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Dissertation Thesis Abstract

Numerical methods and their use in acoustical simulations

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Abstrakt

Dizertačná práca sa zaoberá implementáciou numerických metód do oblasti akustických simulácií. Začiatok je venovaný hlavným princípom tzv. lúčových metód. Táto skupina metód je v dnešnej akustickej praxi bežne využívaná, avšak najmä pri nízkych frekvenciách môžu lúčové metódy čeliť špecifickým problémom. Hlavná časť práce pozostáva z riešenia Helmholtzovej rovnice, ktorá predstavuje časovo-nezávislú formu vlnovej rovnice, pomocou metódy okrajových prvkov a metódy konečných objemov. Práca prezentuje numerickú schému, vybrané výsledky kódu a možné aplikácie. Ďalej je pomocou metódy konečných objemov je riešená vlnová rovnica, kde sú predmetom štúdia taktiež odhady stability a konvergencia.

Kľúčové slová: akustické simulácie, lúčová metóda, vlnová rovnica, Helmholtzova rovnica, metóda okrajových prvkov, metóda konečných objemov

Abstract

Dissertation thesis deals with the implementation of numerical methods into the area of acoustic simulations. At the start, the main principles of so called ray-based methods are described. This group of methods is nowadays commonly used in acoustic praxis, but they face particular problems especially when dealing with low frequencies. The main part of the work consists of solving the Helmholtz equation, which represents the time-independent form of the wave equation, by the Boundary element method and the Finite volume method. Here, the numerical scheme, chosen results of the code, and possible applications are presented. Further, the wave equation is solved by the Finite volume method, where the stability estimates and the convergence of the scheme are studied as well.

Keywords: acoustic simulations, ray-based method, wave equation, Helmholtz equation, Boundary element method, Finite volume method

Contents

Introduction

The main goal of the thesis is to contribute to the area of acoustic simulations, which are nowadays an important tool in engineering praxis. Simulations of acoustic features of spaces represent very efficient aid in designing new buildings with appropriate acoustic properties, as well as in restoration of already existing ones. They can lead to so called auralization of chosen space (i.e. acoustical visualization). That means that one could gain the idea of propagation of sound in the room, even without being there.

There are many connections between room acoustics and other fields. From these, we have to mention architecture, other similar fields like concert hall acoustics, classroom acoustics, and acoustics of health care facilities. This reveals the connections to music, health care, and also psychology.

One of the most important requirements for acoustic simulations is their preciseness. To achieve this, many methods have been developed in this field. Today, many softwares are using ray-based methods, which are described in the section 1. Using these methods brings various problems, and that's why the thesis focuses on models based on differential equations (section 2). Two numerical methods, particularly the Boundary element method (section 3) and the Finite volume method (section 4), have been chosen to solve the Helmholtz equation, which is the time-independent form of the wave equation. Last section 5 is dedicated to the solution of the wave equation by the Finite volume method, whereas the stability estimates and convergence of the scheme are investigated as well.

1. Ray-based methods

The group of methods known as ray-based methods is very well known in the field of room acoustics. Their main point consists of the fact, that they consider sound waves as rays. These methods are based on classical geometrical principles. They work in time domain, which means that they analyze the data with respect to time. Advantages of ray-based methods are reasonable calculation time, convenient model creation, and no meshing problems. Many of modern acoustic softwares are using these methods including their combinations. Briefing about ray-based methods can be found in [8] and [19], which were also used to elaborate this section.

With use of ray-based methods, it is possible to calculate various parameters, e.g. sound pressure level, speech transmission index, reverberation time etc. In context of the thesis, the sound pressure level is the most important one.

1.1. Ray-based methods and lower frequencies

Ray-based methods are very accurate in the prediction of higher and middle frequencies, but when it comes to lower frequencies, results from their simulations are less solid. This phenomenon is caused by the fact, that these methods handle the sound waves as rays.

Wavelength of the sinusoidal wave (which represents the sound wave) is the distance between repeating periods of the wave motion. In other words it is the spatial period of the wave. Frequency indicates the number of repeating of periodical action (wavelength) per time unit (usually one second)

$$f = \frac{c}{\lambda}. \tag{1.1.1}$$

Here f is the frequency in Hertz (or s^{-1}) and λ is the wavelength in meters. That means that the lower the frequency is, the bigger is the wavelength. For example when the frequency is 1000 Hz, the wavelength is about 34 centimeters long. But when we lower the frequency to 10 Hz the wavelength goes up to 34 meters.

The value c in (1.1.1) is the phase speed, which in the field of acoustics represents the speed of sound. This value depends on the temperature and composition of medium, in which the sound wave propagates. For example at the temperature of 20 °C, the sound travels at 343 m/s in air, at 1480 m/s in water, and at 5120 in iron.

Sound wave with bigger wavelength doesn’t behave according to the principles described above (e.g. the angle of incidence equals the angle of reflection). The size of this wavelength can’t be neglected and this wave can’t be handled as ray. Behavior of this sound wave is even more complicated, when it occurs in the space with objects with comparable size as the wavelength (e.g. columns, stair etc.). When the simulation of lower frequencies is done by ray-based methods, an error occurs, which could even grow in particular type of spaces. That’s why in this thesis author focuses on numerical methods and their contribution into this area.

2. Models based on differential equations in acoustics

2.1. Wave equation

In the area of acoustics, the mathematical models work with the wave equation and its generalizations

$$\frac{\partial^2}{\partial t^2} P = c^2 \Delta P, \tag{2.1.1}$$

where Δ is Laplace operator, P is the pressure and c is already mentioned phase speed mentioned in (1.1.1). This equation describes the behavior of sound, light or water waves.

To solve (2.1.1), the method of separation of variable can be applied. Thus the time-independent form of the original equation is obtained, which is called the Helmholtz equation (described in next subsection).

In the case of time harmonic acoustic propagation and scattering [3], the pressure function is given by

$$P(x,t) = Re\left(A(x)e^{-i\omega t}\right). \tag{2.1.2}$$

Here Re denotes the real part and ω is the angular frequency

$$\omega = 2\pi f, \tag{2.1.3}$$

where f is the frequency measured in Hz (1.1.1). Function A is in general complex valued and we call it complex acoustic pressure. The physical sound pressure is the magnitude of complex acoustic pressure. The real part is the magnitude of sound pressure, and imaginary part is the phase of the time harmonic pressure fluctuation.

2.2. Helmholtz equation

As mentioned in previous subsection, the Helmholtz equation is related to the wave equation (2.1.1). It is in the form

$$\Delta A + k^2 A = 0. \tag{2.2.1}$$

Here $|A(x)|$ is the amplitude of the time harmonic pressure fluctuation at x, and k is the wavenumber. Wavenumber is the spatial frequency of a wave measured in radians per unit distance rad/m. It is also given by the formula

$$k = \frac{\omega}{c}. \tag{2.2.2}$$

The Helmholtz equation is related to the problems of steady-state oscillation. Because of its relation to the wave equation (2.1.1), it has use in various areas of physics such as electromagnetic radiation, elasticity or seismology.

2.3. Impedance boundary conditions

The boundary conditions which we mostly work with when solving (2.2.1) are of Robin type. They are called the impedance boundary conditions [3]

$$ik\beta A + \frac{\partial A}{\partial \mathbf{n}} = g. \tag{2.3.1}$$

Here A is already mentioned complex acoustic pressure and i is imaginary unit. $\frac{\partial A}{\partial \mathbf{n}}$ is the normal derivative, which can be rewritten as $\nabla A \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit vector of outward-pointing normal. The function g on the right side can be generally seen as the function of source. The parameter β is called the relative surface admittance. When this parameter is set to $\beta = 0$, this represents the simulation of acoustically hard wall with maximum energy reflected. When it is set to $\beta = 1$, it simulates the wall with maximal sound absorption, i.e. the free space. It is important to note, that if we do the calculation considering the inward normal, the sign is opposite $\beta = -1$.

Remark: There must be mentioned that the Helmholtz equation with homogenous boundary conditions is an eigenvalue problem for the Laplacian [5]. When the situation $\lambda = k^2$ (where λ is the eigenvalue) occurs, the solution to the equation is not unique. Moreover its existence also depends on compatibility with the source function g.

Remark (taken from [11]): When considering an acoustic application, where u represents the sound pressure, and using the linearized one dimensional Euler equation in acoustics

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{2.3.2}$$

where v is the acoustic particle and ρ_0 the nominal density, (2.3.1) can be written as

$$ik\beta u = \rho_0 \frac{\partial v}{\partial t} + g = i\rho_0 \omega v + g. \tag{2.3.3}$$

Considering a boundary without having a source term g, the ratio between the acoustic pressure u and the acoustic particle velocity v can be written as

$$\frac{v}{u} = \frac{k\beta}{\rho_0 \omega} = \frac{\beta}{\rho_0 c}. \tag{2.3.4}$$

In acoustics the ratio $\frac{u}{v}$ is known as the acoustic impedance, and the ratio $\frac{v}{u}$ as the acoustic admittance. Thus β can be seen as the non-dimensional acoustic admittance [11].

2.4. Numerical methods in acoustics

The numerical methods were originally developed as engineering methods used mostly in the field of solid and fluid mechanics and they are becoming more common in the area of acoustics. We focus on their implementation into acoustic simulations, as we believe that they could bring more precise results into this field.

Numerical methods include e.g. the Finite element method (FEM), the Boundary element method (BEM), the Finite volume method (FVM), the Finite difference method (FDM), etc. They are used to approximate the solution of partial differential equations.

Differential equations are used to describe phenomena of different fields like physics, economics or biology. In many cases, they are not solvable directly [21], and the approximation of solution is sufficient. That’s where numerical methods are very effective tool for solving differential equations.

Unlike ray-based methods (described in first chapter), the numerical methods are, in the area of acoustics, the frequency domain algorithms, which means that the constant frequency in Helmholtz equation or in wave equation is considered.

3. Boundary element method

The part of author's work consisted of solving the Helmholtz equation by the Boundary element method (BEM). As next sections (fourth and fifth) are the core of the thesis as well as the author's main focus, this section is elaborated in brief.

The main principle of this method lies in the transformation of the partial differential equation, which depicts the particular problem, into an integral equation over the boundary of the domain [3, 12, 13, 18, 20, 23, 24]. That brings an advantage of the method, which is the dimension reduction (i.e. 3D problem reduces to 2D – only to the boundary). Thanks to that, it is not necessary to discretize the whole domain, like for example in the Finite element method, but only the boundary.

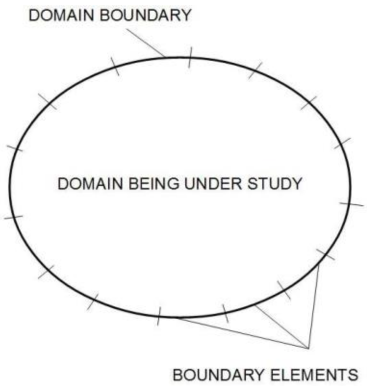


Figure 3.1.1 Discretization of the boundary

3.1.1. Calculus of Boundary element method and the results

BEM can be applicable only to the equations, for whose their fundamental solution is known. For 2D case, the fundamental solution of the Helmholtz equation is the function

$$v(x,y) = \frac{i}{4} H_0^{(1)}(z), \tag{3.1.1.1}$$

where i is the imaginary unit and z is the Euclidean norm $z = k\|x - y\|$. $H_0^{(1)}$ is the Hankel function of the first kind. Computing by BEM includes multiplying the original Helmholtz equation by the function v (3.1.1.1), integrating over the domain and applying the Green's theorem about the relationship between the line and double integral. This way the equation for the interior problem is obtained

$$u(x) = \int_{\partial\Omega} v(x,y) \frac{du}{d\mathbf{n}}(y) ds_y - \int_{\partial\Omega} \frac{dv}{d\mathbf{n}_y}(x,y) u(y) ds_y. \tag{3.1.1.2}$$

Symbol Ω represents the domain, $\partial\Omega$ signifies its boundary, and u is function in demand. The equation was discretized and the BC were approximated by constant functions. For calculating the integrals we have used Simpson's quadrature rule. Thus the system of linear algebraic equations was obtained. There occurred the problem of singularity, which still needs to be solved, and which could be avoided by using better quadrature rule.

4. Finite volume method for the Helmholtz equation

4.1. 2D Finite volume method scheme

In this chapter the numerical scheme based on the 2D Finite volume method is presented [6, 7, 17, 18]. Here the solution is approximated at discrete points of mesh usually called the representative points. Important feature of the method is the local conservativity of numerical fluxes, which means that the flux is conserved from one discretization cell to its neighbor.

The finite volume numerical scheme can be obtained by integrating the differential (2.2.1) equation on each finite volume, using Green's theorem and applying the approximation function. This way we obtain

$$k^2 u_p m(p) + \sum_{q \in N(p)} \frac{u_q - u_p}{d_{pq}} m(\sigma_{pq}) = 0. \tag{4.1.1}$$

where u_p is approximated solution on finite volume p, u_q is the solution in the neighboring volume, $m(p)$ is the size of the finite volume, $m(\sigma_{pq})$ is the size of edge σ_{pq} and d_{pq} is the distance between representative points.

The solution to the Helmholtz equation is based on complex values. From prescribed boundary condition (2.3.1), we obtained the conditions for real and imaginary part of the solution

$$\begin{aligned} \frac{\partial A^r}{\partial \mathbf{n}} - k\beta A^i &= g^r, \\ \frac{\partial A^i}{\partial \mathbf{n}} + k\beta A^r &= g^i. \end{aligned} \tag{4.1.2}$$

This way we create a system of linear algebraic equations, in which the matrix was of order $2nm$. After solving the system, both real and imaginary part, are obtained for each finite volume.

4.1.1. Solving of the system of equations

This subsection is dedicated to solving of the system of equations. As is stated in [16] the iterative methods like e.g. Jacobi method, Gauss-Seidel method or the Successive over-relaxation (SOR) method, are very effective and appropriate tool - mostly in case of very big matrices, when it is disadvantageous to use direct methods (e.g. LU decomposition).

It is important to note, that there are the properties which the matrixes must have when dealing with iterative methods. For example the Jacobi method converges only when the matrix in the system is diagonally dominant, or the Gauss-Seidel method converges when the matrix is symmetric, positive-definite and diagonally dominant. As the matrixes created by schemes described in previous parts have none of these properties, to solve the system of equations one of direct methods was used, particularly the LU decomposition.

4.1.2. Results of numerical experiment

For the results to be presented, the solution of (2.2.1) is with $\beta = 1$ in (2.3.1), so Robin boundary conditions are prescribed for all sides of the domain. We have used the exact solution from [1]

$$u(x, y) = e^{i(k_1 x + k_2 y)} = \cos(k_1 x + k_2 y) + i\sin(k_1 x + k_2 y), \qquad (4.1.2.1)$$

The source function g was set so as to comply with the exact solution (4.1.2.1). The results for $\theta = \frac{\pi}{2}$, for two different wavenumbers are presented. As was expected, the presented scheme is effective in case of lower frequencies. When dealing with high frequencies, finer discretization is required

Table 4.1.2.1 Values of L2 error, $\theta = \frac{\pi}{2}$

	$k = 10rad/m$			$k = 25rad/m$		
n	10	40	60	10	40	60
L2error	0.2653	0.0143	0.0063	1.3518	0.2324	0.1

Table 4.1.2.2 Values of EOC for the wavenumber $10rad/m$, $\theta = \frac{\pi}{2}$

<table>
<tr><th>n</th><th>L2 error</th><th>α</th></tr>
<tr><td>10</td><td>0.265255</td><td rowspan="2">2.17192</td></tr>
<tr><td rowspan="2">20</td><td rowspan="2">0.058864</td></tr>
<tr><td rowspan="2">2.03935</td></tr>
<tr><td rowspan="2">40</td><td rowspan="2">0.01432</td></tr>
<tr><td rowspan="2">2.01011</td></tr>
<tr><td>80</td><td>0.003555</td></tr>
</table>

4.1.3. The rigid piston simulations

This chapter presents the program, which is the result of ideas and consultations with Nicholaas Bert Roozen during the stay at KU Leuven in Belgium. The main motivation was to compare the results obtained by software implementing BEM with ones obtained by previously presented FVM scheme. The code was also described in [11].

The square domain of size 8 metres with rigid piston on left side was considered. The boundary condition of the piston, moving with the velocity v_{piston}, is prescribed as

$$\frac{\partial u}{\partial \mathbf{n}} = -i\rho\omega v_{piston}. \qquad (4.1.3.1)$$

Here $\rho = 1.2\ kg/m^3$ is the density of air, ω is the angular frequency measured in rad/s and given by

$$\omega = 2\pi f. \qquad (4.1.3.2)$$

$v_{piston} = 1\ m/s$ is the starting velocity of the piston, and i is the imaginary unit. f is the frequency measured in Hz, and it is set to different values.

In first case presented, the boundary conditions on other sides than the rigid piston, are prescribed as zero Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0. \qquad (4.1.3.3)$$

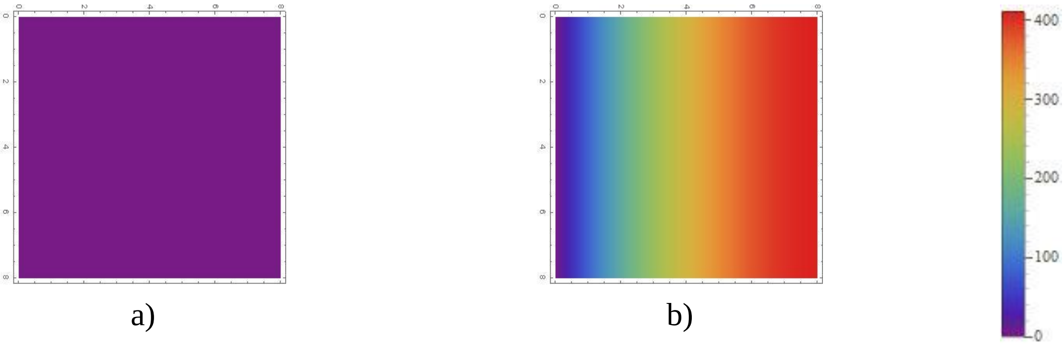
This simulates the wall with $\beta = 0$, which means an acoustically hard wall with maximum energy reflected. In second case we wanted to simulate the free space. The source function is set to $g = 0$, so the boundary conditions are in the form

$$iku - \frac{\partial u}{\partial \mathbf{n}} = 0. \tag{4.1.3.4}$$

And finally in the last case, the boundary conditions on right side of the square domain are given by (4.1.3.4), and on the up and down side by (4.1.3.3). The following subsection shows the results of the simulations.

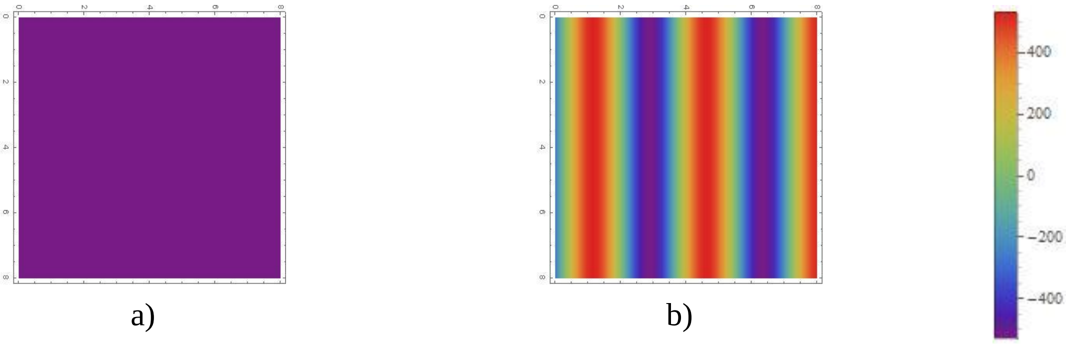
4.1.3.1. Rigid piston simulations results

First results to be presented are with rigid piston (4.1.3.1) on left side, and Neumann boundary conditions (4.1.3.3) on other sides (RPN). Figures 4.1.3.1.1 and 4.1.3.1.2 show the numerical results for two different frequencies (10 and 100 Hz). Number of discretizing points along one side of the domain is $n = 50$ (i.e. 2500 finite volumes).



a) b)

Figure 4.1.3.1.1 RPN for $f = 10\ Hz$, $n = 50$ a) real part b) imaginary part

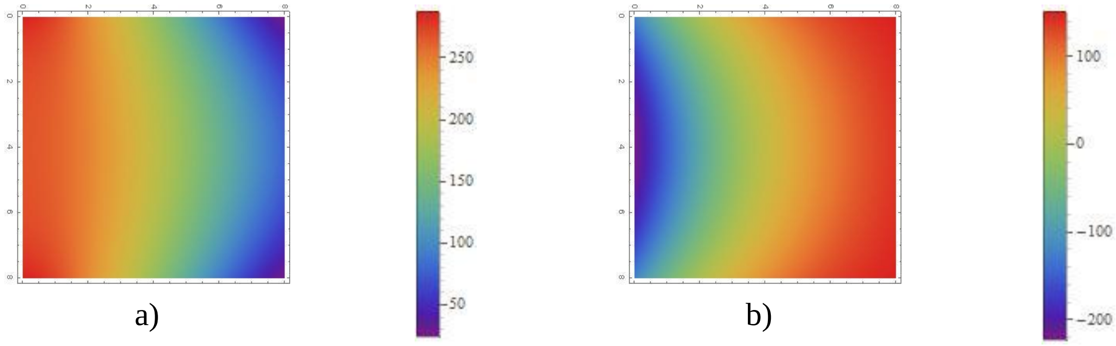


a) b)

Figure 4.1.3.1.2 RPN for $f = 100\ Hz$, $n = 50$ a) real part b) imaginary part

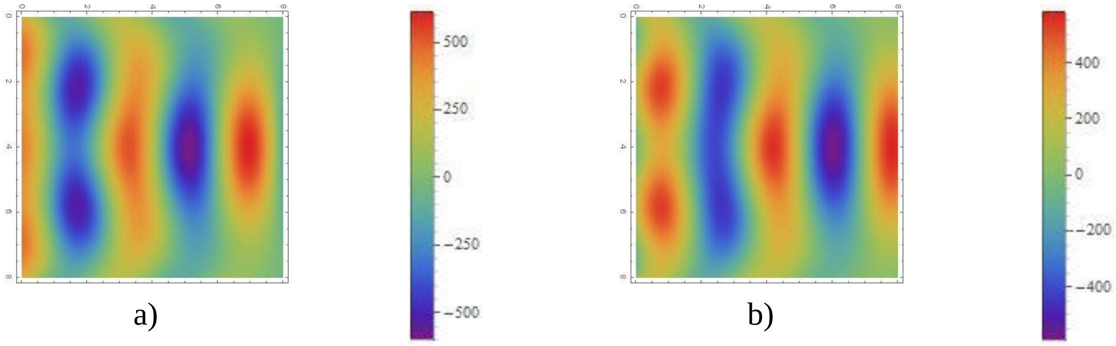
As can be seen, the real part is in both cases zero. It points up that the numerical solution is right, as it is clear that the zero solution of real part is complying the prescribed boundary conditions. Physically, it means that the acoustic pressure in the square cavity is out of phase with $i\rho_0 \omega v_{piston}$, and thus in phase with the prescribed velocity v_{piston} of the piston.

Next the numerical results of the case with $\beta = -1$ (4.1.3.4) on all sides but the left side with piston are presented (RPB). The Figures 4.1.3.1.3 and 4.1.3.1.4 show the computation for same frequencies with $n = 50$.



a) b)

Figure 4.1.3.1.3 RPB for $f = 10\ Hz, n = 50$ a) real part b) imaginary part

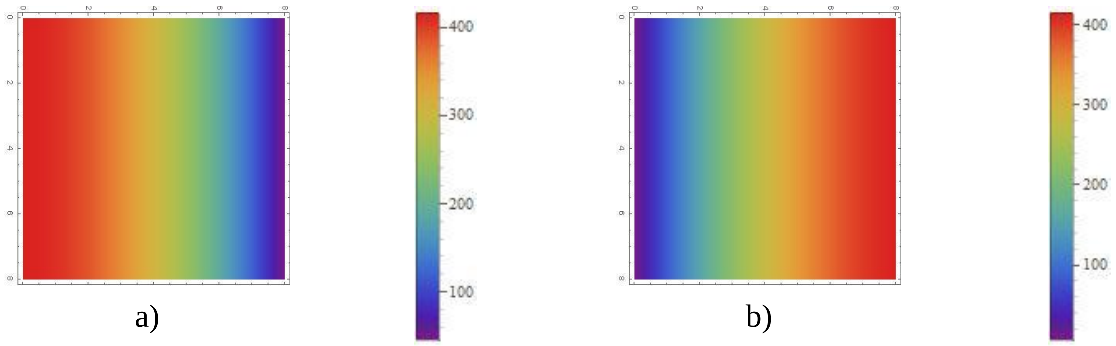


a) b)

Figure 4.1.3.1.4 RPB for $f = 100\ Hz, n = 50$ a) real part b) imaginary part

Obviously, the pressure distribution in the cavity is now more complex. The acoustic wavefronts are not traveling only in horizontal direction, away from the vibrating piston, but also partly in the direction of up and down side of the domain. This is caused by the $\beta = -1$ boundary condition on these sides of the domain. It is necessary to note that due to the fact that the waves are not perpendicular to the up and down side of the domain, there boundaries do not perfectly absorb the acoustic waves impinging on them, but will also be partly reflected.

The last results (figures 4.1.3.1.5 and 4.1.3.1.6) are for the case with piston on left side, Neumann boundary conditions on up and down side, and $\beta = -1$ on right side of the domain (RPNB). This type of boundary condition was chosen, as to prevent the situation that the acoustic wavefronts are non-perpendicular incident upon the boundaries. For the right side of the boundary the acoustic wave will be perpendicular incident, given the fact that the vibrating piston is on the opposite site of the domain. Thus this side will be perfectly absorbent.



a) b)

Figure 4.1.3.1.5 RPNB for $f = 10\ Hz, n = 50$ a) real part b) imaginary part

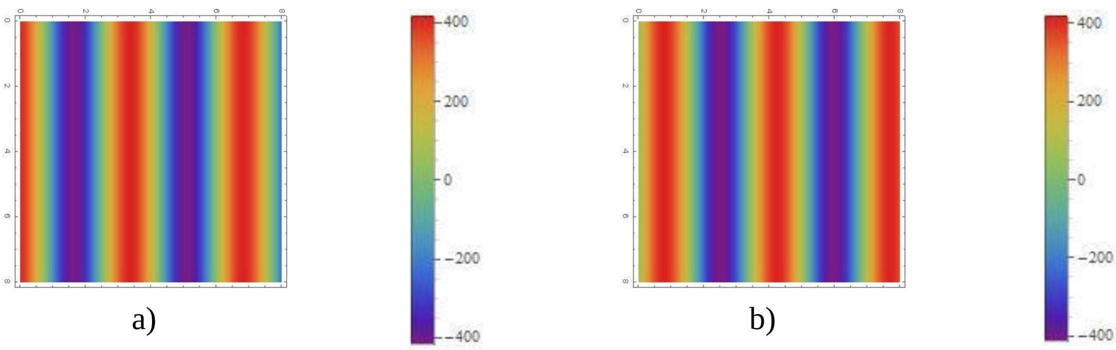


Figure 4.1.3.1.6 RPNB for $f = 100\ Hz$, $n = 50$ a) real part b) imaginary part

It is important to note, that presented numerical results have been also compared to results gained by software implementing the Boundary element method. They are very close, which suggests the assumption that the results are correct [11].

4.1.3.2. Analytical solution of RPNB

The values of the L2 error and the experimental order of convergence (EOC) of last case (RPNB) are presented. The analytical solution used is in the form

$$p(x, y) = \cos(kx) + i\sin(kx). \tag{4.1.3.2.1}$$

The starting velocity of the piston v_{piston} had to be changed, and was calculated as

$$v_{piston} = \frac{-1}{\rho c}. \tag{4.1.3.2.2}$$

Here $c = 340.29\ m/s$ is the speed of sound.

Table 4.1.3.2.1 presents the L2 error EOC for the frequency $f = 1000\ Hz$ in the square domain of size 1 meter.

Table 4.1.3.2.1 L2 error and EOC, $f = 1000\ Hz$

n	L2 error	α
10	1.339592	
		1.50926
20	0.470587	
		2.29863
40	0.095659	

4.2. 3D Finite volume method scheme results

Third part of thesis presents the results of 3D Finite volume method scheme for solving the Helmholtz equation. The scheme is the same as described in section 4.1, and the domain is cube of size 1 meter.

The behavior of the solution is similar as with the 2D scheme, but with the same discretization, more finite volumes is considered as the domain is bigger (third dimension).

4.3. Applications of Finite volume method scheme

4.3.1. Room modes and sound waves behavior

This section is based on ideas and consultations with acoustic measurement expert Ing. Jiří Olša. Sometimes the acoustician wants to attenuate certain frequency in a room by placing e.g. absorptive panel. To do this he needs to know the exact behavior of this frequency, i.e. where are the frequency's nodes and antinodes, as it is mostly effective to place acoustic component in the spot of the node. This behavior of frequencies depends on the room's dimensions, wall surfaces, etc.

To obtain the information about particular frequency's propagation in a room, it is necessary to perform a measurement in many spots of the room. Subsequently, the acoustician musts analyze the measurement data for each spot, perform the Fourier transform of time domain signal, pick the data connected to chosen frequency, and create the "map of frequency's propagation". This process can be very complicated and time-consuming, especially when it comes to larger spaces.

That's why it is believed, that solution of the Helmholtz equation by the Finite volume method would be helpful in this area. The measurement of simple room and comparison with the results of Finite volume method program is planned for future work.

4.3.2. Measurement simulations

The following part is also presented in [9]. The main motivation was to numerically simulate the values obtained by acoustic measurements. The measurements were performed by author in specialized acoustic laboratory at the Faculty of Science at KU Leuven, in Belgium. Main idea was to study the reflection of different frequencies from boards with various openings.

At known positions of the space, the speaker and microphone were placed. These positions were also varying. Exponential sweep, containing all audible frequencies, was used as a test signal, sent from the loudspeaker and recorded by microphone, and impulse response was calculated.

The main assumption was that if there is a full board placed in the room, all frequencies with wavelength smaller that the size of the board will be reflected. In case of a board with opening, a part of the sound energy with higher frequency content won't be reflected and will get through the panel, and the lower frequencies will fully reflect due to diffraction effects. If the opening is smaller, more of the frequencies are reflected. It is because the lower the frequency is, the bigger is its wavelength [18, 19]. When the frequency spectrum of the reflection was studied, our assumption was confirmed.

The main attempt was the comparison of the data obtained from the measurements with numerical simulations implementing the Finite volume method.

4.3.2.1. Numerical simulations and comparison of the data

The simulations of the measurement were performed by the Finite volume method scheme described in section 4.1. The case is three-dimensional, but we decided to simulate only the plane where the speaker and microphone were placed. Thus the problem reduces to two-dimensional which makes the computation easier and faster. The domain considered in the simulations was square, whereas its size was set to match the real situation.

As various problems occurred along the way, it is not possible to compare simulated and measured data at this stage. Main issue is the measurement. Whereas this was the first study experiment, as in the position of mathematicians we were not technically ready for it. As was stated, it is necessary to improve the measurement by placing more than one microphone in the room, which would secure bigger accuracy of the data used for simulations, and thus enable the comparison. This is left for future work.

5. Wave equation

As was stated in section 2., the wave equation plays an important role in many fields, like e.g. acoustics, electro-magnetics or fluid dynamics. In its general form

$$u_{tt} - \Delta u = f(t, x, u, u_t), \tag{5.1}$$

together with prescribed boundary conditions, it has been studied by many authors (see e.g. [4], [15] and references therein). There are also many studies concerning numerical methods for the wave equation (see e.g. [2], [14], [22] and references therein). In this chapter, the attention is focused to the numerical method based on finite volumes (also presented in [10]). In the whole chapter, the notation as presented in [6] will be used.

We are interested in the wave equation for two-dimensional domain namely

$$u_{tt}(t, x) = c^2 \Delta u(t, x), t \in [0, T], x \in \Omega \subset \mathbb{R}^2, \tag{5.2}$$

with zero Dirichlet boundary condition on the boundary of Ω (denoted by $\partial\Omega$) and the initial condition of the type

$$\begin{aligned} u(0, x) &= \varphi(x) \quad x \in \Omega, \\ u_t(0, x) &= \psi(x) \quad x \in \Omega. \end{aligned} \tag{5.3}$$

5.1.1. Classical Finite volume method in 2D

For the space discretization we use the method based on finite volumes. We define our discretization mesh as in [6]. The discretization of our numerical scheme is inspired by the scheme in [2], where the implicit scheme to compute $n + 1$ time step uses in diffusion term the average of diffusion terms in $n + 1$ and $n - 1$ time steps. Using classical approach in finite volume methodology, which includes integrating the equation (5.2) on arbitrary finite volume V_p and applying divergence theorem and approximation, we obtain our numerical scheme for unknown value u_p^{n+1} of the following form:

$$\begin{gathered} \frac{u_p^{n+1} - 2u_p^n + u_p^{n-1}}{\tau^2} m(V_p) = \\ \frac{c^2}{2} \left(\sum_{q \in N(p)} \frac{u_q^{n+1} - u_p^{n+1}}{d_{pq}} + \frac{u_q^{n-1} - u_p^{n-1}}{d_{pq}} \right) m(\sigma_{pq}), \end{gathered} \tag{5.1.1.1}$$

for $n = 1,2,3, \dots, N - 1$.

5.2. Stability estimates

In this section of the thesis, necessary stability estimates for the numerical solution are derived.

Theorem 5.2.1. *Let the hypothesis H hold. Let the time step fulfil the condition* $\tau \leq \tau_0$*. Then the numerical solution on first time step* u_T^1 *of the numerical scheme is bounded independently on discretization parameters in* $L_2(\Omega)$*. Moreover, its discrete* H_1 *norm is also bounded, and the approximation of the first time derivation is bounded in* $L_2(\Omega)$ *as well:*

$$\|u_T^1\| + \left\|\partial u_T^1\right\| + \|u_T^1\|_{1,T} \leq C. \tag{5.2.1}$$

Theorem 5.2.2. *Let the hypothesis H and condition* $\tau \leq \tau_0$ *hold. Then the numerical solution* $u_{T,\tau}$ *of the numerical scheme is bounded independently on discretization parameters in* $L_\infty(I; L_2(\Omega)) \cap L_\infty(I; H_0^1(\Omega))$*, and the approximation of the first time derivation is bounded in* $L_\infty(I; L_2(\Omega))$ *as well:*

$$\|\delta u_T^m\|^2 + \|u_T^m\|^2 + \frac{c^2}{2}\|u_T^m\|_{1,T}^2 \leq C. \tag{5.2.2}$$

for $m = 1,2, \dots, N$.

5.3. Convergence

To prove the convergence for the numerical solution, the stability estimates obtained in the previous section were used, and by passing to the limit the convergence to the weak solution can be obtained.

5.4. Numerical experiments

In this section we present several experiments of the example, where the exact solution is given using proposed scheme. Here we present the experiments for various values of constant c in the wave equation, and for various values of so called Courant number or CFL (Courant – Friedrichs - Lewy) number as a ratio between space and time discretization parameters together with constant c. In the numerical analysis of explicit time integration schemes, this Courant number must fulfil certain condition for convergence of the scheme and this condition is called as CFL condition.

Example

Here we use the example of a problem (5.2)-(5.3) studied in [2] for 2D case. Our computed region is square [0,1] x [0,1], and the computation is done for the time interval [0,1]. We discretize our domain with rectangles with edge size h, so each finite volume V_p has $m(V_p) = h^2$. The exact solution is in the following form:

$$u(x, y, t) = \sin(2\pi y)\cos(2\pi y)\cos\left(2\sqrt{2}ct\right). \tag{5.4.1}$$

We have computed initial functions φ and ψ from the exact solution and we used the Dirichlet boundary conditions as in [2]. To compute the errors and the experimental order of convergence (EOC) we have used the following $L_\infty(I, L_2(\Omega))$ and $L_2(I, L_2(\Omega))$ errors.

In all experiments we have set the time-space discretization in such way, that CFL number $\frac{c\tau}{h} = 1$. We present three experiments for various values of constant $c = 1$, $c = 10$ and $c = 100$. All results are presented in tables. As can be seen from the tables 5.4.1, 5.4.2 and 5.4.3, the computed EOC's are of second order as it was presented in [2] for their method based on boundary integral solutions. After we had set $\frac{c\tau}{h} = 2$, similar results for EOC were obtained. But this is not true for higher values of CFL number. This will be the question for further investigation.

Table 5.4.1 Results of the experiment, $c = 1$

τ	h	*ELN*	*EOC*	*EL2*	*EOC*
0.1	0.1	3.966e-01	-	2.228e-01	-
0.05	0.05	9.843e-02	2.021	6.024e-27	1.918
0.025	0.025	2.470e-02	1.995	1.516e-2	1.991
0.0125	0.0125	6.477e-03	1.931	3.777e-3	2.005
0.00625	0.00625	1.638e-03	1.983	9.408e-4	2.005
0.003125	0.003125	4.105e-04	1.997	2.348e-4	2.003

Table 5.4.2 Results of the experiment, $c = 10$

τ	h	*ELN*	*EOC*	*EL2*	*EOC*
0.1	0.1	3.994-00	-	1.725e-00	-
0.05	0.05	1.741e-01	4.519	9.147e-2	4.238
0.025	0.025	3.905e-02	2.157	2.001e-2	2.192
0.0125	0.0125	9.532e-03	2.035	4.941e-3	2.018
0.00625	0.00625	2.368e-03	2.009	1.228e-3	2.009
0.003125	0.003125	5.909e-04	2.002	3.065e-4	2.002

Table 5.4.3 Results of the experiment, $c = 100$

τ	h	*ELN*	*EOC*	*EL2*	*EOC*
0.1	0.1	9.745e-00	-	4.697e-00	-
0.05	0.05	1.777e-01	5.777	9.030e-27	5.700
0.025	0.025	3.952e-02	2.169	1.991e-2	2.182
0.0125	0.0125	9.671e-03	2.031	4.882e-3	2.028
0.00625	0.00625	2.411e-03	2.004	1.217e-3	2.005
0.003125	0.003125	6.036e-04	1.999	3.040e-4	2.001

Conclusion

The dissertation thesis is dedicated to mathematical models in acoustics, which are based on differential equations solved by numerical methods.

In section 1, we have given brief description of main principles of so called ray-based methods (e.g. image source method, ray-tracing method etc.), which are nowadays commonly used in acoustic praxis. These methods handle sound waves as rays, which can lead to inaccurate results when dealing with low frequencies. Examples of use of softwares implementing ray-based methods from author's today praxis are shown.

As ray-based methods face particular problems, the thesis focuses on implementation of numeric methods into the area of acoustic simulations. Section 2 is dedicated to models based on differential equations, which in acoustics work with the wave equation, and in case of steady-state oscillations with the Helmholtz equation. The last part of the section is devoted to impedance boundary conditions, with which we have worked with later on.

Section 3 is dedicated to the solution of Helmholtz equation by the Boundary element method. Here, the calculus and the results of 2D scheme are presented. The main problem of the scheme was the occurrence of singularity, which could be avoided by using better quadrature rule.

In section 4, we have worked with the Finite volume method for solving the Helmholtz equation. Again, the calculus and the results of 2D and 3D scheme are presented, including simulations of rigid piston. Further, the possible applications are shown, e.g. obtaining of room modes and sound wave behavior idea in space with use of Finite volume method scheme instead of performing complicated and time-consuming measurement.

Last section 5 is devoted to the Finite volume method scheme for solving the wave equation. Scheme and results are presented. Further, the stability estimates and convergence are proven.

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