes from flow past the Rankine Oval (see Fig. 5.9). On the other hand, airfoils which need to satisfy the Kutta Condition (cusped trailing edge) have been obtained from different approaches in which we implement the fundamental geometric parameters like airfoil thickness and chord length (see Fig.5.13).

Furthermore, we introduced potential flow with circulation (section 5.5) and utilized the effect for Joukovsky type airfoils in section 5.6 with three types of circulation: constant, linearly dependent on angle of attack α (introducing drag), and proportional to $\sin\alpha$ for airfoils under full inversion or rotation.

Afterwards the effects of lift have been examined in section 5.7 where we derived the Blasius-Chaplygin formula (5.16) and the Kutta-Joukovsky theorem (5.17) both of which were compared in terms of magnitude and direction exerted on the airfoil with varying angle of attack.

Although much of the methods used in this work have been developed in the early to mid 20th century, with the use of computational methods modern aerodynamics and aerospace engineering is a constantly developing field involving some of the most complicated engineering projects in history.

As for the tools used for this work, the major tool for numerical computation and visualization has been Wolfram Mathematica©. Enhanced phase portraits and related visualizations of complex-valued functions have been provided by an online tool: [15], and mostly by a Matlab© script [16] called the *Complex Function Explorer* by Wegert. Most figures have undergone additional adjustments in multiple graphics programs.

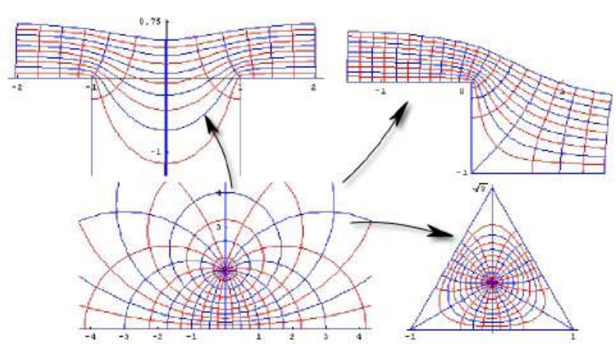


Figure 6.1: Schwarz-Christoffel mappings of the upper-half plane $\mathbb{H}$ onto various domains with polygonal boundaries.

It can be concluded that the approach through conformal mappings yields very realistic results for some intrinsic parameters (especially in examples from sections 5.6 and 5.7), while other values produce overlapping analytic branches of the complex potential and cannot be used for most problems. However, it might be useful for future research to study the behavior of individual analytic branches for complex potentials like the ones used in Chapter 5, perhaps analytic continuations can be carried out along different paths to obtain a global complex potential, represented on a Riemann Surface, for example.

Other mappings on $\mathbb{C}$ producing suitable airfoil shapes, such as the *Kármán-Trefftz profiles* can be utilized as well.

Another possibility for future development is to make use of the Schwarz-Christoffel mapping (see Wegert [2], section 6.8., p.302) which maps the upper half plane $\mathbb{H} = \{z \in \mathbb{C}\, |\mathrm{Im}(z) > 0\}$ conformally onto a polygonal region (see Fig.6.1 for examples). Using this type of conformal map one could model flow in 2-dimensional sections of a pipeline, or even regions with more complicated geometry [18]. The results of these methods can then be compared with numerical solutions and/or experimental measurements. The SC Toolbox for Matlab © (see Driscoll [17]) is a versatile tool for Schwarz-Christoffel maps on a wide range of domains.

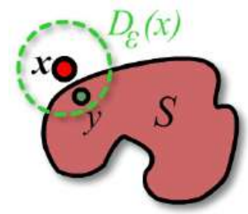
There is, of course, a wide variety of topics from complex analysis one can bridge into. Overlapping with major problems in topology and differential geometry (many of which are still unsolved), complex analysis is still a developing field with wide applications, ranging from quantum theory, signal processing to partial differential equations.

Appendices

Appendix A

Some Topology in $\mathbb{C}$

Because of the use of some fundamental notions in previous chapters, it is necessary to properly define them. This appendix gives definitions of basic terms in algebraic topology, as well as plane-geometric terminology regarding paths in the complex plane that are essential for complete understanding of complex integration and harmonic function theory.



Definition A.0.1. A *metric space* is an ordered pair (X, d) where X is a set and map $d : X \times X \to \mathbb{R}$ is called a *metric*, where for all $x, y, z \in X$:
(I) $d(x, y) \geq 0$ with $d(x, y) = 0 \iff x = y$.
(II) $d(x, y) = d(y, x)$.
(III) $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

Definition A.0.2. Set $\mathbb{D}_\epsilon(a) = \{x \in X \mid d(x, a) < \epsilon\}$ is called an *open disk* with radius $\epsilon > 0$ in X. And any set N such that $x \in N$ and $\mathbb{D}_\epsilon(x) \subseteq N$ is called a *neighborhood* of x.

Definition A.0.3. Let X be a set, then a *topology* of X is a family τ_X of subsets of X (also called *open sets*) such that:
(I) $X, \varnothing \in \tau_X$.
(II) $\forall\, \mathcal{U}_i \in \tau_X$, $\bigcap_{i=1}^{n} \mathcal{U}_i \in \tau_X$, $n \in \mathbb{N}$.
(III) $\forall\, \mathcal{U}_i \in \tau_X$, $\bigcup_{i=1}^{\infty} \mathcal{U}_i \in \tau_X$.

A pair (X, τ_X) is also called a *topological space*. If for each $\mathcal{U} \in \tau_X$ there exists $\epsilon > 0$ such that $\mathbb{D}_\epsilon(x) \subseteq \mathcal{U}$, the topology $\tau_d = \{\mathcal{U} \subseteq X \mid \forall x \in \mathcal{U}, \exists \epsilon > 0; \mathbb{D}_\epsilon(x) \subseteq \mathcal{U}\}$ is called topology on X induced by a metric d, or also a *standard topology* (or also *Euclidean*, i.e.: induced by Euclidean metric).

Given $S \subseteq X$ of a metric space (X, d), there are two common topologies on S:
(a) $\tau_d(S) = \{\mathcal{U} \subseteq S \mid \forall x \in \mathcal{U}, \exists \epsilon > 0; D_\epsilon(S, x) \subseteq \mathcal{U}$ where $\mathbb{D}_\epsilon(S, x) = \mathbb{D}_\epsilon(x) \cap S\}$
(b) $\tau_{sub}(S) = \{\mathcal{U} \subseteq S \mid \mathcal{U} = \mathcal{V} \cap S$ for some $\mathcal{V} \in \tau_X\}$ a.k.a: *subspace (relative) topology* on S.

A set $E \subseteq X$ is *closed* if it is a complement of an open set: $E = S^{\mathrm{c}} = X \backslash S = \{x \in S \mid x \notin S\}$ where $S \in \tau_X$. A *limit point* x of S is a point such that for any $\epsilon > 0$, there exists $y \in \mathbb{D}_\epsilon(x)$ where $x \neq y$ and $y \in S$, i.e.: x is a limit point of S if and only if $\mathbb{D}_\epsilon(x) \cap S \neq \varnothing$ for all $\epsilon > 0$. Also, x is a limit point of S if every neighborhood of x contains uncountably many elements of S. A set $S \subseteq X$ is called *dense in* X if every $x \in X$ is either inside S or its limit point.

Set $\mathrm{cl}(S) = S \cup \{x \in X \mid x$ is a limit point of $S\}$ is called *closure* of S (also denoted as $\overline{S}$, not to be confused with complex conjugation). Also S is closed if $S = \mathrm{cl}(S)$. A *boundary* of S is $\mathrm{bd}(S) = \{x \in X \mid \mathbb{D}_\epsilon(x) \cap S \neq \varnothing$ & $\mathbb{D}_\epsilon(x) \cap S^{\mathrm{c}} \neq \varnothing\}$, and the *interior*: $\mathrm{int}(S) = \{x \in S \mid \exists\ \epsilon > 0,\ \mathbb{D}_\epsilon(x) \subseteq S\}$ (with shorter notation: ∂S). Naturally: $\mathrm{cl}(S) = S \cup \partial S$, and if S is dense in X, then $X = \mathrm{cl}(S)$.

Definition A.0.4. Let X be a topological space. An *open cover* $\mathcal{C}$ of X is a family of sets $\mathcal{U} \in \tau_X$ such that $X = \bigcup_{\mathcal{U}\in\mathcal{C}} \mathcal{U}$. X is called *compact* if there exists a finite subset $\mathcal{F} \subset \mathcal{C}$ such that $X = \bigcup_{\mathcal{U}\in\mathcal{F}} \mathcal{U}$

Definition A.0.5. Let (X, τ_X) and (Y, τ_Y) be topological spaces, then map $f : X \to Y$ is *continuous* if for every $\mathcal{V} \in \tau_Y$, the preimage $f^{-1}[\mathcal{V}] \in \tau_X$. If there exists a continuous bijection f (*homeomorphism*) between spaces X and Y, they are called *homeomorphic*.

Homeomorphism is an equivalence relation between topological spaces. The existence of the continuous bijection ensures the possibility of continuous transformation of space X into space Y.

Definition A.0.6. In a metric space (X, d), for $A \subseteq X$ and $x \in X$, the *distance*:
$\operatorname{dist}(x, A) = \inf\{d(x, a) \mid \forall\, a \in A\}$.
For $A, B \subseteq X$: $\operatorname{dist}(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

It should be noted that dist is not a metric, since for $A = \{0, 1\}$ and $B = \{1, 2\}$, $\operatorname{dist}(A, B) = 0$ even though $A \neq B$.

Definition A.0.7. Topological space X is *connected* if for non-empty partitions $A, B \subseteq X$ such that $A \cap B = X$, $A \cap B = \varnothing$, the closures of these sets are not disjoint: $\operatorname{cl}(A) \cap \operatorname{cl}(B) \neq \varnothing$.

Equivalently, S is connected if it cannot be partitioned into two non-empty subsets, open in the relative topology induced on $S \subseteq X$.

Remark. The *domain* $\Omega \subseteq \mathbb{C}$ of every complex-valued function $f : \Omega \to \mathbb{C}$ is non-empty, open and connected.

Definition A.0.8. Let $I \subseteq \mathbb{R}$ be an interval. A *curve* in $\mathbb{C}$ is a continuous map $\gamma : I \to \mathbb{C}$. The *trace* (or *image*) of γ is $[\gamma] = \gamma[I] \subseteq \mathbb{C}$. γ is *regular* if $\gamma'(t) \neq 0$, $\forall t \in I$ and *smooth* if $\mathrm{Re}\gamma$ and $\mathrm{Im}\gamma$ have continuous derivatives for all $t \in I$.

Definition A.0.9. A curve $\gamma_1 : I \to \mathbb{C}$ is a *reparametrization* of $\gamma_0 : J \to \mathbb{C}$ if there exists a continuous bijection $\phi : I \to J$ such that $\gamma_1 = \gamma_0 \circ \phi$.

Consequently, their traces (images) are equal: $[\gamma_0] = [\gamma_1]$.

Definition A.0.10. A *reversed* (*negative*) curve is $\gamma^- : [\alpha, \beta] \to \mathbb{C} : t \mapsto \gamma^-(t) = \gamma(\alpha + \beta - t)$. If $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ and $\gamma_2 : [\alpha_2, \beta_2] \to \mathbb{C}$ are curves in $\mathbb{C}$ and $\gamma_1(\beta_1) = \gamma_2(\alpha_2)$, then $\gamma = (\gamma_1 \oplus \gamma_2) : [\alpha_1, \beta_1 + \beta_2 - \alpha_2] \to \mathbb{C}$, such that

$$\gamma(t) = \begin{cases} \gamma_1(t) & \alpha_1 \leq t \leq \beta_1 \\ \gamma_2(t + \alpha_2 - \beta_2) & \beta_1 < t \leq \beta_1 + \beta_2 - \alpha_1 \end{cases}$$

is called a *concatenation* of curves γ_1 and γ_2. Generally, a *contour* or a *path* is $\Gamma = \gamma_1 \oplus ... \oplus \gamma_n$. A path is called *simple* if γ is injective. A path γ for which $\gamma(\alpha) = \gamma(\beta)$, where $\alpha = \alpha_1$ and $\beta = \beta_n$ is called a *closed path* or *loop*.

The $\oplus$-operator from the previous definition is associative, but not commutative. A simple example of a curve in $\mathbb{C}$ is a circle $\gamma(t) = z_0 + re^{2\pi i t}$, centered at $z_0 \in \mathbb{C}$.

Remark. To make sure all the results for notions like *path integrals* hold independently of the given parametrization, the term "*curve*" is often used to denote the equivalence class of all the paths that are reparametrizations of one another.

Definition A.0.11. Let $\gamma : I \to \mathbb{C}$ be a path. A *chain of disks covering* γ is a finite sequence $D_0, D_1, ..., D_n$ of open disks D_k satisfying
(1) There exists a partition of $I = [0, 1]$ such that $0 = t_0 < t_1 < ... < t_n = 1$ and $D_k = D_r(\gamma(t_k))$, $r > 0$.
(2) The section of the trace $[\gamma]$ between $\gamma(t_{k-1})$ and $\gamma(t_{k+1})$ is containted in D_k (see Fig.A.1).

Lemma A.0.1. *(Path Covering Lemma): Let γ be a path in $\Omega \subseteq \mathbb{C}$, then there exists a chain of disks $D_k \subseteq \Omega$ covering γ. Moreover, the radii of the disks can be chosen to be of the same size and arbitrarily small.*

Proof. Since γ is continuous on $[0, 1]$, its trace is a compact subset of Ω. $\mathbb{C}\backslash\Omega$ is closed and hence $d = \text{dist}([\gamma], \mathbb{C}\backslash\Omega) > 0$. If $0 < r < d$, then all disks are contained in Ω. Because γ is uniformly continuous, there exists $\delta > 0$ such that $s, t \in [0, 1]$ and $|s - t| < \delta$ implies that $|\gamma(s) - \gamma(t)| < r$. So all requirements are satisfied if the partition $0 = t_0 < t_1 < ... < t_n = 1$ is chosen so that $t_k - t_{k-1} < \delta$. $\square$

The Path Covering Lemma will be important later when showing some essential properties of winding numbers in $\mathbb{C}$.

Now we define one of the major notions in algebraic topology:

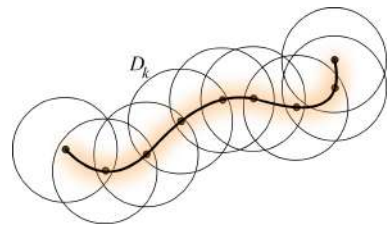


Figure A.1: a chain of disks D_k covering a path.

Definition A.0.12. Let $\Omega \subset \mathbb{C}$. Consider paths $\gamma_0 : I \to \Omega$ and $\gamma_1 : J \to \Omega$ with a and b being the starting and ending points of both γ_0 and γ_1. Then these paths are said to be *homotopic* if there exists a continuous map $h : I \times J \to \Omega$ such that

(I) $h(0, t) = \gamma_0(t)$, for $0 \leq t \leq 1$
(II) $h(1, t) = \gamma_1(t)$, for $0 \leq t \leq 1$
(III) $h(s, 0) = a$, for $0 \leq s \leq 1$
(IV) $h(s, 1) = b$, for $0 \leq s \leq 1$.

Map h is called a *homotopy* from path γ_0 to path γ_1. The homotopy relation between paths is denoted as $\gamma_0 \simeq \gamma_1$ (see.Fig.A.2 (up)).

If $\gamma \circ \phi$ is a reparametrization of γ, then $\gamma \circ \phi \simeq \gamma$. Clearly if we take $h(s, t) = \gamma((1 - s)t + s\phi(t))$, then h is a homotopy. Note that every path is homotopic to itself $\gamma \simeq \gamma$, and the relation is symmetric, and if $\gamma_0 \simeq \gamma_1$ and $\gamma_1 \simeq \gamma_2$, then also $\gamma_0 \simeq \gamma_2$, thus $\simeq$ is an equivalence relation.

Definition A.0.13. A domain Ω is *path-connected* if any two points $a, b \in \Omega$ can be joined by a path.

Surely, path-connectedness is stronger than connectedness. Because a connected set may be a union of disjoint sets whose closures are not disjoint, not all connected sets are also path-connected.

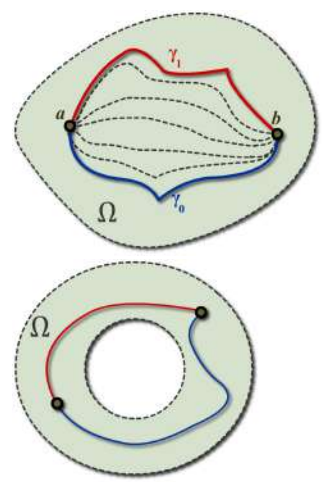


Figure A.2: A simply connected domain (above) with homotopic paths γ_0 and γ_1 and a multiply connected domain (below)

Definition A.0.14. A domain Ω is *simply-connected* if any two paths γ_0 and γ_1 with $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$, and $[\gamma_0], [\gamma_1] \subset \Omega$, are homotopic in Ω. Domains that are not simply connected are called *multiply-connected* (see.Fig.A.2 (down)).

For example, a disk $\mathbb{D}_\epsilon(z_0)$ is a simply-connected domain, but a punctured disk $\mathbb{D}_\epsilon(z_0)\backslash\{z_0\}$ or a ring domain $\Omega = \{z \in \mathbb{C} \mid \epsilon_1 < |z| < \epsilon_2\}$ are multiply-connected.

For closed pahths (loops) there is a second notion of homotopy which is more general because it does not require the endpoints to be fixed.

Definition A.0.15. Loops γ_0 and γ_1 in Ω are *freely homotopic* if there exists a continuous map $h : I \times J \to \Omega$ such that

(I) $h(0, t) = \gamma_0(t)$, $0 \leq t \leq 1$
(II) $h(1, t) = \gamma_1(t)$, $0 \leq t \leq 1$

(III) $h(s,0)=h(s,1)$, $0\le s\le 1$

(see.Fig.A.3 (a)). A loop which is freely homotopic to a constant path $\gamma_a : t\mapsto a$, $a\in\Omega$ is said to be *null-homotopic* (*contractible*)(Fig.A.3 (b)).

Lemma A.0.2. *For any path γ in Ω the loop $\gamma\oplus\gamma^-$ is null-homotopic in Ω to its base point $z_0=\gamma(0)$.*

Proof. If $\gamma\oplus\gamma^-$ is contracted along its trace $[\gamma]$ to its base point $\gamma(0)$ by $h:[1,0]\times[1,0]\to\Omega$ such that

$$h(s,t)=\begin{cases}\gamma(2st) & 0\le t\le 1/2\\ \gamma(2s(1-t)) & 1/2<t\le 1\end{cases}$$

then h is continuous on $[1,0]\times[1,0]$ with its range is contained in $[\gamma]$, and it satisfies $h(0,.)=\gamma(0)$ and $h(1,.)=\gamma\oplus\gamma^-$ (see.Fig.A.3 (c)). □

Lemma A.0.3. *If a loop with base point z_0 is null-homotopic in Ω then it is also homotopic with fixed endpoints to the constant path $t\mapsto z_0$.*

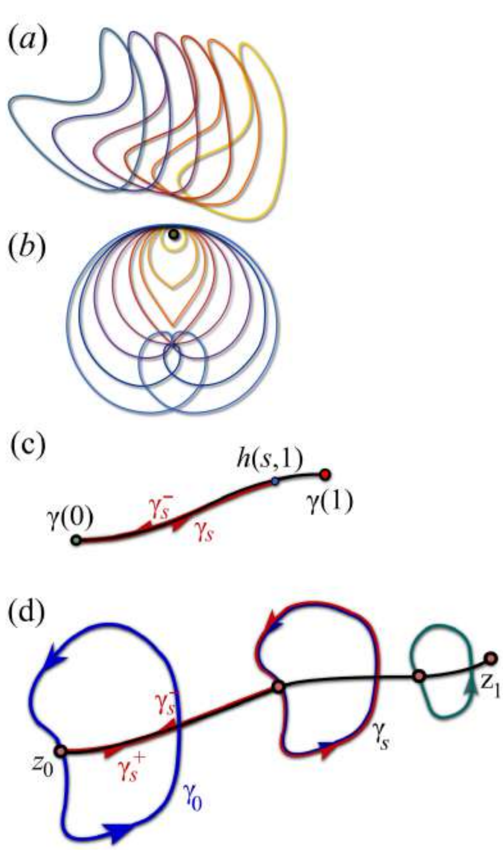


Figure A.3

Proof.

(see.Fig.A.3 (d)) Let γ_0 be a given loop and h be a homotopy that contracts γ_0 to a point z_1. Define γ_s and γ_s^+ by $\gamma_s(t)=h(s,t)$, $\gamma_s^+(t)=h(st,0)$, so γ_s is a family of loops transforming under h into point z_1 and γ_s^+ and γ_s^- be mutually negative paths such that $\gamma_s^+\oplus\gamma_s^-=z_0$ (a constant path). Then the path γ_s^+ connects z_0 with the moving base point $z_s=h(s,0)=h(s,1)$ of loop γ_s. Take a family of paths (parametrized by s) $\gamma_s^*=\gamma_s^+\oplus\gamma_s\oplus\gamma_s^-$ which has a fixed base point z_0. Obviously, all of them are homotopic to one another. So $\gamma_0\simeq\gamma_0^*$, $\gamma_1\simeq\gamma_1^+\oplus\gamma_1^-$. And by Lemma A.0.2: $\gamma_1^+\oplus\gamma_1^-\simeq z_0$, thus by the fact that $\simeq$ is an equivalence relation: $\gamma_0\simeq z_0$. □

Lemma A.0.4. *A domain Ω is simply connected if and only if any loop in Ω is null-homotopic.*

Proof. ($\Longrightarrow$): Assume that Ω is simply connected and $\gamma:[0,1]\to\Omega$ is a loop. Decomposing $\gamma=\gamma_1\oplus\gamma_2$ gives the fact that paths γ_2 and γ_1^- have the same initial point and the same terminal point, and hence $\gamma_2\simeq\gamma_1^-$. Then $\gamma\simeq\gamma_2\oplus\gamma_1^-$ and by Lemma A.0.3: γ is null-homotopic.

($\Longleftarrow$): Let Ω be a domain with any loop null-homotopic. If γ_0 and γ_1 are paths with $\gamma_0(0)=\gamma_1(0)=a$ and $\gamma_0(1)=\gamma_1(1)=b$ then $\gamma=\gamma_0\oplus\gamma_1^-$ is a loop with base point a and $\gamma_0=\gamma\oplus\gamma_1$. By the assumption and by Lemma A.0.3: γ is homotopic with fixed endpoints to the constant path $t\mapsto a$ via a family of paths γ^s which induces a homotopic family $\gamma_s=\gamma^s\oplus\gamma_1$ from γ_0 to γ_1, which means Ω is simply connected. □

Lemma A.0.5. *Let $\gamma:I\to\mathbb{C}\backslash\{0\}$ be a path. Then there exist continuous maps $\theta:I\to\mathbb{R}$ and $r:I\to\mathbb{R}^+$ such that $\gamma(t)=r(t)e^{i\theta(t)}$.*

Proof. $r(t) = |\gamma(t)|$ is continuous and positive (Def.A.0.8), so the proof reduces to finding an appropriate continuous map θ.

At the initial point of γ choose $\theta(0) = \mathrm{Arg}(\gamma(0))$. If $t \in [0, t_1]$ all points $\gamma(t)$ lie in the disk $\mathbb{D}_0$. Since $\mathbb{D}_0$ does not contain the origin, it is contained in a sector with vertex at 0 and opening angle $\alpha < \pi$. Consequently $\theta(t) = \mathrm{Arg}(\gamma(t))$ can be chosen such that $|\theta(t) - \theta(0)| < \pi/2$ which yields a continuous map θ on $[0, t_1]$.

Suppose that such a map has already been constructed on $[0, t_k]$. Then it can be prolongated to $[0, t_{k+1}]$ by choosing θ on $[t_k, t_{k+1}]$ such that $|\theta(t) - \theta(t_k)| < \pi/2$, which is possible by the *Path Covering Lemma*A.0.1. So $\gamma(t) \in \mathbb{D}_k$ and $0 \notin \mathbb{D}_k$. By induction θ can be extended to all of $I = [0, 1]$. □

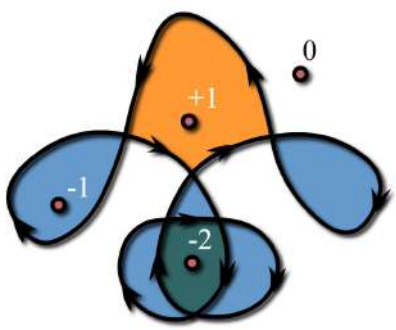


Figure A.4: Winding numbers of a loop γ for points z_0 in different components of $\mathbb{C}\backslash[\gamma]$.

The path is chosen such that $\gamma(t) \neq 0$ for all $t \in [0, 1]$ so that it can be written in a product form: $\gamma(t) = r(t)\mathrm{e}^{i\theta(t)}$ and $r(t) \neq 0$, $t \in [0, 1]$.

A map θ satisfying Lemma A.0.5 is called a *continuous branch of the argument along path* γ. The difference of such functions θ_1 and θ_2 is a constant integral multiple of 2π on $[0, 1]$. If θ is a continuous branch of the argument along a loop, then $\theta(1) - \theta(0)$ is a constant integral multiple of 2π independent of the choice of θ.

Definition A.0.16. Let γ be a loop in $\mathbb{C}\backslash\{0\}$ and θ a continuous branch of the argument along γ. Then the integer

$$\mathrm{wind}\gamma = \frac{1}{2\pi}(\theta(1) - \theta(0)) \tag{A.1}$$

is called a *winding number* (*index*) of γ. If $z_0 \in \mathbb{C}$ and γ is a loop in $\mathbb{C}\backslash\{z_0\}$ then the *winding number* of γ around z_0 is $\mathrm{wind}(\gamma, z_0) = \mathrm{wind}(\gamma - z_0)$.

Winding numbers are useful tools for complex integration and also, for example, in computer graphics, where they can be used, to find out whether a given point is inside a loop.

Lemma A.0.6. *Let γ be a piecewise-smooth closed path (loop). Then for any $z_0 \in \mathbb{C}\backslash[\gamma]$*

$$\mathit{wind}(\gamma, z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z - z_0} \tag{A.2}$$

Proof. Assume that $\gamma : [0, 1] \to \mathbb{C}$ is smooth. Map $t \mapsto \gamma(t) - z_0$ is continuous and non-vanishing on $[0, 1]$. Thus by Lemma A.0.5 there exist maps θ and r such that $\gamma(t) = z_0 + r(t)\mathrm{e}^{i\theta(t)}$ for all $t \in [0, 1]$. Since γ is smooth, and θ and r are continuously differentiable: $\gamma' = (r' + ir\theta')\mathrm{e}^{i\theta}$. Then

$$\oint_\gamma \frac{\mathrm{d}z}{z - z_0} = \int_\gamma \frac{\gamma'(t)}{\gamma(t) - z_0}\mathrm{d}t = \int_0^1 \left[\frac{r'(t)}{r(t)} + i\theta'(t)\right]\mathrm{d}t =$$

$$= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(\log r + i\theta\big)(t)\mathrm{d}t = \log r(1) - \log r(0) + i(\theta(1) - \theta(0)) = i(\theta(1) - \theta(0))$$

Where $r(1) = |\gamma(1)| = |\gamma(0)| = r(0)$ form the fact that γ is a loop. The result can be extended for a piecewise smooth path $\gamma = \gamma_1 \oplus ... \oplus \gamma_n$ by adding integrals of individual smooth components. □

Theorem A.0.7. *Let $\gamma, \gamma_0, \gamma_1 : I \to \mathbb{C}$ be loops. Then*

(1) If $z \notin [\gamma]$ then wind$(\gamma, z) =$ wind$(\gamma + w, z + w)$ for any $w \in \mathbb{C}$.

(2) If $z \in [\gamma]$ then wind$(\gamma, z) =$ wind$(\gamma - z, 0)$.

(3) If $0 \notin [\gamma_0] \cup [\gamma_1]$ then wind$(\gamma_0\gamma_1, 0) =$ wind$(\gamma_0, 0) +$ wind$(\gamma_1, 0)$ and wind$(\gamma_0/\gamma_1, 0) =$ wind$(\gamma_0, 0) -$ wind$(\gamma_1, 0)$.

(4) If $[\gamma] \subseteq \mathbb{D}_r(z_0)$ then wind$(\gamma, z) = 0$ for all $z \in \mathbb{D} \backslash \mathbb{D}_r(z_0)$, $r > 0$.

(5) If $|\gamma_0(t) - \gamma_1(t)| < |\gamma_0(t)|$ for all $t \in I$ then $0 \notin [\gamma_0] \cup [\gamma_1]$ and wind$(\gamma_0, 0) =$ wind$(\gamma_1, 0)$

Proof. (1): Suppose $z \notin [\gamma]$. Fix $w \in \mathbb{C}$ and let $\theta : I \to \mathbb{R}$ be a continuous branch of the argument of $\gamma - z$. Since $\gamma + w : I \to \mathbb{C}$ is a closed loop with $z + w \notin [\gamma + w]$ and since $(\gamma + w) - (z - w) = \gamma - z$ it follows that θ is also a continuous branch of the argument of $(\gamma + w) - (z - w)$. Thus

$$\mathrm{wind}(\gamma + w, z + w) = \frac{1}{2\pi}[\theta(1) - \theta(0)] = \mathrm{wind}(\gamma, z)$$

(2): This follows from (1) by choosing $w = -z$.

(3): Suppose $0 \notin [\gamma_0] \cup [\gamma_1]$, so $\gamma_0(t) \neq 0$ and $\gamma_1(t) \neq 0$ for any $t \in I$, which shows that $(\gamma_0\gamma_1)(t) \neq 0$ for any $t \in I$ and hence $\gamma_0\gamma_1$ is a closed curve with $0 \notin [\gamma_0\gamma_1]$. Let $\theta_0\theta_1 : I \to \mathbb{R}$ be a continuous branch of the argument of $\gamma_0\gamma_1$. Then

$$(\gamma_0\gamma_1)(t) = |\gamma_0(t)||\gamma_1(t)|\mathrm{e}^{i(\theta_0+\theta_1)(t)}$$

and by definition

$$\mathrm{wind}(\gamma_0\gamma_1, 0) = \frac{(\theta_0 + \theta_1)(1) - (\theta_0 + \theta_1)(0)}{2\pi} = \frac{\theta_0(1) - \theta_0(0)}{2\pi} + \frac{\theta_1(1) - \theta_1(0)}{2\pi} = \mathrm{wind}(\gamma_0, 0) + \mathrm{wind}(\gamma_1, 0)$$

and similarly for γ_0/γ_1:

$$(\gamma_0/\gamma_1)(t) = \frac{|\gamma_0(t)|}{\gamma_1(t)}\mathrm{e}^{i(\theta_0-\theta_1)(t)} \implies \mathrm{wind}(\gamma_0/\gamma_1, 0) = \frac{(\theta_0 - \theta_1)(1) - (\theta_0 - \theta_1)(0)}{2\pi} = \mathrm{wind}(\gamma_0, 0) - \mathrm{wind}(\gamma_1, 0)$$

(4): Suppose $[\gamma] \subset \mathbb{D}_r(z_0) = \mathbb{D}$ and fix $z \in \Omega = \mathbb{C} \backslash \mathbb{D}$. Define $f : \mathbb{D} \to \mathbb{C} : w \mapsto w - z$. Clearly if f is analytic and non-vanishing on $\mathbb{D}$, by fundamental theorem (2.3.4): $\oint_\gamma f(z)\mathrm{d}z = 0$. f has an *analytic branch of the logarithm*: $\lambda : \Omega \to \mathbb{C}$ such that $f = \exp(\lambda)$ (see Def.B.0.11 and Lemma B.0.15). And $\mu = \mathrm{Im}\{\lambda\}$ has a continuous branch of the argument of f on Ω, i.e.: $f(w) = |f(w)|\mathrm{e}^{i\mu(w)}$ for any $w \in \mathbb{D}$ and thus $f(\gamma(t)) = |f(\gamma(t))|\mathrm{e}^{i\mu(\gamma(t))}$ for $t \in I$ and by $\theta = \mu \circ \gamma$: $(\gamma - z)(t) = \gamma(t) - z = |f(\gamma(t))|\mathrm{e}^{i\mu(\gamma(t))} = |(\gamma - z)(t)|\mathrm{e}^{i\theta(t)}$. And so θ is a continuous branch of the argument of $\gamma - z$. Since $z \notin [\gamma]$ and $\gamma(0) = \gamma(1)$:

$$\mathrm{wind}(\gamma, z) = \frac{\theta(1) - \theta(0)}{2\pi} = 0$$

(5): Suppose $|\gamma_0(t) - \gamma_1(t)| < |\gamma_0(t)|$ for any $t \in I$. If $0 \in [\gamma_0]$, so that $\gamma_0(\tau) = 0$ for some $\tau \in I$, then $|\gamma_1(\tau)| < 0$ which is a contradiction. If $0 \in [\gamma_1]$, so that $\gamma_1(\tau) = 0$ for some $\tau \in I$, then $|\gamma_0(\tau)| < |\gamma_0(\tau)|$ which is also a contradiction. Hence $0 \notin [\gamma_0] \cup [\gamma_1]$. Now, $|\gamma_0(t) - \gamma_1(t)| < |\gamma_0(t)| \implies |(\gamma_0/\gamma_1)(t) - 1| < 1$ for any $t \in I$ and so $[\gamma_0/\gamma_1] \subset \mathbb{D}_1(1)$. Since $0 \in \mathbb{C} \backslash \mathbb{D}_1(1)$ by (4) we obtain $\mathrm{wind}(\gamma_0/\gamma_1, 0) = 0$ and hence $\mathrm{wind}(\gamma_0, 0) - \mathrm{wind}(\gamma_1, 0) = 0$. □

The corollary of statement (5) in Theorem A.0.7 is that winding numbers are invariant under "small" perturbations in path γ. The winding number of a loop will change when the path changes orientation, creates another loop on top of the former ones, and/or it becomes null-homotopic on the punctured plane $\mathbb{C} \backslash \{0\}$ (i.e.: all rotations of the continuous branch of the argument of γ "cancel out"). Loops with zero winding number are called *null-homologous*.

Thus, all loops that have the same winding number form an equivalence class. All loops in a single equivalence class are homotopic (with a fixed basepoint $\gamma(0) = \gamma(1)$) to each other. Consider a binary operation $\circledast$ that "adds" (concatenates) loops from these equivalence classes such that they share the same basepoint. The set of all the equivalence classes of loops corresponding to a single winding number forms an abelian group under $\circledast$ operation. This group is called the *fundamental group* of $\partial\mathbb{D}_1 = \mathbb{S}^1$ as a topological space (any loop γ with $\text{wind}(\gamma) = \pm 1$ is homotopic to a circle $\mathbb{S}^1$). It is denoted as $\pi(\mathbb{S}^1)$. This group is an important object in algebraic topology, and can be extended to an arbitrary topological space. The fundamental group of a circle $\mathbb{S}^1$ is isomorphic to $(\mathbb{Z}, +)$, which is a corollary to the fact that winding numbers can only assume integer values.

Lemma A.0.8. *Let $\Omega \subseteq \mathbb{C}$ be a simply-connected domain. Then loops γ_0 and γ_1 are homotopic in the punctured domain $\Omega \backslash \{z_0\}$ if and only if they have the same winding number for $z_0 \in \Omega$.*

Proof. The fact that homotopic loops have the same winding number follows from part (5) in Theorem A.0.7. The converse is intuitive, but the proof requires more theoretical background from algebraic topology (Hatcher [4]). $\square$

The trace of a simple loop is called a *Jordan curve.*

Theorem A.0.9. *(Jordan Curve Theorem) Let γ be a simple loop in $\mathbb{C}$. Then $[\gamma]\backslash\mathbb{C} = int(\gamma) \cup ext(\gamma)$ such that $int(\gamma) \cap ext(\gamma) = \varnothing$.*

The proof the theorem is not easy since γ can potentially be a complicated fractal entity like the boundaries of *Koch snowflake* or the *Mandelbrot set* (general proof in Hatcher [4]).

It follows from the Jordan Curve theorem that a *Jordan domain* Ω (a domain bounded by a Jordan curve) is homeomorphic to a unit disk $\mathbb{D}_1$.

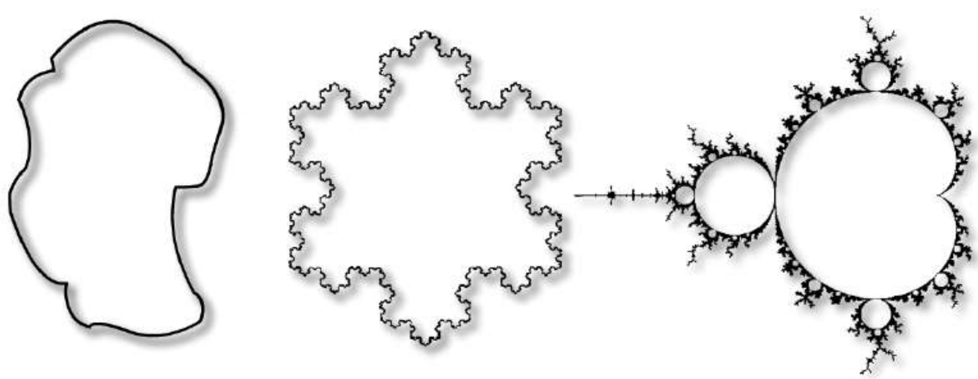


Figure A.5: Jordan curves can have complicated fractal shapes, like the *Koch snowflake* in the middle and the boundary of the Mandelbrot set on the right.

Appendix B

Power Series and Interesting Properties of Analytic Functions

Power series have been a rigorous and versatile tool for complex function theory (and function theory in general) ever since Weierstrass. As was introduced in Section 2.2, analytic functions are by definition those, that can be represented by power series: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. For "most" complex functions, this is generally true only within a certain subset of their domain. The most important notions related to the convergence of power series are introduced in this appendix.

The nature of convergence depends, of course, on the behavior of the a_n sequence of complex numbers.

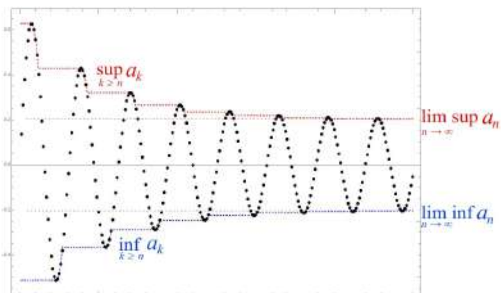


Figure B.1: Limit superior and limit inferior of a sequence $a_n = (\frac{1}{2}\mathrm{e}^{-n/50} + \frac{1}{5}) \sin \frac{n}{5}$.

Definition B.0.1. Let $\{a_n\}_{n=0}^{\infty}$ be a bounded sequence in $\mathbb{R}$. Then $A, B \in \mathbb{R}$ such that:

$$A = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} \big(\sup_{k\geq n} a_n\big) = \lim_{n\to\infty} A_n\,, \qquad B = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \big(\inf_{k\geq n} a_n\big) = \lim_{n\to\infty} B_n \tag{B.1}$$

are called *limit superior* and *limit inferior* of $\{a_n\}$. If a_n is unbounded from below, $A_n \to -\infty$ and if a_n is unbounded from above, $B_n \to \infty$. (see Fig.B.1). If $A = B$ and $\{a_n\}$ is bounded, then it is a *Cauchy* sequence.

Definition B.0.2. A sequence $\{a_n\}$ *converges* to $a \in \mathbb{C}$ if $\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}; |a_n - a| < \varepsilon$ for any $n \geq n_0$.

Recall from elementary analysis that a necessary condition for the convergence of an infinite series of real numbers $\sum_{n=0}^{\infty} a_n$ is that $\lim\limits_{n\to\infty} a_n = 0$.

Definition B.0.3. Series $\sum_{n=0}^{\infty} a_n$ *converges absolutely* if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Theorem B.0.1. *(Cauchy Criterion) Series* $\sum_{n=0}^{\infty} a_n$ *converges if and only if* $\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}$ *such that* $|\sum_{i=0}^{k} a_{n+i}| < \varepsilon$ *for any* $n \geq n_0$ *and* $k > 0$.

Proof. Let $\{s_n\}_{n=0}^{\infty}$ be a sequence of partial sums: $s_n = \sum_{k=0}^{n} a_k$. Then for any $m, n \in \mathbb{N}$ and $\varepsilon > 0$: $|s_m - s_n| = |\sum_{k=n}^{m} a_k| < \varepsilon$. Which means that this sequence is Cauchy and thus convergent. The converse is trivial. $\square$

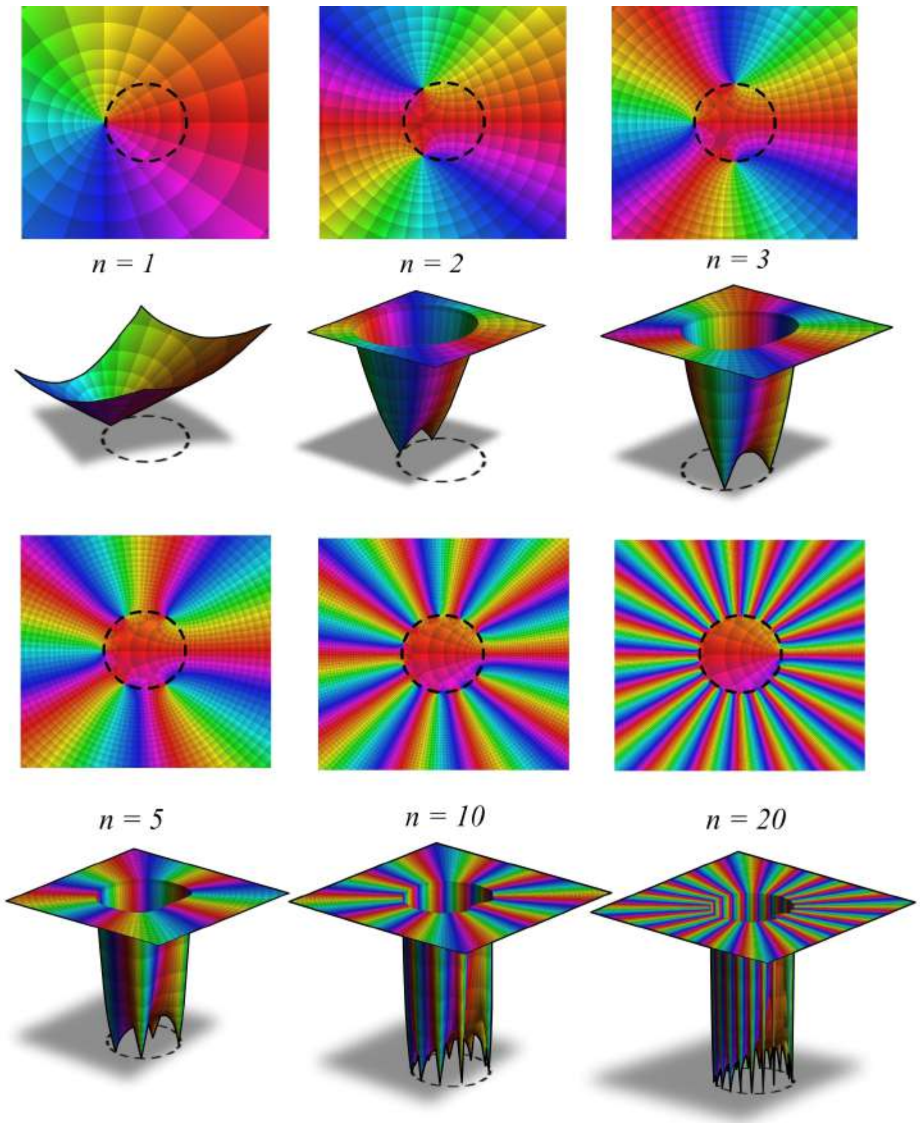


Figure B.2: Enhanced phase portraits of partial sums of power series $s_n(z) = \sum_{k=0}^{n} z^k$ with their *analytic landscape* below, where the modulus $|s_n(z)|$ is shown on the z-axis. One notices that the series converges inside the unit disk $\mathbb{D}_1 = \{z \in \mathbb{C} \mid |z| < 1\}$.

Definition B.0.4. A sequence of functions $\{f_n\}$ such that $f_n : \Omega \to \mathbb{C}$ *converges* (*pointwise*) for $z_0 \in \Omega$ to $f : \Omega \to \mathbb{C}$ if $f_n(z_0)$ converges to $f(z_0)$. And converges *uniformly* on Ω if $\forall\, \varepsilon > 0\ \exists\, n_0 \in \mathbb{N}; |f_n(z) - f(z)| < \varepsilon$ for any $n \geq n_0$ and $z \in \Omega$.

Definition B.0.5. Let $\{f_n\}$ be a sequence of functions on $\Omega \subseteq \mathbb{C}$ such that $\forall m, n \in \mathbb{N}$: $|f_m(z) - f_n(z)| \leq |a_m - a_n|$ where $\{a_n\}$ is a series of real numbers, then $\{a_n\}$ is called a *majorant* of $\{f_n\}$ if $\exists M > 0$ such that $|f_n(z)| \leq M a_n$ for any $z \in \Omega$ and $n \in \mathbb{N}$. Conversely series $\{b_n\}$ such that $|f_n(z)| \geq L b_n$ is called a *minorant* of $\{f_n\}$.

Theorem B.0.2. *(Weierstrass M-Test) Let f_n be a sequence of functions defined on $\Omega \subseteq \mathbb{C}$. If there exists a sequence of positive real numbers M_n such that $|f_n(z)| \leq M_k$ for all $z \in \Omega$ and $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} M_n$ converges, then $\sum_{n=0}^{\infty} f_n(z)$ converges absolutely and uniformly on Ω.*

Proof. Let $S_n(z) = \sum_{k=1}^{n} f_k(z)$. Since $\sum_{n=1}^{\infty} M_n$ converges and $M_n \geq 0$, for any $n \in \mathbb{N}$ then by the Cauchy criterion: for any $\varepsilon > 0$ there exist $n_0, m, n \in \mathbb{N}$ such that $m > n > n_0$ and $\sum_{k=n+1}^{m} M_k < \varepsilon$. Then for all $z \in \Omega$ and $m > n > n_0$: $|S_n(z) - S_m(z)| = \left|\sum_{k=n+1}^{m} f_k(z)\right|$. Then by triangle inequality: $\left|\sum_{k=n+1}^{m} f_k(z)\right| \leq \sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=m}^{n} M_k < \varepsilon$ which means that the sequence of partial sums converges uniformly, and thus $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly as well. □

The most trivial example of a power series is $1 + z + z^2 + z^3 + ... = \sum_{n=0}^{\infty} z^n$. This is a special case of series in expression (2.10) with $z_0 = 0$ and all coefficients $a_k = 1$ for $k = 0, 1, ...$ One can write partial sums $s_n(z) = \sum_{k=0}^{n} z^k = \frac{z^n - 1}{z - 1}$ using the formula for the sum of geometric series. Letting $n \to \infty$ one notices that the value of $s_n(z)$ "explodes" for certain $z \in \mathbb{C}$ because of the term in the numerator. But for $|z| < 1$ the numerator becomes 1, and we get $s_\infty(z) = \frac{1}{z-1}$ (see Fig.B.2).

Surprisingly power series of any function at a given point converges inside a disk with some radius, even if the radius is infinite and the domain of convergence covers the entire $\mathbb{C}$ (*entire functions*). These results are summarized in the following theorem:

Theorem B.0.3. *(Cauchy-Hadamard) For every power series (2.10) there exists R such that $0 \leq R \leq \infty$, called the radius of convergence with the following properties:*

(1) The series converges absolutely for all $z \in \mathbb{D}_R$, and if $0 \leq \rho < R$ the convergence is uniform for $z \in \mathbb{D}_\rho$.

(2) If $|z| > R$, the terms of the series are unbounded, and the series is consequently divergent.

(3) Inside $\mathbb{D}_R$, the sum of the series is differentiable, and the derivative has the same radius of convergence.

Proof. (1)

Assume $0 < R < \infty$. Choose $1/R = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ (Hadamard formula). If $|z - z_0| < R$ find ρ such that $|z - z_0| < \rho < R$. Then $1/\rho > 1/R$ and by the Hadamard formula and the definition of limit superior, there exists $n_0 \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < 1/\rho \Rightarrow |a_n| < 1/\rho^n$ for $n \geq n_0$.

It follows that $|a_n(z - z_0)^n| < (|z - z_0|/\rho)^n$, so the power series (2.10) has a convergent geometric series (with quotient $|z - z_0|/\rho$) as a majorant, and consequently is convergent.

To show the uniform convergence for $|z - z_0| \leq \rho < R$ choose ρ' such that $\rho < \rho' < R$ and find $|a_n(z - z_0)^n| < (|z - z_0|/\rho)^n \leq (\rho/\rho')^n$ for $n \geq n_0$. Since the majorant is convergent and has constant terms, it is concluded by the Weierstrass M-Test that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is uniformly convergent.

Now suppose $R = 0$, then $1/R = \infty$, so for any $z \neq z_0$:

$$\limsup_{n\to\infty} \sqrt[n]{|a_n(z - z_0)^n|} = |z - z_0| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$$

and thus (2.10) diverges. For $z = z_0$ the convergence is trivial since (2.10) becomes just one term a_0.

Finally, suppose $R = \infty$ then $1/R = 0$ and for any $z \in \mathbb{C}$ $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$. It follows that z can be chosen arbitrarily far from z_0 and the series (2.10) would still converge.

(2)

If $|z - z_0| > R$ choose ρ so that $R < \rho < |z - z_0|$. Since $1/\rho < 1/R$ for arbitrarily large n $\sqrt[n]{|a_n|} > 1/\rho \Rightarrow |a_n| > 1/\rho^n$. Thus $|a_n(z - z_0)^n| > (|z - z_0|/\rho)^n$ for infinitely many n, and the terms are unbounded.

(3)

Let $\sum_{n=1}^{\infty} na_n(z - z_0)^n$ be the derivative of series (2.10). For $|z - z_0| < R$ write

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = s_n(z) + R_n(z) \text{ ,where } s_n(z) = \sum_{k=0}^{n-1} a_k(z - z_0)^k \text{ , } R_n(z) = \sum_{k=n}^{\infty} a_k(z - z_0)^k \quad \text{(B.2)}$$

And also

$$f_1(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^n = \lim_{n\to\infty} s'_n(z) \quad \text{(B.3)}$$

Now it remains to be shown that $f'(z) = f_1(z)$. Consider

$$\frac{f(z) - f(w)}{z - w} - f_1(w) = \left(\frac{s_n(z) - s_n(w)}{z - w} - s'_n(w)\right) + \left(s'_n(w) - f_1(w)\right) + \left(\frac{R_n(z) - R_n(w)}{z - w}\right) \quad \text{(B.4)}$$

where $z \neq w$ and $|z - z_0|, |w - z_0| < \rho < R$. The last term in (B.4) can be re-written as

$$\frac{R_n(z) - R_n(w)}{z - w} = \sum_{k=n}^{\infty} a_k\left((z - z_0)^{k-1} + (z - z_0)^{k-2}(w - z_0) + ... + (z - z_0)(w - z_0)^{k-2} + (w - z_0)^{k-1}\right)$$

and we conclude that

$$\left|\frac{R_n(z) - R_n(w)}{z - w}\right| \leq \sum_{k=n}^{\infty} k|a_k|\rho^{k-1} \quad \text{(B.5)}$$

The right-hand side in (B.5) is the remainder term in a convergent series, hence for any $\varepsilon/3 > 0$ there exists $n_0 \in \mathbb{N}$ such that $\left|\frac{R_n(z)-R_n(w)}{z-w}\right| < \frac{\varepsilon}{3}$ for any $n \geq n_0$. Similarly there also exists $n_1 \in \mathbb{N}$ such that $|s'_n(w) - f_1(w)| < \frac{\varepsilon}{3}$ for any $n \geq n_1$. Now fix $n \geq n_0, n_1$ and by the definition of a derivative find $\delta > 0$ such that $0 < |z - w| < \delta$ implies

$$\left|\frac{s_n(z) - s_n(w)}{z - w} - s'_n(w)\right| < \frac{\varepsilon}{3}$$

Combining all three terms into (B.4) it follows that

$$\left|\frac{f(z) - f(w)}{z - w} - f_1(w)\right| < \varepsilon \text{ whenever } 0 < |z - w| < \delta \quad \text{(B.6)}$$

And so for $z \to w$ equality holds without change in R. □

The last result of Theorem B.0.3 has unexpected consequences. The reasoning in the proof of part (3) of the theorem can be repeated indefinitely. Thus, the existence of a single derivative and a power series expansion implies the existence of the derivatives of all orders:

$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$

$f'(z) = a_1 + 2a_2(z - z_0) + 3a_3(z - z_0)^2 + ...$

$f''(z) = 2a_2 + 6a_3(z - z_0) + 12a_4(z - z_0)^2 + ...$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$

$f^{(k)}(z) = k!a_k + \frac{(k+1)!}{1!}(z - z_0) + \frac{(k+2)!}{2!}(z - z_0)^2 + ...$

That is, of course, provided that the power series expansion converges on the domain or its subdomains. Dividing the last expression by $k!$ and putting $z = z_0$ we get $a_k = \frac{f^{(k)}(z_0)}{k!}$ which are the coefficients of a Taylor series development. However, this result was proved under the assumption that f already has a power series development, which means that other more advanced tools have to be used to prove the existence of power series for a holomorphic function (see Theorem 2.3.9).

Theorem B.0.4. *(Local Normal Form) Let $f : \Omega \to \mathbb{C}$ be analytic on its domain. If f is non-constant in a neighborhood of $z_0 \in \Omega$ then there exists $m \in \mathbb{N}$ and an analytic function $g : \Omega \to \mathbb{C}$, such that $g(z_0) \neq 0$ and $f(z) = f(z_0) + (z - z_0)^m g(z)$.*

Proof. Assume that the Taylor series of f converges in some disk $\mathbb{D} \subseteq \Omega$. Denoting a_m the first non-zero coefficient from $a_1, a_2, a_3, ...$ then

$$f(z) = f(z_0) + (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m} \quad , z \in \mathbb{D}$$

The sum in the second term is an analytic function in $\mathbb{D}$ with $g(z_0) = a_m \neq 0$. Define $g : \Omega \to \mathbb{C}$ so that

$$g(z) = \begin{cases} (z - z_0)^{-m}(f(z) - f(z_0) & \text{if } z \in \Omega \backslash \{z_0\} \\ a_m & \text{if } z = z_0 \end{cases}$$

Then g is analytic in $\Omega \backslash \{z_0\}$. And since it coincides with $g(z_0) = a_m$ in $\mathbb{D}$, it is also analytic at z_0.

To show the uniqueness, assume $(z - z_0)^n g_1(z) = (z - z_0)^m g_2(z)$ with $n > m$ for all $z \in \Omega$. Then $(z - z_0)^{n-m} g_1(z) = g_2(z)$ and the left-hand side vanishes at z_0 while $g_0(z_0) \neq 0$. So $m = n$ and $g_1 = g_2$. ☐

The normal form describes the local behavior of an analytic function in the neighborhood of some point z_0. The integer m is a crucial parameter.

Definition B.0.6. $m \in \mathbb{N}$ from Theorem B.0.4 is called the *order* (or the *multiplicity*) of f at z_0 and is denoted $\text{ord}(f, z_0)$. If f is constant then $\text{ord}(f, z_0) = \infty$. If $f(z_0) = 0$ then m is the order (multiplicity) of the zero of f at z_0.

The order of a function at a given point can be easily read from its phase portrait. If ξ are *isochromatic lines* of f, i.e.: the curves along which $\text{Arg} f(z) = $ *const.*, then if $f(z_0) \neq 0$ the order of f at z_0 is the number of isochromatic lines passing through z_0.

Take the example in Fig.B.3: $f(z) = \frac{z-1}{z^2+z+1}$. By finding the roots of the expression's denominator one finds two 1-st order poles at $p_0 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (where f fails to be analytic) and $p_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$, and obviously $z_2 = 1$ is a zero. Take also points z_0 and z_1. z_0 has a single isochromatic line passing through and one can substitute specific value to verify that the order of this point is actually $m = 1$. z_1 is an interesting example of a chromatic *saddle point*, where two isochromatic lines meet, but the function does not change its argument in any direction from z_1. And finally, the zero at $z_2 = 1$

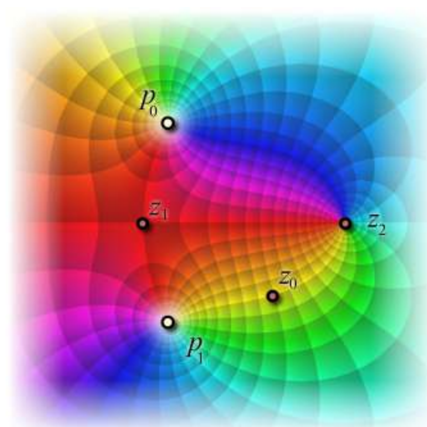


Figure B.3: An enhanced phase portrait of $f(z) = \frac{z-1}{z^2+z+1}$ with poles at $p_0 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $p_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and a zero at $z_2 = 1$.

has all the local isochromatic lines ξ passing through. Notice that the argument revolves once around z_2 and the poles p_0 and p_1 as well. One may suspect that the amount of "revolutions" of the argument around a zero or a pole is closely related to its order.

Definition B.0.7. A zero (or a singularity) z_0 is *isolated* if for $\epsilon > 0$ (arbitrarily small) $\mathbb{D}_\epsilon(z_0)$ contains no other zeros (or singularities) of f.

The following lemma states that if f assumes multiple values on its domain, in a sufficiently small neighborhood of a given point, the value is unique.

Lemma B.0.5. *If f is analytic at z_0 and $f(z) = a$, then there exists $\mathbb{D}(z_0)$ such that either $f(z) = a$ for all $z \in \mathbb{D}(z_0)$ or $f(z) \neq a$ for all $z \in \mathbb{D}(z_0)\backslash\{z_0\}$.*

Proof. If f is constant, then the result is obvious. If f is non-constant and $f(z_0) = a$ then by the definition of function g in Theorem B.0.4 $f(z) = a$ only for $z = z_0$ and for no other value. $\square$

Essentially, what this lemma states is that all zeros of non-constant analytic functions are isolated. Note that this statement does not claim that the set of zeros of an analytic function has no limit points (lying in the closure of the domain $\mathrm{cl}(\Omega)$). It merely states that the limit points of zeros cannot be zeros themselves.

Theorem B.0.6. *(Identity Theorem, Uniqueness Principle) Let f and g be analytic on (a connected domain) Ω. If there exists a sequence $\{z_n\} \subset \Omega\backslash\{z_0\}$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = g(z)$ for all $z \in \Omega$.*

Proof. (1) Let $h = f - g$, then h has a sequence of zeros converging to $z_0 \in \Omega$. The continuity of h implies that $h(z) = 0$. Then z_0 is also a zero which is not isolated. Since h is analytic in Ω, by Lemma B.0.5 h must be constantly zero on $\mathbb{D}(z_0)$.

(2) Choose any $z_1 \in \Omega$ and show that $h(z_1) = 0$. Since Ω is connected, there exists a path $\gamma : I \to \Omega$ such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Now let $\mathcal{S} = \{s \in [0,1] \mid h(\gamma(t)) = 0, \forall t \in [0,s]\} \neq \varnothing$. Set $\mathcal{S}$ is non-empty because not only $h(\gamma(0)) = h(z_0) = 0$, but also there is an arbitrary infinite sequence of points $\{z_n\}$ converging to z_0. Denote $s_0 = \sup \mathcal{S}$. Since h is continuous $h(\gamma(s_0)) = 0$ and for all $t \in [0, s_0]$: $h(\gamma(t)) = 0$ as well. By Lemma B.0.5 is then $h(z) = 0$ in a neighborhood of $\gamma(s_0)$. This is only possible if $s_0 = 1$ because otherwise $h(\gamma(t)) = 0$ for all $t \in [0, s_1]$ where $s_1 > s_0$. $\square$

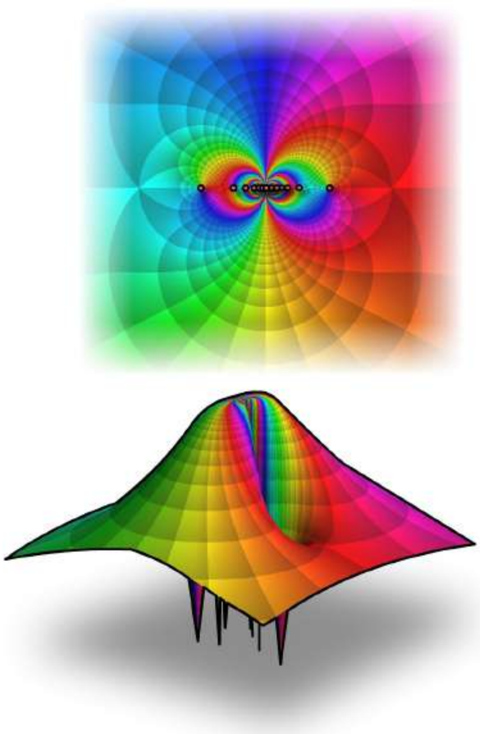


Figure B.4: An enhanced phase portrait (top) and a compressed analytic landscape (bottom) of $f(z) = \sin 1/z$ with infinitely many zeros in the neighborhood of $z = 0$.

Corollary B.0.1. *If f is non-constant and analytic on Ω then the number of zeros of f on a compact subset $\mathcal{C} \subseteq \Omega$ is finite.*

If f had infinitely many zeros on $\mathcal{C}$, there would exist a sequence of these zeros converging to $z_0 \in \mathcal{C}$ which would, of course, mean that $f(z_0) = 0$. Then by Theorem B.0.6 $f \equiv 0$ on Ω.

The existence of infinitely many zeros in a neighborhood of some point is typical of singularities, as can be seen in Fig. B.4. Function $f(z) = \sin 1/z$ has infinitely many zeros in the neighborhood of 0, but it is not analytic at $z = 0$.

Theorem B.0.7. *(Maximum and Minimum Modulus Principle) Let $f : \Omega \to \mathbb{C}$ be non-constant and analytic. Then $|f|$ has no local maximum in Ω and every local minimum of $|f|$ is a zero of f.*

Proof. (1) Let $r > 0$ such that $\mathbb{D}_r(z_0) \subseteq \Omega$. By the Mean Value Theorem (Corollary 2.3.1) and the Standard Integral Estimate (Lemma 2.3.3): $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\mathrm{d}t$. Then

$$|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\mathrm{d}t \leq \max_{t\in[0,2\pi]} |f(z_0 + re^{it})|$$

So there exists $\omega \in \mathbb{S}_r(z_0)$ such that $|f(z_0)| \leq |f(\omega)|$. Equality holds when $|f|$ is constant on $\mathbb{S}_r(z_0)$.

However, since this holds for arbitrarily small $r > 0$, $|f|$ would have to be constant inside $\mathbb{D}_r(z_0)$ as well. And hence f is constant in the entire domain Ω, contradicting the assumption.

(2) Assume that $f(z) \neq 0$ for all $z \in \mathrm{cl}(\Omega)$, then of course $1/f$ is analytic on Ω and continuous in $\mathrm{cl}(\Omega)$. By the Maximum Modulus Principle (Part 1), $|1/f|$ attains its maximum on $\partial\Omega$, i.e.: there exists $w \in \partial\Omega$ such that $|\frac{1}{f(z)}| \leq |\frac{1}{f(w)}|$ for all $z \in \mathrm{cl}(\Omega)$. So indeed for all $z \in \mathrm{cl}(\Omega)$: $|f(w)| \leq |f(z)|$.

On the other hand, if $|f|$ does not assume a minimum on $\partial\Omega$, then there exists $v \in \Omega$ such that $|f(v)| \leq |f(z)|$ for $z \in \Omega$, which implies that $|\frac{1}{f(z)}| \leq |\frac{1}{f(v)}|$ for all $z \in \Omega$. But since $v \notin \partial\Omega$, the Maximum Modulus Principle gives that f must be constant (since the maximum is not attained on the boundary). This contradicts the assumption, and since supposing that $f(v) \neq 0$ implies $f = const.$ and because $f \neq const.$ the conclusion is that $f(v) = 0$. $\square$

Now let $\mathrm{wind}_\gamma f = \mathrm{wind}(f \circ \gamma)$ where $f : \Omega \to \mathbb{C}$ and $\gamma : I \to \Omega$ be a piecewise smooth path. $\mathrm{wind}_\gamma f$ is the winding number of the resulting path after the trace $[\gamma]$ gets mapped by f (see Fig.B.6).

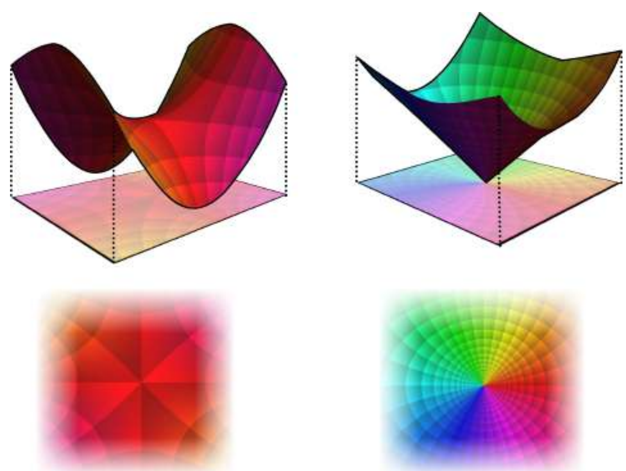


Figure B.5: Maxima and minima of moduli of functions $z \mapsto \cos z$ and $z \mapsto \sin z$.

Theorem B.0.8. *(Argument Principle) Let $f : \Omega \to \mathbb{C}$ be analytic and let $\gamma : I \to \Omega$ be a positively oriented Jordan Curve. If f has no zeros on $[\gamma]$, then the number of zeros in $\mathrm{Int}\gamma$ counted with multiplicity is equal to $\mathrm{wind}_\gamma f$.*

Proof. Let $\mathcal{K} = \mathrm{Int}\gamma \cup [\gamma] \subseteq \Omega$. Since $\mathcal{K}$ is compact, the number of zeros inside it is finite. Let $z_1, z_2, ..., z_k \in \mathrm{Int}\gamma$ be the zeros with multiplicities $m_1, m_2, ..., m_k$. Applying the Local Normal Form (Theorem B.0.4) for each zero we get

$$f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2}...(z - z_k)^{m_k} g(z)$$

where g is analytic and non-zero on $\mathcal{K}$. Let $f_j : z \mapsto (z - z_j)^{m_j}$ for $j = 1, 2, ..., k$. Functions g and all f_j are non-vanishing on $[\gamma]$ and hence the winding numbers of $g \circ \gamma$ and $f_j \circ \gamma$ are well defined. Using part (3) of Theorem A.0.7:

$$\mathrm{wind}_\gamma f = \mathrm{wind}_\gamma g + \mathrm{wind}_\gamma f_1 + ... + \mathrm{wind}_\gamma f_k$$

Now for each $f_j(z) = (z-z_j)^{m_j}$ the winding number $\mathrm{wind}_\gamma f_j = \sum_{i=1}^{m_j} \mathrm{wind}_\gamma(z-z_j) = m_j$ because $z \mapsto (z-z_j)$ is merely a translation (see part (2) of Theorem A.0.7). All that remains now is to show that $\mathrm{wind}_\gamma g = 0$.

Let $z_0 \in \mathrm{Int}\gamma$. γ is freely homotopic to a constant path $t \mapsto z_0$ in $\mathcal{K}$, so there exists a continuous family of paths $\gamma_s (0 \leq s \leq 1)$ such that $\gamma_0 = \gamma$ and $\gamma_1 : t \mapsto z_0$ and $[\gamma_s] \subset \mathcal{K}$, $\forall s \in [0, 1]$. Since $[g \circ \gamma_s] \subset \mathbb{C}\backslash\{0\}$, by Lemma A.0.8 we get that $\mathrm{wind}_{\gamma_s} g = 0$ for all $s \in [0, 1]$. $\square$

After some consideration, the Argument Principle becomes an elegant tool for locating zeros of a function, or solutions of a general equation $f(z) = a$. Recall the interval-splitting algorithm, an iterative method for finding the zeros of a continuous real-valued function f on an interval. In the algorithm one splits the given interval $[a, b]$ into half-intervals $I_1 = [a, a+0.5(b-a)[$ and $I_2 = [b - 0.5(b - a), b]$. If $f(a) > 0$ and $f(a + 0.5(b - a)) < 0$ by continuity of f we know that the a zero lies in I_1, and so on. For complex-valued functions construct a rectangle (as a Jordan curve γ), split it into two rectangles γ_1 and γ_2, calculate the winding number for each, and if $\mathrm{wind}_{\gamma_j} f > 0$ for $j = 1, 2$ then $\mathrm{Int}\gamma_j$ contains at least one zero. If z_0 is a zero and $z_0 \in [\gamma]$, then the zeros can again be found iteratively solving for the parameter $t \in [0, 1]$ of the Jordan curve γ (in 1D).

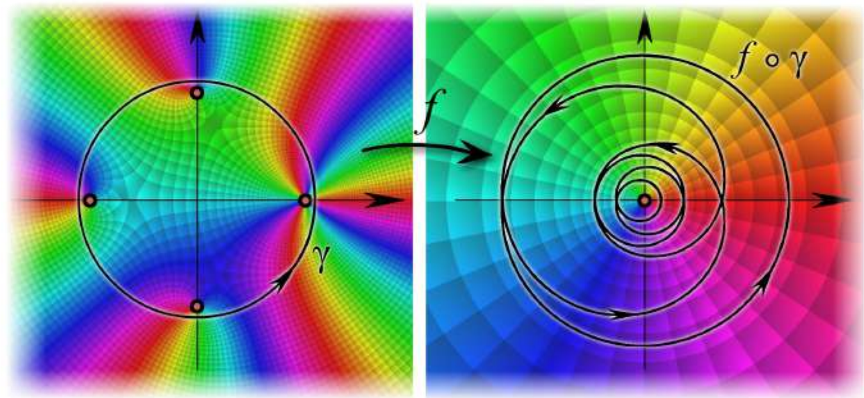


Figure B.6: The domain of polynomial $f(z) = (z^2+1)(z+1)(z-1)^3$ with zeros $i, -i, -1$ and 1 all inside circle $\gamma : t \mapsto 1.2\ \mathrm{e}^{it}$ with the image $f \circ \gamma$ of the circle under f, winding exactly 6 times around 0 (as is the sum of all multiplicities of the roots).

The number of zeros of f does not change under "small" perturbations of f on $[\gamma]$. Exactly how much can the function change on the trace of the Jordan curve is given by the following theorem:

Theorem B.0.9. *(Rouché) Let Ω be a domain and assume that the trace of a positively oriented Jordan curve γ and its interior are contained in Ω. If f and g are analytic in Ω and*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \ , \ z \in [\gamma] \tag{B.7}$$

then f and g have the same number of zeros in $\mathrm{Int}\gamma$, counting multiplicities.

Proof. The inequality (B.7) is a weaker version of the condition in part (5) of Theorem A.0.7, where one works with curves $f \circ \gamma$ and $g \circ \gamma$. Under the assumption of (B.7) there exists a continuous map $h : [0, 1]\times[0, 1] \to \mathbb{C}\backslash\{0\} : (s, t) \mapsto f(z)+s(g(z)-f(z))$ where $z \in [\gamma]$, hence $f\circ\gamma$ and $g\circ\gamma$ are freely homotopic, so $\mathrm{wind}_\gamma f = \mathrm{wind}_\gamma g$, and the assertion follows from the Argument Principle. $\square$

Recall that if $f : X \to Y$ is a continuous map between topological spaces X and Y, then the preimage $f^{-1}[\mathcal{V}]$ of an open set $\mathcal{V} \subseteq Y$ is open in X. The converse, i.e.: that the image $f[\mathcal{U}]$ of an open set $\mathcal{U} \subseteq X$ is open, is not generally true, because f could be a constant map and $f[\mathcal{U}]$ would be a singleton which is not an open set. If, however, f is non-constant and analytic, the following holds:

Theorem B.0.10. *(Open Mapping Principle) Suppose $\mathcal{U} \subseteq \Omega$ is open and connected, and $f : \Omega \to \mathbb{C}$ is non-constant and analytic. Then $f[\mathcal{U}]$ is open.*

Proof. The result follows intuitively from the Argument Principle: If a closed path has a positive winding number about some w_0, then this also holds for the points in a sufficiently small neighborhood of the point. If $w_0 \in f[\mathcal{U}]$, then there exists $z_0 \in \Omega$ such that $w_0 = f(z_0)$. Since $\mathcal{U}$ is open, it contains a closed disk $\mathbb{D}_r(z_0)$. By the corollary of the Identity Theorem (B.0.6) z_0 is an isolated zero of $f - w_0$ because $r > 0$ can be chosen small enough to contain only one point for which $f(z) = w_0$ and that point would be z_0. Then,

$|f - w_0|$ has a positive minimum δ on the boundary $\partial\mathbb{D}_r(z_0) = [\gamma]$. If $w \in \Omega$ satisfies $|w - w_0| < \delta$, then by Rouché's Theorem (B.0.9), functions $f - w_0$ and $f - w$ have the same number of zeros in $\mathbb{D}_r(z_0)$. Since $f(z_0) = w_0$ this number is positive, and thus all w such that $|w - w_0| < \delta$ are contained in $f[\mathcal{U}]$. $\square$

The Open Mapping Principle reveals a new connection with the Maximum and Minimum Modulus Principles (Theorem B.0.7): for any $z_0 \in \Omega$ the image $f[\mathcal{U}]$ of an open neighborhood $\mathcal{U}$ of z_0 contains a neighborhood of $f(z_0)$, so $|f|$ cannot have a local maximum, nor a positive local minimum on $\mathcal{U}$.

Theorem B.0.11. *(Schwarz Lemma) Let f be analytic on the unit disk $\mathbb{D}$ and assume that $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then $|f(z)| \leq |z|$, for all $z \in \mathbb{D}$, and if $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}\backslash\{0\}$ then there exists a (unimodular) constant $c \in \mathbb{S}^1$ such that $f(z) = cz$, $z \in \mathbb{D}$.*

Proof. Define $g(z) = f(z)/z$, where g is analytic on $\mathbb{D}\backslash\{0\}$. In order to extend g to an analytic function on $\mathbb{D}$, denote $a_0 + a_1 z + a_2 z^2 + ...$ the Taylor series of f at 0. Since $f(0) = 0$: $a_0 = 0$. Setting $g(0) = a_1$ we get $g(z) = a_1 + a_2 z + ...$ for all z in a neighborhood of 0. So g is also analytic at 0.

$|f(z)| \leq 1$ guarantees that $|g(z)| \leq 1/|z|$ for $0 < |z| < 1$. Referring to the Maximum Modulus Principle (Theorem B.0.7), conclude that $|g(z)| \leq 1/r$ for $|z| \leq r < 1$. When $r \to 1$ we get $|g(z)| \leq 1$ for all $z \in \mathbb{D} \;\Rightarrow\; |f(z)| \leq |z|$.

If $|f(z_0)| = |z_0|$ for some $z_0 \in \mathbb{D}\backslash\{0\}$ then $|g(z_0)| = 1$. Since $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, $|g|$ has a maximum at $z_0 \in \mathbb{D}$. By the Maximum Modulus Principle, g must be a constant (in this case unimodular) $c \in \partial\mathbb{D} = \mathbb{S}^1$, such that $f(z) = cz$, $z \in \mathbb{D}$. $\square$

Analytic Continuation

Generally a function f defined on some set can be extended beyond its domain in many ways. Even if one requires that the extension inherits properties like continuity or differentiability, the new function is usually not unique.

Strangely enough, for analytic functions, one encounters a completely different situation: Due to the Uniqueness Principle (Theorem B.0.6), there is only one possibility (if any) to extend an analytic function to a larger domain. If f is analytic on a given domain, say an open disk $\mathbb{D}_r$, the behavior of f beyond $\mathbb{D}_r$ is completely determined. In a sense, an analytic function is "already there" even when one knows it only inside $\mathbb{D}_r$.

The following technique is referred to as the *Weierstrass disk chain method* [2] and it can be described as "exploring" or "revealing" the hidden functional landscape by gluing function elements with intersecting disk domains.

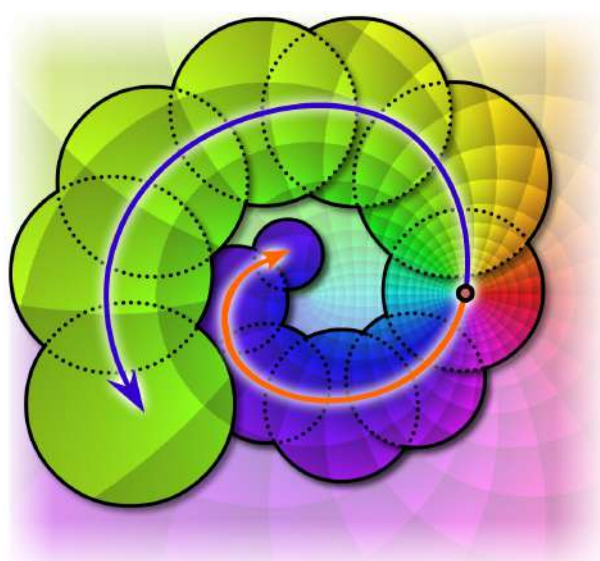


Figure B.7: Analytic continuation of $f : z \mapsto \log z$ by a chain of function elements defined on disks.

Consider $f_1 : D_1 \to \mathbb{C}$ and $f_2 : D_2 \to \mathbb{C}$ where $D_1 \cap D_2 \neq \varnothing$ and $f_1 = f_2$ on $D_1 \cap D_2$. By the Identity Theorem (Theorem B.0.6) there exists $f : D_1 \cap D_2 \to \mathbb{C}$ such that

$$f(z) = \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2 \end{cases}$$

A pair (f_2, D_2) is called a *direct analytic continuation* of (f_1, D_1). Domains D_1 and D_2 may be any connected open sets, but from now on they will be considered as open disks.

Definition B.0.8. A pair (f, D), where $f : D \to \mathbb{C}$ is analytic, is called a *function element*. A sequence of function elements $(f_0, D_0), (f_1, D_1), ..., (f_n, D_n)$ is called a *chain* if any function element (except the first) is a direct analytic continuation of its predecessor, denoted:

$$(f_0, D_0) \varpi (f_1, D_1) \varpi ... \varpi (f_n, D_n) \tag{B.8}$$

If γ is a path, then if disks $D_0, D_1, D_2, ..., D_n$ cover $[\gamma]$ (as in the sense of Definition A.0.11), then (B.8) is called the analytical continuation of (f_0, D_0) along γ.

"ϖ" is an equivalence relation between function elements. The proof of this is trivial. Taking an analytic continuation of f along a path γ, the resulting function does not depend on the choice of individual function elements. This is shown in the following lemma:

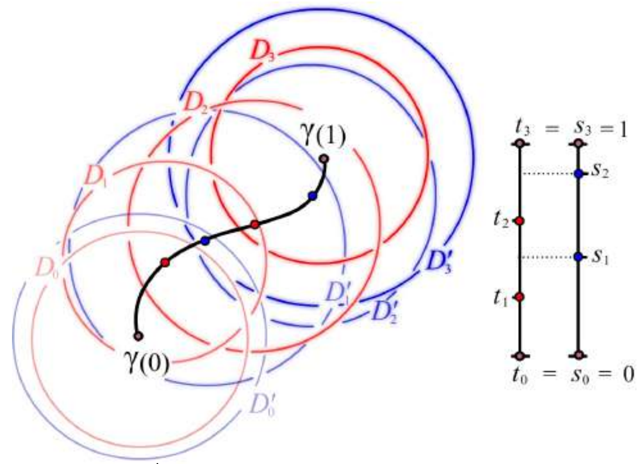


Figure B.8: Visualization of a possible disk chain for the proof of Lemma B.0.12, with $n = m = 3$

Lemma B.0.12. *Let* $(f_0, D_0) \varpi ... \varpi (f_n, D_n)$ *and* $(g_0, D_0^{'}) \varpi ... \varpi (g_n, D_m^{'})$ *be two chains of function elements along a path* γ*. If* $(f_0, D_0) \varpi (g_0, D_0^{'})$ *then* $(f_n, D_n) \varpi (g_m, D_m^{'})$

Proof. Let $\gamma : [0, 1] \to \mathbb{C}$. Partition $[0, 1]$ into $0 = t_0 < t_1 < ... < t_n = 1$ and $0 = s_0 < s_1 < ... < s_m = 1$, so that $\gamma([t_{k-1}, t_k]) \subset D_k$ and $\gamma([s_{j-1}, s_j]) \subset D_j^{'}$ for $k = 1, ..., n$ and $j = 1, ..., m$.

Define a set of index pairs:

$$S = \{(k, j) \big| 1 \leq k \leq n, 1 \leq j \leq m, [t_{k-1}, t_k] \cap [s_{j-1}, s_j] \neq \varnothing \,\&\, (f_k, D_k) \varpi (g_j, D_j^{'})\}$$

describing intersecting partition intervals with corresponding equivalent function elements. If $k = j = 1$, $(1, 1) \in S$ because $D_1 \cap D_1^{'} \neq \varnothing$ and $(f_1, D_1) \varpi (g_1, D_1^{'})$.

Now proceeding with induction hypothesis for $(k, j) \in S$ assuming $k + j < m + n$. Without loss of generality, suppose that $t_k \leq s_j$ because $[t_{k-1}, t_k] \cap [s_{j-1}, s_j] \neq \varnothing \Rightarrow t_k \in [s_{j-1}, s_j]$.

If $k = n$, then $s_j \geq t_n \geq 1$, and hence $j = m$ and $k + j = m + n$ gives a contradiction.

Alternatively if $k + 1 \leq n$: $[t_{k-1}, t_k] \cap [s_{j-1}, s_j] \neq \varnothing$ and $\gamma(t_k) \in D_k \cap D_{k+1} \cap D_j^{'} \neq \varnothing$. Since "$\varpi$" is an equivalence relation, it is transitive, so based on the fact that all three disks intersect: $(f_{k+1}, D_{k+1}) \varpi (f_k, D_k)$ with $(f_k, D_k) \varpi (g_j, D_j^{'})$ imply $(f_{k+1}, D_{k+1}) \varpi (g_j, D_j^{'})$, so $(k + 1, j) \in S$ as well.

Among all $(k, j) \in S$ there exists one pair with the largest sum $k + j$. This pair must be (m, n) because for all other pairs at least one of the pairs $(k + 1, j)$ or $(k, j + 1)$ is in S too. $\square$

This means that as long as the conditions of the disk chain (Definition A.0.11) are met, the radii of the disks D_k may be almost arbitrary. The radii are, of course, bounded from above, so that the disks fit inside the domain Ω.

One can then render all function elements defined on a set of concentric disks centered at some point $z_0 \in \Omega$ as equivalent

Definition B.0.9. (f_1, D_1) and (f_2, D_2) centered at $z_0 \in \Omega$ are said to be *equivalent* if $f_1(z) = f_2(z)$ for any $z \in D_1 \cap D_2$. A *germ* at z_0 is an equivalence class of function elements centered at z_0, denoted f^*. A *canonical representative* of a germ f^* at z_0 is a function element $(f, D) \in f^*$ such that D has the largest possible radius. Denote *value* $f^*(z_0) = f(z_0)$ for any $f \in f^*$ (see Fig.B.9).

The reason why the representative elements may have a finite largest possible radius, seems obvious when one thinks of a function which has power series that converge inside a disk with finite radius. The radius of the canonical representative is then the distance to the closest singularity.

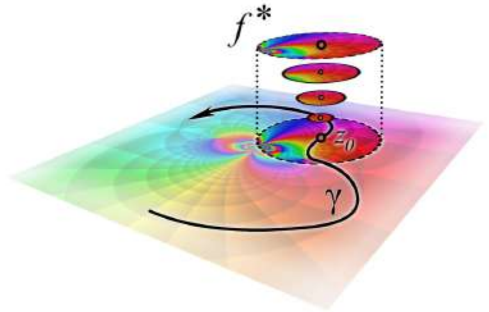
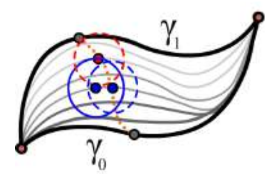


Figure B.9: A germ of function elements centered at z_0.

Whenever an analytic continuation of a germ along γ exists, there is a unique terminal germ which does not depend on the choice of function elements along γ (recall Lemma B.0.12). Compared to analytic continuation by function elements, the germ analog of the process has the further advantage that it does not only yield a unique terminal germ, but also a well defined *family of germs*: $f^*(\gamma_t)$.

The question that remains unanswered is: under what conditions is the analytic continuation independent of the choice of γ? Intuitively, the first hint that comes into mind is that it is bound to depend on the topology of the domain Ω.

Theorem B.0.13. *(Monodromy Principle I.) Let γ_s with $s \in [0,1]$ be a family of homotopic paths with fixed endpoints. If f^* admits an analytic continuation $f^*(\gamma_s)$ along any path γ_s, then $f^*(\gamma_0) = f^*(\gamma_1)$.*

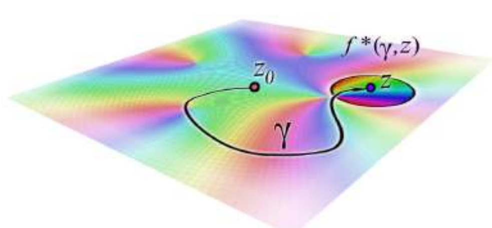


Proof. (1) First prove a *local* result: if $s \in [0,1]$ is fixed, and $\exists\ \delta > 0$, $\sigma \in [0,1]$ such that $|\sigma - s| < \delta$ (sufficiently small), then $f^*(\gamma_\sigma) = f^*(\gamma_s)$.

Fix a representative (f_0, D_0) of f^* and a chain $(f_0, D_0) \infty \ldots \infty (f_n, D_n)$ of function elements covering γ_s. Denote the centers z_k of disks D_k with radii r_k. The corresponding partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ satisfies $\gamma_s(t_k) = z_k$ and $\gamma_s([t_{k-1}, t_k]) \subset D_k$ for $k = 1, 2, \ldots, n$.

Now shift the disks D_k so that their centers lie on γ_σ, denoted as: $D_k^\sigma = D_k - \gamma_s(t_k) + \gamma_\sigma(t_k)$. Then for all $\sigma \in [0,1]$ with $|\sigma - s| < \delta$ (δ sufficiently small), the chain of disks $D_0^\sigma, D_1^\sigma, \ldots, D_n^\sigma$ covers γ_σ and $D_k \cap D_{k+1} \cap D_k^\sigma \neq \varnothing$ for $k = 0, 1, \ldots, n-1$. Then using the transitive property of ∞, we get that germs $f^*(\gamma_s)$ and $f^*(\gamma_\sigma)$ have a common representative, that is: $f^*(\gamma_\sigma) = f^*(\gamma_s)$.

(2) So far, it has been shown that $s \mapsto f^*(\gamma_s)$ is a locally constant map, i.e.: for all $s \in [0,1]$ there exists an open interval I_s such that $f^*(\gamma_\sigma) = f^*(\gamma_s)$ for all $\sigma \in I_s \cap [0,1]$. Because $[0,1]$ is compact, i.e.: covered by a finite collection of such overlapping intervals $s \mapsto f^*(\gamma_s)$ must be constant on all $[0,1]$. $\square$



Fix a germ f^* at z_0 and consider the analytic continuations $f^*(\gamma)$ along all paths γ with fixed initial point z_0 (for which, of course, such continuation exists). Denote $f^*(\gamma, z)$ the value of the germ f^* at the terminal point $z \in \Omega$ of path γ.

Definition B.0.10. A germ f^* at z_0 is said to have an *unrestricted analytic continuation* on domain Ω, if it admits an analytic continuation $f^*(\gamma)$ along any path γ in Ω with initial point z_0. Moreover, if there exists an analytic function $f : \Omega \to \mathbb{C}$ such that $f^*(\gamma, z) = f(z)$ for every $z \in \Omega$, then it is said that f^* *generates* the analytic function f in Ω.

As we will soon find out, a germ of a function can have an unrestricted analytic continuation on its domain, but one may encounter "conflicting elements" (see Fig.B.7). The exact pattern that needs to be followed in order to avoid such conflicts is described in an updated version of the Monodromy Principle:

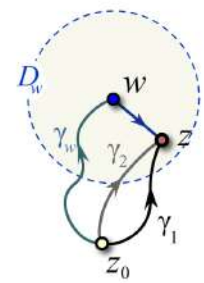
Theorem B.0.14. *(Monodromy Principle II.) Let f^* be a germ at $z_0 \in \Omega$ which has an unrestricted analytic continuation inside the domain. Then f^* generates an analytic function in Ω in each of the following cases:*

(I.) The analytic continuation of f^* along any closed path γ in Ω is trivial, i.e.: $f^*(\gamma, z_0) = f^*(z_0)$.

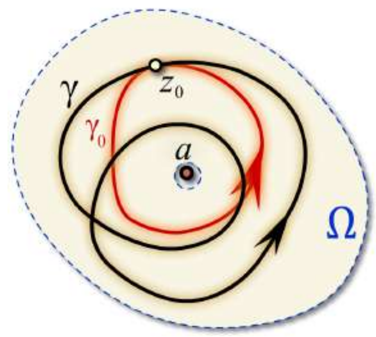
(II.) Ω is simply-connected.

(III.) Ω is a multiply-connected domain, punctured at point a and the analytic continuation of f^* along some closed path γ_0 with winding number 1 around a is trivial.

(I.)



(III.)



(I.): Let γ_1 and γ_2 be two paths from z_0 to z in Ω. $\gamma_1 \oplus \gamma_2^-$ is a closed path. Thus by assumption $f^*(\gamma_1 \oplus \gamma_2^-) = f^*$ and consequently $f^*(\gamma_1) = f^*(\gamma_1 \oplus \gamma_2^- \oplus \gamma_2) = f^*(\gamma_2)$. Since the analytic continuation of f^* is independent of the choice of the path from z_0 to z, $f(z) = f^*(\gamma, z)$ is well defined in Ω.

Now to show that f is analytic, choose an arbitrary point w and a path γ_w from z_0 to w. The germ $f^*(\gamma_w, w)$ is represented by a function element (f_w, D_w). Assume $z \in D_w$. Then the path $\gamma = \gamma_w \oplus [w, z]$ connects z_0 with z and since the line segment $[w, z] \subset D_w$, the value of the analytic continuation of f_w^* along $[w, z]$ coincides with $f_w(z)$. So we get: $f(z) = f^*(\gamma, z) = f^*(\gamma_w \oplus [w, z], z) = f^*([w, z], z) = f_w(z)$ and hence f is analytic in D_w.

(II.): If Ω is simply-connected, any closed path γ in Ω with $\gamma(0) = \gamma(1) = z_0$ is homotopic to the constant path $\gamma_0 : t \mapsto z_0$, $t \in [0, 1]$ (see Lemma A.0.4). By the Monodromy Principle I. (Theorem B.0.13) f^* is trivial in $\Omega :$ $f^*(\gamma, z_0) = f^*(\gamma_0, z_0) = f^*(z_0)$.

(III.): With $n \in \mathbb{N}$ denote $n\gamma_0 = \overbrace{\gamma_0 \oplus ... \oplus \gamma_0}^{n \text{ times}}$, set $-n\gamma_0 = n\gamma_0^-$, and let $0\gamma_0 : t \mapsto z_0$ be a constant path. Then $f^*(\gamma_0) = f^*$ implies $f^*(n\gamma_0) = f^*$ for all $n \in \mathbb{Z}$.

According to Lemma A.0.3 , any closed path γ in $\Omega = \Omega_0 \backslash \{a\}$ with initial point z_0 and $\text{wind}(\gamma, a) = n$ is homotopic with fixed endpoints to $n\gamma_0$. The Monodromy Principle I. (Theorem B.0.13) then says that $f^*(\gamma) = f^*(n\gamma_0)$, so the assumptions of part (I.) are satisfied. □

Consider, for example, the complex logarithm $f : \mathbb{C}\backslash\{0\} \to \mathbb{C} : z \mapsto \log z$. With the initial function element: (f_0, D_0) where $D_0 = \{z \in \mathbb{C} \mid |z - 1| < 1\}$ and $f_0 : D_0 \to \mathbb{C} : z \mapsto \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(z-1)^k$ (which is a Taylor series expansion of *log* at $z_0 = 1$). Now consider a path $\gamma : [0, 1] \to \mathbb{C}\backslash\{0\}$ with $\gamma(0) = 1$ and an arbitrary terminal point $\gamma(1) = z_1 \in \mathbb{C}\backslash\{0\}$. Take $D_t = \{z \in \mathbb{C} \mid |z - z_t| < |z_t|\}$ which is the largest disk around z_t contained in $\mathbb{C}\backslash\{0\}$. Also $\log z = \log|z| + i\text{Arg}(z)$. Now denote $\arg_\gamma(z_t) = \theta_\gamma(t) = -i \log \frac{z_t}{|z_t|}$ the continuous branch of the argument of log along γ, with $\arg_\gamma(z_0) = 0$. By substituting z/z_t into $f_0(z)$ instead of z we get

$$f_t(z) = \log|z_t| + i\arg_\gamma(z_t) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k z_t^k}(z - z_t)^k \quad , t \in [0, 1] \tag{B.9}$$

Since $|\gamma| > 0$ there exists a finite partition $0 = t_0 < t_1 < ... < t_n = 1$ of $[0, 1]$ such that disks D_{t_j} cover γ. Thanks to the additional term: $\log|z_t| + i\arg_\gamma(z_t)$, any two neighboring elements are direct analytic continuations of each other. The function element (f_1, D_1) at the terminal point z_1 of γ is an analytic continuation of (f_0, D_0) along γ. Denote $f_1 = \log(\gamma, .)$, so then $\log(\gamma, z) = \log|z| + i \arg_\gamma(z)$, where $\arg_\gamma(z) = \arg_\gamma(z_1) + \text{Arg}(z/z_1)$. Note that $\exp(\log(\gamma, z)) = z$.

Definition B.0.11. A function $g : \Omega \to \mathbb{C}$ such that $f = \exp g$ is called an *analytic branch of the logarithm* in Ω.

Lemma B.0.15. *If $f : \Omega \to \mathbb{C}\backslash\{0\}$ is analytic on a simply-connected domain, it has an analytic branch of the logarithm g such that $f = \exp g$ on Ω.*

Proof. Fix $z_0 \in \Omega$ and let D_0 be a sufficiently small disk centered at z_0. Then $f[D_0] \subseteq G$ where G is a disk centered at $w_0 = f(z_0)$ such that $0 \notin G$. Set $g_0 = \log_G f$. If γ is a path in Ω with $\gamma(0) = z_0$, then $f \circ \gamma$ is a path in $\mathbb{C}\backslash\{0\}$ with $(f \circ \gamma)(0) = w_0$. Then $\log_G$ has an analytic continuation $\log(f \circ \gamma, .)$ along $f \circ \gamma$. Write $g(\gamma, z) = \log(f \circ \gamma, z),\ z \in \Omega$ which is an analytic continuation of (g_0, D_0) along γ. So $f(z) = \exp(g(\gamma, z)),\ z \in D_\gamma$. Since (g_0, D_0) has an unrestricted analytic continuation in Ω and the domain is simply-connected, by the Monodromy Principle II. (Theorem B.0.14) g exists. $\square$

Remark. Because of the difference in arguments analytic branches of $\log f$ can differ by a constant function $z \mapsto 2k\pi i,\ k \in \mathbb{Z}$. If $0 \notin \Omega$ applying Lemma B.0.15 to the identity function $f = \mathrm{id}_{\mathbb{C}} : z \mapsto z$ we obtain analytic branches of the log function: $\log : \Omega \to \mathbb{C} : z \mapsto \log|z| + i \arg z$, where arg is a continuous branch of the argument in Ω.

If $f : \Omega \to \mathbb{C}\backslash\{0\}$ is assumed to be continuous on its domain, there exists a continuous function $g : \Omega \to \mathbb{C}$ such that $f(z) = \exp(g(z))$, for $z \in \Omega$. g is then called a *continuous branch of the logarithm of f*.

Returning to the example of log function we find out that the conditions for part (III.) of the Monodromy Principle II. (Theorem B.0.14) were not satisfied. Take $\gamma = \mathbb{S}^1 : [0,1] \to \mathbb{C} : t \mapsto \mathrm{e}^{2\pi i t}$, then by (B.9): $f_1(z) - f_0(z) = 2\pi i$. Taking a path integral along $\mathbb{S}^1$ of the derivative $z \mapsto \frac{\mathrm{d}}{\mathrm{d}z}\log(z) = 1/z$ yields the same result. Even if a function element (f_0, D_0) happens to have an unrestricted analytic continuation on Ω, any attempt to define an analytic function globally on Ω fails because $f(\gamma, z)$ is not just a function of z, but also of γ.

There are two possible solutions to this problem:
(1): define *multiple-valued* functions on Ω.
(2): define a single-valued "global" analytic function on a *Riemann Surface*.

Definition B.0.12. When the conditions of the Monodromy Principle II. (Theorem B.0.14) are not satisfied, the analytic continuation of a function element (f_0, D_0) or a germ f_0^* onto Ω yields a set of functions:

$$F(z) = \{f_0(\gamma, z) \mid \gamma \in \Gamma_\Omega(z)\} \tag{B.10}$$

where $\Gamma_\Omega(z)$ is the set of all paths starting at z_0 (the center of D_0 or the point of the germ f_0^*) and end at $z \in \Omega$. Set $F(z)$ is called a *multiple-valued* function.

The analytic continuation of the log function from the previous example can be written as

$$\mathrm{LOG}(z) = \{\log|z| + i(2k\pi + \mathrm{Arg}(z)),\ k \in \mathbb{Z}\} \tag{B.11}$$

with branches of the logarithm: $\mathrm{Log}_k(z) = \log|z| + i(2k\pi + \mathrm{Arg}(z)),\ k \in \mathbb{Z}$ as individual elements.

Similarly the analytic continuation of the square root function $f : z \mapsto \sqrt{z}$ can be written as:

$$\mathrm{SQRT}(z) = \left\{(-1)^k \sqrt{z},\ k \in \{0, 1\}\right\} \tag{B.12}$$

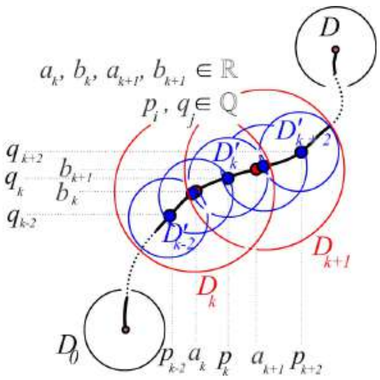
It can be shown that an analytic continuation along a path γ with $\mathrm{wind}(\gamma, 0) = \pm 2$ yields the same value (see Wegert [2], p. 129). The values $k \in \mathbb{Z}$ correspond to individual function elements that are also called *branches*. A branch corresponding to $k = 0$ is referred to as the *principal branch*.

Definition B.0.13. A *global analytic function* in Ω is a non-empty set $\mathcal{F}$ of analytic function elements such that:

(I.) For any $(f_1, D_1), (f_2, D_2) \in \mathcal{F}$: $(f_1, D_1) \text{ ⚭ } (f_2, D_2)$.

(II.) If $(f_0, D_0) \in \mathcal{F}$ and $(f_0, D_0) \text{ ⚭ } (f, D)$, then $(f, D) \in \mathcal{F}$.

Theorem B.0.16. *(Poincaré-Volterra) For every global analytic function $\mathcal{F}$ and every $z \in \mathbb{C}$ the set $\mathcal{F}_z^*$ of germs in $\mathcal{F}$ is at most countable.*



Proof. Let $(f_0, D_0) \in \mathcal{F}$ be fixed with center z_0. Then any element $(f, D) \in \mathcal{F}$ centered at z is an analytic continuation of (f_0, D_0) along some path γ from z_0 to z. According to Lemma B.0.12 the germ f_z^* represented by (f, D) is independent of the choice of the chain of disks $D_0, D_1, ..., D_{n-1}, D$ covering γ for analytic continuation. If $D_0, D'_1, ..., D'_{n-1}, D$ is another such chain of disks, where $D'_k \subset D_k$ for $k = 1, ..., n-1$ then the analytic continuations along both chains coincide. Now the chain $D_0, D'_1, ..., D'_{n-1}, D$ of smaller disks is chosen with rational radii and rational coordinates of centers (which can certainly be done thanks to the *density of rationals* theorem), which means that the set of all chains formed from such disks is countable. Thus the number of different germs in $\mathcal{F}$ at z is at most countable. $\square$

The set $\mathcal{F}_z^*$ of all germs at z corresponds to the number of possible values any analytic continuation can attain at z. And by the Poincaré-Volterra theorem, any function can attain at most countably many values at a given point. The square root function admits two possible values for each $z \in \mathbb{C}\backslash\{0\}$ and one value for $z = 0$. Whereas the complex logarithm admits countably many for each $z \in \mathbb{C}\backslash\{0\}$.

It was Riemann who laid foundations for a tool used for constructing such analytic functions (in his dissertation). Instead of working with a set of functions that admit different values for a given point, he mapped the them onto a surface:

Definition B.0.14. The *Riemann surface* of the global analytic function $\mathcal{F}$ is the set:

$$S(\mathcal{F}) = \left\{(z, g) \in \mathbb{C} \times \mathcal{F}^* \;\middle|\; z \in \mathbb{C},\ g \in \mathcal{F}_z^*\right\} \tag{B.13}$$

With this tool, even operations like path integration along a closed path with a non-zero winding number around a singularity point make geometric sense. One can now apply the Extended Fundamental Theorem (2.33) to a curve *lifted* from its domain onto the Riemann surface.

A point $a \in \mathbb{C}$ is called an *isolated singularity* of the global analytic function $\mathcal{F}$ if there exist a punctured disk $\dot{D} = D\backslash\{a\}$ and a function element $(f_0, D_0) \in \mathcal{F}$ such that $D_0 \subset \dot{D}$ which admits an unrestricted analytic continuation in $\dot{D}$, but not in D (see Def.B.0.7 for comparison).

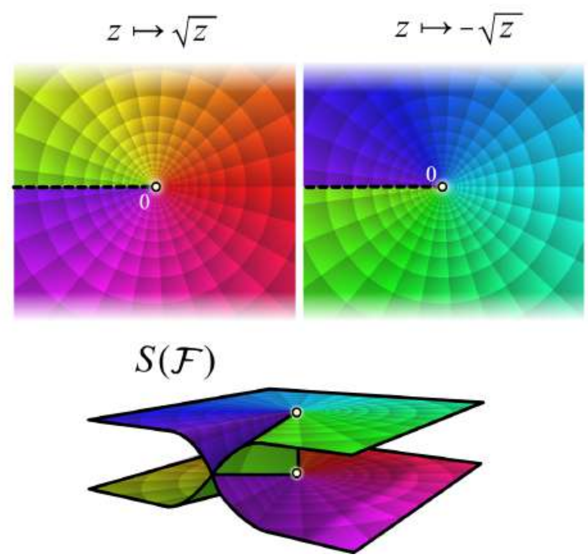


Figure B.10: The Riemann Surface of a global square root function with its 0 (*principal*) and 1 branches.

Let $\mathcal{F}_0$ be a global analytic function in a disk D and assume that $a \in D$ is the only isolated singularity of $\mathcal{F}_0$ in D. Then a is said to be a *branch point* of $\mathcal{F}_0$ if one (and then any) germ in $\mathcal{F}_0^*$ has a non-trivial analytic continuation (see Theorem B.0.14 for the definition of trivial) along some closed path in $\dot{D} = D \backslash \{a\}$.

Let $\mathcal{F}_0$ be generated by a germ f_0^* at $z_0 \in \dot{D}$. Because the analytic continuation of f_0^* along any closed path γ in $\dot{D}$ depends only on the winding number k of γ about a, denote f_k^* such continuation of f_0^*, and consider two possible cases:

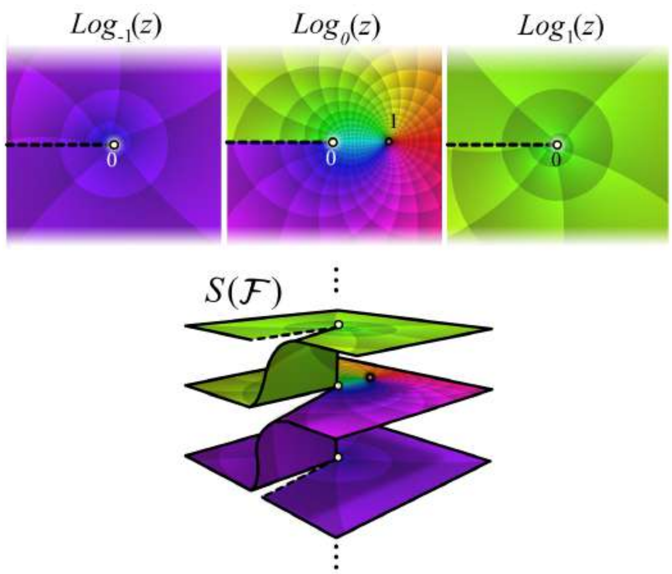


Figure B.11: A part of the Riemann Surface of a global log function with its principal branch and branches $k = \pm 1$.

(1): $f_k^* \neq f_m^*$ whenever $k \neq m$.
(2): $f_k^* = f_m^*$ for some $m > k$.

In the first case a is a branch point of *infinite order*, which is often referred to as a *logarithmic branch point* because such case occurs for 0 in the log function (see Fig. B.11).

In the second case a is a branch point of *order* $n = \min\{i \geq 2 | f_i^* = f_0^*\}$, or also an *algebraic branch point*. For the square root function 0 is an algebraic branch point of order $n = 2$ since analytic continuation along a path about 0 has a winding number periodicity of 2 (see Fig. B.10).

More about singularities will be shown in Section 2.4 on *residues*. So the fundamentals of analytic function theory have been laid in this appendix. All of the theorems (as well as their corollaries) stated here will later be applied in the chapters.

Appendix C

Normal Convergence

The sums of power series have already been provided as a tool for generating analytic functions. Nonetheless, we have only stated two related concepts of convergence (see Appendix B).

Pointwise convergence (see Def. B.0.4) is too weak, since it generally does not even preserve the continuity of the function sequence $\{f_n\}$. On the other hand, uniform convergence seems too strong because a "very small" subset of function sequences (on non-compact (see Def. A.0.4) subsets of $\mathbb{C}$) converges uniformly, too small, in fact, to provide the necessary building blocks of some analytic functions.

A reasonable "middle ground" is provided by a variant of uniform convergence that is often referred to as *normal*, or *compact* convergence:

Definition C.0.1. A sequence $\{f_n\}$ of continuous functions $f_n : \Omega \to \mathbb{C}$ converges *normally* in Ω to the *limit function* $f : \Omega \to \mathbb{C}$ if $\{f_n\}$ converges uniformly on any compact subset of Ω.

Notice that the analyticity of the functions f_n is not required. Obviously, if f_n converges to f normally, then it also converges pointwise for any $z \in \Omega$.

Lemma C.0.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set. Then a sequence of functions $f_n : \Omega \to \mathbb{C}$ converges normally to $f : \Omega \to \mathbb{C}$ if and only if for any $z \in \Omega$ there exists a (closed or open) disk D_z centered at z such that f_n converges to f uniformly on D_z.*

Proof. If D_z is closed then f_n converges to f normally if and only if it converges on any compact subset $K \subseteq \Omega$ containing D_z. On the other hand, since any compact subset $K \subseteq \Omega$ can be covered by a finite collection of open disks, the result holds for both open and closed D_z. $\square$

The result of Lemma C.0.1 which (again) has no counterpart in real analysis, says that normal convergence of analytic functions f_n implies the convergence of their derivatives $f_n^{(k)}$ of arbitrary order k:

Theorem C.0.2. *Let $\{f_n\}$ be a sequence of functions $f_n : \Omega \to \mathbb{C}$ that converges normally on $\Omega \subseteq \mathbb{C}$ to the limit function f. Then f is analytic on Ω, and if $k \in \mathbb{N}$ then the sequence of k-th order derivatives $\{f_n^{(k)}\}$ converges normally to $f^{(k)}$.*

Proof. (1): Let $K \subseteq \Omega$ be a closed disk. Since Ω is open one can find a larger disk $\widetilde{K} \subseteq \Omega$ such that $K \subseteq \widetilde{K}$. Denote γ the standard parametrization of $\partial\widetilde{K}$ and $G = \mathrm{Int}(\widetilde{K})$. Then by the Cauchy Integral Formula:

$$f_n(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f_n(z)}{z - z_0} \mathrm{d}z \qquad \text{for all} \;\; z_0 \in G$$

Since $\{f_n\}$ converges uniformly on $[\gamma] = \partial\widetilde{K}$, f is continuous on $[\gamma]$ and

$$\lim_{n\to\infty} f_n(z_0) = \frac{1}{2\pi i} \lim_{n\to\infty} \oint_\gamma \frac{f_n(z)}{z - z_0} \mathrm{d}z = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \mathrm{d}z = f(z_0) \quad , \ z_0 \in G$$

By Theorem 2.3.8, the limit function f is analytic in G. Now using the Cauchy Integral Formula for derivatives (2.36):

$$f_n^{(k)}(z_0) - f^{(k)}(z_0) = \frac{k!}{2\pi i} \oint_\gamma \frac{f_n(z) - f(z)}{(z - z_0)^{k+1}} \mathrm{d}z$$

and because f_n converges uniformly on $[\gamma]$ and since $\mathrm{dist}([\gamma], K) > 0$ the integrand converges uniformly to zero with respect to $z_0 \in K$ and by the Standard Integral Estimate (Lemma 2.3.3):

$$|f_n^{(k)}(z_0) - f^{(k)}(z_0)| = \frac{k!}{2\pi i} \left| \oint_\gamma \frac{f_n(z) - f(z)}{(z - z_0)^{k+1}} \mathrm{d}z \right| \leq \frac{k!}{2\pi i} L(\gamma) M(f - f_n)$$

thus $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on K. □

Another useful property of normal convergence is that it locally preserves the number of zeros. More precisely:

Theorem C.0.3. *(Hurwitz Theorem) Let $\{f_n\}$ be a sequence of analytic functions $f_n : \Omega \to \mathbb{C}$ converging normally in Ω to some $f \not\equiv 0$ (not identically zero). Then for each $a \in \Omega$ there exists a disk $\mathbb{D}(a) \subseteq \Omega$ centered at a and $N_a \in \mathbb{N}$ such that for all $n \geq N_a$, f_n and f have the same number of zeros in $\mathbb{D}(a)$ (counted with multiplicity).*

Proof. If f is analytic, by the Identity Theorem (Theorem B.0.6) its zeros are isolated. Consequently for any $a \in \Omega$ there exists $\mathbb{D}(a)$ such that $\mathrm{cl}(\mathbb{D}(a)) \subseteq \Omega$ and this open disk either contains no zero of f (when $f(a) \neq 0$) or exactly one zero, namely a. In both cases $|f|$ has a positive minimum M_a on the compact boundary $\partial\mathbb{D}(a)$.

By the compactness of $\partial\mathbb{D}(a)$, f_n converges normally to f, and there exists $N_a \in \mathbb{N}$ such that $|f_n(z) - f(z)| < M_a \leq |f(z)|$, for $z \in \partial\mathbb{D}(a)$ and $n \geq N_a$. By Rouché's Theorem (B.0.9), f_n and f have the same number of zeros in $\mathbb{D}(a)$ (counted with multiplicity). □

In addition, it is of great importance that normal convergence also preserves univalence (see Def. 4.1.3), with the exception for constant maps:

Corollary C.0.1. *Let $\{f_n\}$ be a sequence of univalent functions $f_n : \Omega \to \mathbb{C}$ converging normally to f in Ω. Then the limit function f is either constant or univalent.*

Proof. Assuming that f is not univalent, there exist $a, b \in \Omega$ such that $a \neq b$ and $f(a) = f(b)$. Hence $f - f(a)$ has at least two zeros. A sequence of functions $f_n - f(a)$ converges normally to $f - f(a)$. Now if f is non-constant then $f - f(a)$ is clearly not identically zero. Thus by the Hurwitz Theorem (Theorem C.0.3) $f_n - f(a)$ must also have at least two distinct zeros for sufficiently large n which is impossible because functions f_n are supposed to be univalent. □

Compactness, being an essential concept in real and complex analysis, is frequently used in many existence proofs. For example, in order to find solutions ot an extremal problem, one might begin with a set of "almost extremal" elements. If a set is compact, it contains a converging sequence. The limit of such sequence is then usually a suitable candidate for solution.

If one wishes to apply this principle to problems involving analytic functions, appropriate "compactness criteria" need to be provided. For families of continuous functions these conditions are provided by the

Arzela-Ascoli Theorem on boundedness under uniform convergence (which can be found in Rudin [11], p. 158).

However, if a family $\mathcal{F}$ consists of analytic functions and uniform convergence is replaced by normal convergence, the boundedness can be limited only to compact subsets of the domain.

Definition C.0.2. A family $\mathcal{F}$ of functions $f : \Omega \to \mathbb{C}$ on an open set $\Omega \subseteq \mathbb{C}$ is *locally bounded* if for any $a \in \Omega$ there exist: a idsk $\mathbb{D}(a) \subseteq \Omega$ centered at a, and a positive constant C_a such that

$$|f(z)| \le C_a \ , \quad z \in \mathbb{D}(z) \ , \ f \in \mathcal{F} \tag{C.1}$$

Lemma C.0.4. *Let $\mathcal{F}$ be a locally bounded family of functions on $\Omega \subseteq \mathbb{C}$ and let $K \subseteq \Omega$ be compact. Then $\mathcal{F}$ is uniformly bounded and uniformly Lipschitz continuous on K, that is: there exist $C, L > 0$ such that*

$$|f(z)| \le C \ , \quad |f(z_1) - f(z_2)| \le L|z_1 - z_2| \quad \textit{for} \ \ z, z_1, z_2 \in K \ , \ f \in \mathcal{F} \tag{C.2}$$

Proof. (1): Cover K by disks $\mathbb{D}(a)$ with $a \in K$ so that there exists $C_a > 0$ with $|f(z)| \le C_a$, $z \in \mathbb{D}(a)$ and $f \in \mathcal{F}$. Given a finite covering $\bigcup_{i=1}^{n} \mathbb{D}(a_i) \supset K$ and let $C = \max\{C_{a_1}, C_{a_2}, ..., C_{a_n}\}$.

(2): The union $U = \bigcup_{i=1}^{n} \mathbb{D}(a_i)$ is an open set containing K and $\operatorname{dist}(K, \partial U) > 0$. Since $K \subset U$, the radii of disks $\mathbb{D}(a_i)$ can be reduced simultaneously by the same amount δ so that $K \subseteq \widetilde{U}$, where $\widetilde{U} = \bigcup_{i=1}^{n} \widetilde{\mathbb{D}}(a_i)$. We can choose δ such that the boundary circles $\partial\widetilde{\mathbb{D}}(a_i)$ are in such position that there are no tangent pairs of circles and no triplets of circles with the same intersection.

(3): In step (2) we obtained an open set $\widetilde{U}$ as the union of disks $\widetilde{\mathbb{D}}(a_i)$ such that: $K \subseteq \widetilde{U} \subset \operatorname{cl}(\widetilde{U}) \subset U \subseteq \Omega$. According to step (1) of the proof, $\operatorname{cl}(\widetilde{U}) \subset U$ implies that $|f(z)| \le C$, $z \in \operatorname{cl}(\widetilde{U})$ and $f \in \mathcal{F}$.

The boundary $\partial\widetilde{U}$ consists of a finite collection curves $\gamma_1, ..., \gamma_m$ composed of circular arcs. Given suitably oriented parametrizations of these paths, we obtain a cycle $\Gamma = \gamma_1, ..., \gamma_m$, null-homologous in Ω, with $\operatorname{wind}(\Gamma, z) = 1$ about any $z \in \widetilde{U}$.

(4): Applying the generalized Cauchy Integral Formula for null-homologous cycles (Theorem 2.38) we get:

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \ , \quad z_0 \in \widetilde{U} \ , \ f \in \mathcal{F}$$

And for any $z_1, z_2 \in K$:

$$f(z_1) - f(z_2) = \frac{1}{2\pi i} \oint_{\Gamma} \left(\frac{f(z)}{z - z_1} - \frac{f(z)}{z - z_2} \right) \mathrm{d}z = \frac{z_1 - z_2}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_1)(z - z_2)} \mathrm{d}z$$

Since $\operatorname{dist}(\Gamma, K) > 0$ and $|f(z)| \le C$ for $z \in \operatorname{cl}(\widetilde{U})$ and $f \in \mathcal{F}$, by the Standard Integral Estimate (Lemma 2.3.3) we get the Lipschitz condition: $|f(z_1) - f(z_2)| \le L|z_1 - z_2|$. □

As it turns out, pointwise convergence and local boundedness of a sequence of analytic functions already imply normal convergence. In fact, a stronger result can be shown:

Theorem C.0.5. *Let $\{f_n\}$ be a locally bounded sequence of analytic funcions $f_n : \Omega \to \mathbb{C}$ on $\Omega \subseteq \mathbb{C}$. If $\{f_n\}$ converges pointwise on a dense subset $S \subseteq \Omega$ then it converges normally in Ω.*

Proof. Let $K \subseteq \Omega$ be compact and let L be the Lipschitz constant in (C.2) for the family $\mathcal{F} = \{f_n\}$ on K. Fix $\varepsilon > 0$ and cover K by a finite collection of open disks $D_1, D_2, ..., D_n$ with radii $r_k \le \varepsilon/(2L)$ such that $D_k \cap K \ne \varnothing$.

Since S is dense in Ω (that is: $\mathrm{cl}(S) = \Omega$), any disk D_k will contain a point $z_k \in S$. Then for any $z \in K$ there exists z_k such that $|z - z_k| < \varepsilon/L$. Then by the triangle inequality:

$$|f_n(z) - f_m(z)| \leq |f_n(z) - f_n(z_k)| + |f_n(z_k) - f_m(z_k)| + |f_m(z_k) - f_m(z)| \tag{C.3}$$

with $z_k \in S$ satisfying $|z - z_k| < \varepsilon/L$. By estimate (C.2) the first and the last term of the right-hand side of (C.3) are less than ε. Since $\{f_n\}$ converges for all points in S, the second term is getting smaller than ε. So for sufficiently large $m, n \in \mathbb{N}$:

$$|f_n(z) - f_m(z)| < 3\varepsilon \quad , \quad \text{for } \ z \in K \text{ , and , } \ m, n \geq n_0 \in \mathbb{N}$$

which by Cauchy's criterion (Theorem B.0.1) means that $\{f_n\}$ converges uniformly on K. □

Definition C.0.3. A family $\mathcal{F}$ of analytic functions on $\Omega \subseteq \mathbb{C}$ is said to be *normal* if any sequence $\{f_n\} \subseteq \mathcal{F}$ contains a normally convergent subsequence.

In retrospect, Theorem C.0.2 tells us that the limit of a normally convergent sequence of analytic functions is analytic. This yields a famous result shown by Paul Montel (*1876 - †1975):

Theorem C.0.6. *(Montel's Theorem): A family $\mathcal{F}$ of analytic functions $f : \Omega \to \mathbb{C}$ is normal if and only if $\mathcal{F}$ is locally bounded.*

Proof. First we show that any locally bounded sequence $\{f_n\}$ of analytic functions $f_n : \Omega \to \mathbb{C}$ contains a normally convergent subsequence.

(1): Denote S the set of all points in Ω with rational coordinates. Then S is dense in Ω, that is: $\mathrm{cl}(S) = \Omega$.

(2): Since S is countable, it can be arranged in a sequence $S = \{z_1, z_2, ...\}$. By the assumption, the sequence $\{f_n(z_1)\}$ is bounded. So we can select a subsequence:

$$\left\{f_{1,1} \ , f_{1,2} \ , \dots \ , f_{1,n} \ , \dots\right\} \tag{C.4}$$

convergent at z_1. Analogously, sequence (C.4) contains a subsequence: $\{f_{2,1}, f_{2,2}, ..., f_{2,n}, ...\}$ which not only converges at z_1, but also at z_2. Continuing in this manner, we obtain a family of sequences:

$$\left\{f_{k,1} \ , f_{k,2} \ , \dots \ , f_{k,n} \ , \ \dots\right\} \subset \left\{f_{k-1,1} \ , f_{k-1,2} \ , \ \dots \ , f_{k-1,n} \ , \dots\right\}$$

for any $k \in \mathbb{N}$ converging at all points $z_1, z_2, ..., z_k$.

(3): Now form a diagonal sequence

$$\left\{f_{1,1}, \ f_{2,2}, \ \dots, \ f_{n,n}, \ \dots\right\} \tag{C.5}$$

Because any "tail" of such diagonal sequence, that is: a subsequence $\{f_{k,k} \ , f_{k+1,k+1} \ , ...\}$ is also a subsequence of $\{f_{k,1} \ , f_{k,2} \ , ..., \ f_{k,n}, \ ...\}$, $k \in \mathbb{N}$. It converges at all points $z_k \in S$. So the diagonal sequence (C.5) satisfies the assumptions of Theorem C.0.5 and it converges normally in Ω.

To show the converse assume that $\mathcal{F}$ fails to be locally bounded. Then there exist: a closed disk $K \subseteq \Omega$ and a sequence $\{f_n\} \subseteq \mathcal{F}$ such that $\max\limits_{z \in K} |f_n(z)| \geq n$. Because $\mathcal{F}$ is normal, a subsequence $\{f_{n_k}\}$ converges uniformly on K to a function f analytic on Ω. Then by the Maximum Modulus Principle (Theorem B.0.7) and the triangle inequality:

$$n_k \leq \max_{z \in K} |f_n(z)| \leq \max_{z \in K} |f_{n_k}(z) - f(z)| + \max_{z \in K} |f(z)| \leq C \ , \quad k \in \mathbb{N}$$

which is a contradiction. □

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