Study of a finite volume scheme for the regularised mean curvature flow level set equation

Robert Eymard[*], Angela Handlovičová[†] and Karol Mikula[‡]

Abstract

We propose a new finite volume numerical scheme for the approximation of regularised mean curvature flow level set equations, which ensures the maximum principle, and which is shown to converge to the solution of the problem. The convergence proof uses the monotonicity of the operator, in order to get the strong convergence of the approximation of the gradient. Numerical examples provide indications about the accuracy of the method. Applications to noisy image filtering show less diffusive behaviour than the classical finite difference scheme.

Keywords: Regularised mean curvature flow level set equation, convergence of finite volume method, Leray-Lions operators, image filtering.

1 Introduction

We consider the following problem: find an approximate solution to the equation

$$u_t - g(|\nabla u|)\text{div}\left(\frac{\nabla u}{f(|\nabla u|)}\right) = r, \text{ a.e. } (x,t) \in \Omega \times (0,T) \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \text{ a.e. } x \in \Omega, \tag{2}$$

and the boundary condition

$$u(x,t) = 0, \text{ a.e. } (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{3}$$

under some hypotheses on the real functions f, g, the initial data u_0, the right hand side r, and on the domain Ω, which are detailed below. Note that the case of Neumann boundary conditions on a part of the boundary or on the whole boundary, instead of (3), is interesting as well, and that it does not add specific difficulties to the present study. The standard mean curvature flow level set equation, which is obtained by setting $r = 0$ and

$$f(x) = g(x) = x, \ \forall x \in \mathbb{R}_+ \tag{4}$$

in (1), has numerous applications in science, engineering and technology, ranging from free boundary problems in material sciences and computational fluid dynamics to filtering and segmentation algorithms in image processing and computer vision. We refer to [29] for the original mean curvature flow level set equation, to [1, 3, 6, 23, 32] for some generalisations in various frameworks; in image processing applications, equation (1-4), called the curvature filter, is generalised and used in adaptive image filtering [9], image segmentation by the geodesic active contours [6, 23] and the (generalised) subjective surfaces method [31, 26, 10, 24].

[*]Université Paris-Est, France, robert.eymard@univ-mlv.fr
[†]Slovak University of Technology, angela@math.sk
[‡]Slovak University of Technology, mikula@math.sk

The analysis of numerical algorithms for solving (1-4) and related problems is a difficult task due to its nonlinear character and non-divergent form. In [11, 12, 13], the error estimates for geometric quantities like the regularised normal to the level set of solution and its normal velocity have been established using the finite element method. Such estimates are very useful for the free boundary problems when dealing with the motion of one particular level curve or level surface.

On the other hand, e.g. in image processing applications when the evolution of the whole level set function representing an image intensity is used in practice, the convergence of a numerical approximation to the solution u itself is an important point. In this way, the classical fully explicit finite difference scheme, which is widely used in image processing applications [33, 28], is significantly modified in [27] for getting a convergence property to the viscosity solution of (1-4).

Note that the smoothness of the numerical solution is implicitly assumed by classical finite difference schemes, which are based on second order Taylor's expansions with respect to the space variable. This is no longer the case with finite volume schemes: such schemes are defined by first order Taylor's expansions at the boundary of the finite volumes, corresponding to the pixels of the image, thus respecting the structure of digital images. This is shown in [25], where such a finite volume approach to image processing is introduced for the regularised Perona-Malik equation [30, 7], and a convergence analysis is proposed in [14] for the coherence enhancing nonlinear tensor diffusion.

The mathematical analysis of finite volume methods for mean curvature flow level set equation is partly proposed in [22, 26, 21], applied to the co-volume scheme initially proposed by Walkington [34]. In this scheme, one first replaces (4) by

$$f(x) = g(x) = \sqrt{x^2 + a^2},\ \forall x \in \mathbb{R}_+, \tag{5}$$

for a small value $a > 0$. This regularisation, known as the Evans-Spruck regularisation of the problem (1-4), is used to prevent from the occurrence of zero denominators in the numerical scheme (note that it has first been used in [16, 8] to show the existence of the viscosity solution to (1-4)). Walkington's initial scheme is nonlinear and its linear semi-implicit variant is suggested in [22]. Such semi-implicit scheme is proved to be efficient, as keeping all theoretical properties of Walkington's scheme. It is used in solving various practical 2D and 3D (large-scale) image analysis problems [10, 24, 5]. In [22, 26] the L^∞ stability of the solution and the L^1 stability of its gradient are given. Moreover, in [21], the consistency of the scheme is proved using the Barles and Souganidis [4] approach for solving nonlinear PDEs. However, the convergence of the co-volume semi-implicit scheme to the exact solution remains an open problem.

Note that the convergence of finite volume methods for the solution of the stationary version of (1), has been proved in [2, 15, 19], under the assumptions

(LL1) the function $x \mapsto x/f(x)$ is strictly increasing on $\mathbb{R}_+$,

(LL2) $\dfrac{dx}{c + \ x^{p-1}} \le f(x) \le Cx^{2-p}$ for given $c, d, C > 0$, $p > 1$ and all $x \in \mathbb{R}_+$,

(LL3) g constant.

We get under assumptions (LL1)-(LL2) that the function $u \mapsto -\text{div}\ (\nabla u / f(|\nabla u|))$ is a Leray-Lions operator, whose monotonicity properties allow for the use of Minty and Leray-Lions tricks for the proof of the convergence. Note that property (LL1) holds for the choice (5) for f, but not (LL2). On the contrary, (LL1)-(LL2) hold if we consider for example

$$f(x) = g(x) = \min(\sqrt{x^2 + a^2}, b),\ \forall x \in \mathbb{R}_+, \tag{6}$$

for given reals $0 < a \le b$, setting $p = 2$, $c = 1$, $d = a$ and $C = b$. In the choice (6), the use of the bound b is in accordance with image processing applications. Indeed, on discrete grids, the gradient norms are lower than $\frac{Q}{h}$, where Q is a quantisation parameter and h is the side length of a pixel. This acts in a similar way to the convolution used in [7] for regularising the Perona-Malik equation. Nevertheless, the problem approximated in [2, 15, 19] is stationary and conservative; new difficulties arise in approximating (1), which is transient and non-conservative. In order to be able to overcome these difficulties, we consider in this paper the following hypotheses, called hypotheses (H) in the following.

1. Ω is a finite connected open subset of $\mathbb{R}^d$, $d \in \mathbb{N}^\star$, with boundary $\partial\Omega$ defined by a finite union of subsets of hyperplanes of $\mathbb{R}^d$,

2. $u_0 \in H_0^1(\Omega)$,

3. $r \in L^2(\Omega \times (0,T))$ for all $T > 0$,

4. $g \in C^0(\mathbb{R}_+; [a,b])$, with $0 < a < b$,

5. $f \in C^0(\mathbb{R}_+; [a,b])$ is a Lipschitz continuous (non-strictly) increasing function, such that the function $x \mapsto x/f(x)$ is strictly increasing on $\mathbb{R}_+$.

It is worth noticing that the functions f and g given by (6) satisfy (H4-5). The hypothesis (H2) is restrictive when considering image processing applications where e.g. characteristic functions of shapes can be used. However, this theoretical restriction can be relaxed in practice and the finite volume schemes can also be used in such cases, see Section 5.

Definition 1.1 (Weak solution of (1)-(2)-(3)) *Under hypotheses (H), we say that u is a weak solution of (1)-(2)-(3) if, for all $T > 0$,*

1. $u \in L^2(0,T; H_0^1(\Omega))$ *and* $u_t \in L^2(\Omega \times (0,T))$ *(hence* $u \in C^0(0,T; L^2(\Omega))$*).*

2. $u(\cdot,0) = u_0$

3. *the following holds*

$$\int_0^T \int_\Omega \left(\frac{u_t(x,t)v(x,t)}{g(|\nabla u(x,t)|)} + \frac{\nabla u(x,t) \cdot \nabla v(x,t)}{f(|\nabla u(x,t)|)} \right) \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \frac{r(x,t)v(x,t)}{g(|\nabla u(x,t)|)} \mathrm{d}x\mathrm{d}t, \quad \forall v \in L^2(0,T; H_0^1(\Omega)). \tag{7}$$

Since any function u weak solution of (1)-(2)-(3) in the sense of Definition 1.1 satisfies div $\left(\frac{\nabla u}{f(|\nabla u|)}\right) \in L^2(\Omega \times (0,T))$, we immediately get the following lemma.

Lemma 1.1 (Property of weak solutions of (1)-(2)-(3)) *Under Hypotheses (H), u is a weak solution of (1)-(2)-(3) in the sense of Definition 1.1 if and only if u satisfies, for all $T > 0$:*

1. $u \in L^2(0,T; H_0^1(\Omega))$ *and* $u_t \in L^2(\Omega \times (0,T))$ *(hence* $u \in C^0(0,T; L^2(\Omega))$*),*

2. $u(\cdot,0) = u_0$,

3. $\mathrm{div} \left(\frac{\nabla u}{f(|\nabla u|)}\right) \in L^2(\Omega \times (0,T))$,

4. $u_t - g(|\nabla u|)\mathrm{div} \left(\frac{\nabla u}{f(|\nabla u|)}\right) = r$ *a.e. in* $\Omega \times (0,T)$.

Remark 1.1 *The framework of this paper does not easily allow for using uniqueness results, deduced from the viscosity solution sense. Difficulties arise, when considering general functions f and g, initial data only in $H_0^1(\Omega)$, Lipschitz-continuous boundary for Dirichlet boundary condition. Moreover, the monotonicity of the problem has to be checked, and the relation between a solution in the viscosity sense and the sense of Definition 1.1 is not straightforward. Hence the uniqueness of the weak solution of (1)-(2)-(3) in the sense of Definition 1.1 is at this time an open problem.*

We consider in this paper two different time discretisations of a new finite volume scheme for solving (1) under Hypotheses (H). The main result of this paper, i.e. the strong convergence of both schemes to a solution of (7), is proved thanks to the following property. Let F be the function defined by

$$\forall s \in \mathbb{R}_+, \ F(s) = \int_0^s \frac{z}{f(z)} \mathrm{d}z \in \left[\frac{s^2}{2\ b}, \frac{s^2}{2\ a}\right]. \tag{8}$$

Then, for any sufficiently regular function u, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(|\nabla u(x,t)|)\mathrm{d}x = \int_\Omega \frac{\nabla u(x,t)\cdot\nabla u_t(x,t)}{f(|\nabla u(x,t)|)}\mathrm{d}x\mathrm{d}t. \tag{9}$$

Therefore, assuming that this function u is solution of (1) with $r = 0$ for the sake of simplicity, we get, by taking $v = u_t$ in (7), that $\nabla u \in C^0([0,T]; L^2(\Omega))$ and

$$\int_0^T \int_\Omega \left(\frac{u_t(x,t)^2}{g(|\nabla u(x,t)|)}\right)\mathrm{d}x\mathrm{d}t + \int_\Omega F(|\nabla u(x,T)|)\mathrm{d}x = \int_\Omega F(|\nabla u_0(x)|)\mathrm{d}x. \tag{10}$$

The discrete equivalent of this property is shown in Lemma 3.2 for the semi implicit scheme (using the fact that f is increasing). The hypothesis that $x \mapsto x/f(x)$ is strictly increasing is used by Minty and Leray-Lions tricks; unfortunately, although it is possible to extend some of these properties to the case $f(x) = x$, the convergence study provided in this paper does not hold in this framework, nor Lemma 1.1.

This paper is organised as follows. In Section 2, we present the discretisation tools. Then in Section 3, we show some estimates that are crucial in the convergence proof, given in Section 4. Numerical results are given in Section 5. Conclusions concerning the compared properties of finite volume schemes and the finite difference scheme proposed in [33] are drawn in Section 6. Finally, an appendix containing a few classical technical results is proposed.

2 The finite volume schemes

In order to describe the schemes, we now introduce some notations for the space discretisation.

Definition 2.1 (Space discretisation) *Let Ω be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}\setminus\{0\}$, and $\partial\Omega = \overline{\Omega}\setminus\Omega$ its boundary. A discretisation of Ω, denoted by $\mathcal{D}$, is defined as the triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:*

1. *$\mathcal{M}$ is a finite family of nonempty connected open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{p\in\mathcal{M}}\overline{p}$. For any $p \in \mathcal{M}$, let $\partial p = \overline{p}\setminus p$ be the boundary of p; let $|p| > 0$ denote the measure of p and let h_p denote the diameter of p and $h_\mathcal{D}$ denote the maximum value of $(h_p)_{m\in\mathcal{M}}$.*

2. *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, σ is a nonempty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$-dimensional measure $|\sigma|$ is strictly positive. We also assume that, for all $p \in \mathcal{M}$, there exists a subset $\mathcal{E}_p$ of $\mathcal{E}$ such that $\partial p = \cup_{\sigma\in\mathcal{E}_p}\overline{\sigma}$. For any $\sigma \in \mathcal{E}$, we denote by $\mathcal{M}_\sigma = \{p \in \mathcal{M}, \sigma \in \mathcal{E}_p\}$. We then assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, is denoted by $\mathcal{E}_{\mathrm{ext}}$) or $\mathcal{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, is denoted by $\mathcal{E}_{\mathrm{int}}$). For all $\sigma \in \mathcal{E}$, we denote by x_σ the barycentre of σ. For all $p \in \mathcal{M}$ and $\sigma \in \mathcal{E}_p$, we denote by $\mathbf{n}_{p,\sigma}$ the unit vector normal to σ outward to p.*

3. *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_p)_{p\in\mathcal{M}}$, such that for all $p \in \mathcal{M}$, $x_p \in p$ and p is assumed to be x_p-star-shaped, which means that for all $x \in p$, the inclusion $[x_p, x] \subset p$ holds. Denoting by $d_{p\sigma}$ the Euclidean distance between x_p and the hyperplane including σ, one assumes that $d_{p\sigma} > 0$. We then denote by $D_{p,\sigma}$ the cone with vertex x_p and basis σ.*

4. *We make the important following assumption:*

$$d_{p\sigma}\mathbf{n}_{p,\sigma} = x_\sigma - x_p,\ \forall p \in \mathcal{M},\ \forall\sigma \in \mathcal{E}_p. \tag{11}$$

Remark 2.1 *The preceding definition applies to triangular meshes if $d = 2$, with all angles acute, and to meshes build with orthogonal parallelepipedic control volumes (rectangles if $d = 2$).*

We denote

$$\theta_{\mathcal{D}} = \min_{p\in\mathcal{M}} \min_{\sigma\in\mathcal{E}_p} \frac{d_{p\sigma}}{h_p}. \tag{12}$$

Definition 2.2 (Space-time discretisation) *Let Ω be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ (where $\mathbb{N}^\star$ denotes the set $\mathbb{N}\setminus\{0\}$) and let $T > 0$ be given. We say that $(\mathcal{D},\tau)$ is a space-time discretisation of $\Omega\times(0,T)$ if $\mathcal{D}$ is a space discretisation of Ω in the sense of Definition 2.1 and if there exists $N_T \in \mathbb{N}$ with $T = (N_T+1)\tau$.*

Let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times(0,T)$. We define the set $H_{\mathcal{D}} \subset \mathbb{R}^{\mathcal{M}}\times\mathbb{R}^{\mathcal{E}}$ such that $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\rm ext}$. We define the following functions on $H_{\mathcal{D}}$:

$$N_p(u)^2 = \frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma - u_p)^2,\ \forall p\in\mathcal{M},\ \forall u \in H_{\mathcal{D}}. \tag{13}$$

Let us recall that

$$\|u\|_{1,\mathcal{D}}^2 = \sum_{p\in\mathcal{M}} |p| N_p(u)^2 \tag{14}$$

defines a norm on $H_{\mathcal{D}}$ (see [20]). We then define the set $H_{\mathcal{D},\tau}$ of all $u = (u^{n+1})_{n=0,\ldots,N_T}$ such that $u^{n+1}\in H_{\mathcal{D}}$ for all $n = 0,\ldots,N_T$, and we set

$$\|u\|_{1,\mathcal{D},\tau}^2 = \sum_{n=0}^{N_T} \tau \|u^{n+1}\|_{1,\mathcal{D}}^2,\ \forall u\in H_{\mathcal{D},\tau}. \tag{15}$$

We now define two numerical schemes. The fully implicit scheme is defined by

$$u_p^0 = \frac{1}{|p|}\int_p u_0(x)\mathrm{d}x,\ \forall p\in\mathcal{M}, \tag{16}$$

$$r_p^{n+1} = \int_{n\tau}^{(n+1)\tau}\int_p r(x,t)\mathrm{d}x\mathrm{d}t,\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}, \tag{17}$$

and

$$\frac{|p|}{\tau\ g(N_p(u^{n+1}))}(u_p^{n+1}-u_p^n) - \frac{1}{f(N_p(u^{n+1}))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^{n+1}-u_p^{n+1}) = \frac{r_p^{n+1}}{\tau\ g(N_p(u^{n+1}))},\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}, \tag{18}$$

the following relation is given for the interior edges,

$$\frac{u_\sigma^{n+1}-u_p^{n+1}}{f(N_p(u^{n+1}))\ d_{p\sigma}} + \frac{u_\sigma^{n+1}-u_q^{n+1}}{f(N_q(u^{n+1}))\ d_{q\sigma}} = 0,\ \forall\sigma\in\mathcal{E}_{\rm int} \text{ with } \mathcal{M}_\sigma = \{p,q\},\ \forall n\in\mathbb{N}, \tag{19}$$

and the boundary condition is fulfilled thanks to

$$u_\sigma^{n+1} = 0,\ \forall\sigma\in\mathcal{E}_{\rm ext},\ \forall n\in\mathbb{N}. \tag{20}$$

The semi-implicit scheme is defined by (16),

$$u_\sigma^0 = \frac{1}{|\sigma|}\int_\sigma u_0(x)\mathrm{d}s(x),\ \forall\sigma\in\mathcal{E}, \tag{21}$$

(17), (20) and

$$\frac{|p|}{\tau\ g(N_p(u^n))}(u_p^{n+1}-u_p^n) - \frac{1}{f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^{n+1}-u_p^{n+1}) = \frac{r_p^{n+1}}{\tau\ g(N_p(u^n))},\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}, \tag{22}$$

where the following relation is given for the interior edges

$$\frac{u_\sigma^{n+1}-u_p^{n+1}}{f(N_p(u^n))\ d_{p\sigma}}+\frac{u_\sigma^{n+1}-u_q^{n+1}}{f(N_q(u^n))\ d_{q\sigma}}=0,\ \forall\sigma\in\mathcal{E}_{\rm int}\text{ with }\mathcal{M}_\sigma=\{p,q\},\ \forall n\in\mathbb{N}. \tag{23}$$

In the following, for the sake of shortness and clarity, all properties concerning the fully implicit scheme will be only sketched in remarks, focusing on the semi-implicit scheme. Hence, now considering a family of values $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$, given by (16), (17), (20) and (21), (22), (23), we define the approximate solution $u_{\mathcal{D},\tau}$ in $\Omega\times\mathbb{R}_+$ by

$$u_{\mathcal{D},\tau}(x,0)=u_p^0,\ u_{\mathcal{D},\tau}(x,t)=u_p^{n+1},\ \text{for a.e. } x\in p,\ \forall t\in]n\tau,(n+1)\tau],\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}. \tag{24}$$

We then define $w_{\mathcal{D},\tau}$, $z_{\mathcal{D},\tau}$, $N_{\mathcal{D},\tau}$ and $\widetilde{N}_{\mathcal{D},\tau}$ by

$$\begin{array}{l}w_{\mathcal{D},\tau}(x,t)=-\dfrac{u_p^{n+1}-u_p^n}{\tau\ g(N_p(u^n))}+\dfrac{r_p^{n+1}}{|p|\ \ \tau\ g(N_p(u^n))},\quad z_{\mathcal{D},\tau}(x,t)=\dfrac{u_p^{n+1}-u_p^n}{\tau},\\ N_{\mathcal{D},\tau}(x,t)=N_p(u^{n+1}),\quad \widetilde{N}_{\mathcal{D},\tau}(x,t)=N_p(u^n),\\ \text{ for a.e. } x\in p,\quad\text{for a.e. } t\in]n\tau,(n+1)\tau[,\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}.\end{array} \tag{25}$$

Finally, we define $G_{\mathcal{D},\tau}$, $H_{\mathcal{D},\tau}$ and $\widetilde{H}_{\mathcal{D},\tau}$ by

$$\begin{array}{l}G_{\mathcal{D},\tau}(x,t)=d\dfrac{u_\sigma^{n+1}-u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma},\\ H_{\mathcal{D},\tau}(x,t)=d\dfrac{u_\sigma^{n+1}-u_p^{n+1}}{d_{p\sigma}f(N_p(u^{n+1}))}\mathbf{n}_{p\sigma},\quad \widetilde{H}_{\mathcal{D},\tau}(x,t)=d\dfrac{u_\sigma^{n+1}-u_p^{n+1}}{d_{p\sigma}f(N_p(u^n))}\mathbf{n}_{p\sigma},\\ \text{ for a.e. } x\in D_{p\sigma},\quad\text{for a.e. } t\in]n\tau,(n+1)\tau[,\ \forall p\in\mathcal{M},\ \forall\sigma\in\mathcal{E}_p,\ \forall n\in\mathbb{N},\end{array} \tag{26}$$

(recall that $D_{p\sigma}$ is the cone with vertex x_p and basis σ and $\mathbf{n}_{p\sigma}$ is the normal unit vector to σ outward to p). Note that $u_{\mathcal{D},\tau}$ is the solution of

$$-\frac{1}{f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_\sigma^{n+1}-u_p^{n+1})=|p|\ w_p^{n+1},\forall p\in\mathcal{M},\ \forall n\in\mathbb{N}, \tag{27}$$

$$\frac{u_\sigma^{n+1}-u_p^{n+1}}{f(N_p(u^n))\ d_{p\sigma}}+\frac{u_\sigma^{n+1}-u_q^{n+1}}{f(N_q(u^n))\ d_{q\sigma}}=0,\ \forall\sigma=p|q\in\mathcal{E}_{\rm int},\ \forall n\in\mathbb{N}, \tag{28}$$

and

$$u_\sigma^{n+1}=0,\ \forall\sigma\in\mathcal{E}_{\rm ext},\ \forall n\in\mathbb{N}. \tag{29}$$

The next section is devoted to the study of some estimates satisfies by the discrete solution. These estimates in particular allow for the proof of the existence and uniqueness of the discrete solution. These estimates also give rise to a brief review of a few properties in the case of Crank-Nicolson versions of these schemes, which are confirmed by the numerical tests shown in section 5.

3 Estimates

Let us now state the L^∞ stability of the scheme.

Lemma 3.1 (L^∞ stability of the scheme, existence and uniqueness of the discrete solution) *Under Hypotheses (H), let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times(0,T)$ in the sense of Definition 2.2. We denote by*

$$|u_0|_{\mathcal{D},\infty}=\max_{p\in\mathcal{M}}|u_p^0|, \tag{30}$$

and by

$$|r|_{\mathcal{D},\tau,\infty} = \max\left\{\frac{|r_p^{n+1}|}{\tau\ |p|}, p \in \mathcal{M}, n = 0, \ldots, N_T\right\} \tag{31}$$

(note that, if $u_0 \in L^\infty(\Omega)$ and $r \in L^\infty(\Omega\times\mathbb{R}_+)$, then $|u_0|_{\mathcal{D},\infty} \le \|u_0\|_{L^\infty(\Omega)}$ and $|r|_{\mathcal{D},\tau,\infty} \le \|r\|_{L^\infty(\Omega\times(0,T))}$). Let $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ be a solution of (16), (17), (20) and (21), (22), (23). Then it holds:

$$|u_p^n| \le |u_0|_{\mathcal{D},\infty} + |r|_{\mathcal{D},\tau,\infty}\ n\ \tau \le |u_0|_{\mathcal{D},\infty} + |r|_{\mathcal{D},\tau,\infty}\ T,\ \forall p \in \mathcal{M},\ \forall n = 0, \ldots, N_T.$$

As a straightforward consequence, there exists one and only one solution to the semi-implicit scheme (21), (17), (20), (22), (23).

Proof. Suppose that for fixed time step $(n+1)$ the maximum of all u_p^{n+1} is achieved at the finite volume p. Let us write (22) in the following way:

$$u_p^{n+1} + \frac{\tau\ g(N_p(u^n))}{|p|\ f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(u_p^{n+1} - u_\sigma^{n+1}) = u_p^n + \frac{r_p^{n+1}}{|p|}. \tag{32}$$

Since the value u_σ^{n+1} satisfies the equality

$$u_\sigma^{n+1} = \frac{u_p^{n+1} f(N_q(u^n))d_{q\sigma} + u_q^{n+1} f(N_p(u^n))d_{p\sigma}}{f(N_p(u^n))d_{p\sigma} + f(N_q(u^n))d_{q\sigma}}, \tag{33}$$

which is a convex linear combination of points u_p^{n+1}, u_q^{n+1}, we obtain

$$u_p^{n+1} - u_\sigma^{n+1} = \frac{f(N_p(u^n))d_{p\sigma}(u_p^{n+1} - u_q^{n+1})}{f(N_p(u^n))d_{p\sigma} + f(N_q(u^n))d_{q\sigma}},$$

which is non-negative. This leads to

$$u_p^{n+1} \le u_p^n + |r|_{\mathcal{D},\tau,\infty}\ \tau. \tag{34}$$

Then, we recursively get the estimate (34), similarly reasoning for the minimum values. □

Remark 3.1 *The above proof also applies for the fully implicit scheme, only replacing n by $n+1$ in the arguments of functions f and g, allowing for a proof of existence of at least one discrete solution, thanks to Brouwer's fixed point theorem. Note that the uniqueness of such a discrete solution is not proved.*

Lemma 3.2 **$L^2(\Omega\times(0,T))$ estimate on u_t and $L^\infty(0,T;H_\mathcal{D})$ estimate.** *Let Hypotheses (H) be fulfilled. Let $(\mathcal{D},\tau)$ be a space-time discretisation of $\Omega\times(0,T)$ in the sense of Definition 2.2 and let $\theta \in]0,\theta_\mathcal{D}[$, where $\theta_\mathcal{D}$ is defined by (12). Let $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ be the solution of (16), (17), (20) and (21), (22), (23). Then there exists $C_\theta > 0$, only depending on θ, such that it holds:*

$$\begin{aligned}&\frac{1}{2b}\sum_{n=0}^{m-1}\tau\sum_{p\in\mathcal{M}}|p|\left(\frac{u_p^{n+1}-u_p^n}{\tau}\right)^2 + \sum_{p\in\mathcal{M}}|p|\ F(N_p(u^m))\\ &+\frac{1}{2b}\sum_{n=0}^{m-1}\sum_{p\in\mathcal{M}}|p|\ (N_p(u^{n+1}) - N_p(u^n))^2 \le \frac{C_\theta\|u^0\|_{H^1(\Omega)}^2 + \|r\|_{L^2(\Omega\times(0,T))}^2}{2\ a},\ \forall m = 1, \ldots, N_T.\end{aligned} \tag{35}$$

Proof. We multiply the scheme by $u_p^{n+1} - u_p^n$ and sum over p. We obtain $T_1 + T_2 = T_3$, where

$$T_1 = \sum_{p\in\mathcal{M}}\frac{|p|}{\tau g(N_p(u^n))}(u_p^{n+1} - u_p^n)^2,$$

$$T_2 = \sum_{p\in\mathcal{M}} \frac{1}{f(N_p(u^n))} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma^{n+1} - u_p^{n+1})(u_\sigma^{n+1} - u_p^{n+1} - (u_\sigma^n - u_p^n)),$$

$$T_3 = \sum_{p\in\mathcal{M}} \frac{r_p^{n+1}}{\tau\ g(N_p(u^n))}(u_p^{n+1} - u_p^n)$$

where we have used the properties of the finite volume scheme. We first remark that, thanks to Young's inequality and to the Cauchy-Schwarz inequality

$$T_3 \le \sum_{p\in\mathcal{M}} \frac{(r_p^{n+1})^2}{2|p|\ \ \tau\ g(N_p(u^n))} + \frac{1}{2}T_1 \le \frac{1}{2a}\int_{n\tau}^{(n+1)\tau}\int_\Omega r(x,t)^2 \mathrm{d}x\mathrm{d}t + \frac{1}{2}T_1.$$

Let us turn to the study of T_2. Using Definition (8) of function F, we have:

$$F(N_p(u^{n+1})) - F(N_p(u^n)) = \int_{N_p(u^n)}^{N_p(u^{n+1})} \frac{z\mathrm{d}z}{f(z)}.$$

We remark that, thanks to Hypothesis (H5),

$$\forall c, d \in \mathbb{R}_+, \int_c^d \frac{z\mathrm{d}z}{f(z)} + \frac{(d-c)^2}{2f(c)} \le \frac{d}{f(c)}(d-c). \tag{36}$$

Indeed, we set, for $c, d \in \mathbb{R}_+$, $\Phi_c(d) = \frac{d}{f(c)}(d-c) - \frac{(d-c)^2}{2f(c)} - \int_c^d \frac{z\mathrm{d}z}{f(z)}$. We have $\Phi_c(c) = 0$, and $\Phi_c'(d) = \frac{d}{f(c)} - \frac{d}{f(d)}$, whose sign is that of $d-c$ since f is (non-strictly) increasing. Hence $\Phi_c(d) \ge 0$ and we get

$$F(N_p(u^{n+1})) - F(N_p(u^n)) + \frac{1}{2b}(N_p(u^{n+1}) - N_p(u^n))^2 \le \frac{N_p(u^{n+1})}{f(N_p(u^n))}(N_p(u^{n+1}) - N_p(u^n)).$$

Note that the Cauchy-Schwarz inequality implies

$$\sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}}(u_\sigma^{n+1} - u_p^{n+1})(u_\sigma^n - u_p^n) \le |p|\ \ N_p(u^n)N_p(u^{n+1}),$$

which in turn implies

$$\sum_{p\in\mathcal{M}} |p|\ (F(N_p(u^{n+1})) - F(N_p(u^n)) + \frac{1}{2b}\sum_{p\in\mathcal{M}} |p|\ (N_p(u^{n+1}) - N_p(u^n))^2 \le T_2.$$

Finally we obtain

$$\begin{aligned} &\frac{1}{2b}\tau\sum_{p\in\mathcal{M}} |p| \left(\frac{u_p^{n+1} - u_p^n}{\tau}\right)^2 + \sum_{p\in\mathcal{M}} |p|\ F(N_p(u^{n+1})) + \frac{1}{2b}\sum_{p\in\mathcal{M}} |p|\ (N_p(u^{n+1}) - N_p(u^n))^2 \\ &\le \sum_{p\in\mathcal{M}} |p|\ F(N_p(u^n)) + \frac{1}{2a}\int_{n\tau}^{(n+1)\tau}\int_\Omega r(x,t)^2\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{37}$$

and summing this inequality over $n = 0, \ldots, m-1$ for all $m = 1, \ldots, N_T$, we get that

$$\begin{aligned} &\frac{1}{2b}\sum_{n=0}^{m-1}\tau\sum_{p\in\mathcal{M}} |p| \left(\frac{u_p^{n+1} - u_p^n}{\tau}\right)^2 + \sum_{p\in\mathcal{M}} |p|\ F(N_p(u^m)) \\ &+ \frac{1}{2b}\sum_{n=0}^{m-1}\sum_{p\in\mathcal{M}} |p|\ (N_p(u^{n+1}) - N_p(u^n))^2 \\ &\le \sum_{p\in\mathcal{M}} |p|\ F(N_p(u^0)) + \frac{1}{2a}\int_0^{m\tau}\int_\Omega r(x,t)^2\mathrm{d}x\mathrm{d}t, \end{aligned}$$

where we define u^0_σ by (21). We then use the following inequality, proved in [18]: there exists $C_\theta > 0$, only depending on θ, such that

$$|p|\ N_p(u^0)^2 \le C_\theta \|u_0\|^2_{H^1(p)},\ \forall p \in \mathcal{M}. \tag{38}$$

We thus get (35). □

Remark 3.2 *A property, similar to that stated in Lemma 3.2, can be shown for the fully implicit scheme. One should remark that, in this case, there is no term in $(N_p(u^{n+1}) - N_p(u^n))^2$ in the discrete relation issued from the computations, which rely on the monotonicity of the function $x \mapsto x/f(x)$ and not on that of f. This estimate then also allows for proving the existence of at least one solution to the fully implicit scheme, using the topological degree argument.*

Remark 3.3 (Case of the non-regularised level set equation) *If we use the hypothesis $s/f(s) \le a$ (which holds for $f(s) = s$) instead of $a \le f(s)$, assumed in this paper, the above computations provide an $L^\infty(0, T; L^1(\Omega))$ estimate on the discrete gradient instead of an $L^\infty(0, T; L^2(\Omega))$ estimate. It would nevertheless be possible to get some of the results proved during the convergence study, but not all of them. This is not surprising, since for the level set equation, there is no weak/strong sense, and we should refer to the viscosity solution sense. Hence the convergence study for $f(s) = s$ remains open.*

Consequences on Crank-Nicolson -like versions of the schemes

In this paper, we could as well, for a given $\alpha \in [\frac{1}{2}, 1]$, replace (22) and (23) by

$$\begin{aligned}
&\frac{|p|}{\tau\ g(N_p(u^n))}(u^{n+1}_p - u^n_p) - \frac{1}{f(N_p(u^n))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_p) = \frac{r^{n+1}_p}{\tau\ g(N_p(u^n))},\\
&\widehat{u}^{n+1}_p = \alpha u^{n+1}_p + (1-\alpha)u^n_p,\\
&\forall p \in \mathcal{M},\ \forall n \in \mathbb{N},
\end{aligned} \tag{39}$$

and

$$\frac{\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_p}{f(N_p(u^n))\ d_{p\sigma}} + \frac{\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_q}{f(N_q(u^n))\ d_{q\sigma}} = 0,\ \forall \sigma \in \mathcal{E}_{\rm int} \text{ with } \mathcal{M}_\sigma = \{p, q\},\ \forall n \in \mathbb{N}. \tag{40}$$

We then define the so-called "α-scheme" version of the above semi-implicit scheme, which provides the Crank-Nicolson scheme for $\alpha = \frac{1}{2}$ and (22), (23) for $\alpha = 1$. The convergence properties proved in this paper for $\alpha = 1$ can be immediately generalised to the case $\alpha \in]\frac{1}{2}, 1]$ for the semi-implicit scheme, since the crucial property (36) is modified into

$$\forall c, d \in \mathbb{R}_+, \int_c^d \frac{z{\rm d}z}{f(z)} + (\alpha - \frac{1}{2})\frac{(d-c)^2}{f(c)} \le \frac{\alpha d + (1-\alpha)c}{f(c)}(d-c),$$

which holds under the same hypothesis f increasing.

On the contrary, if, for a given $\alpha \in [\frac{1}{2}, 1]$, we replace (18) and (19) by

$$\begin{aligned}
&\frac{|p|}{\tau\ g(N_p(\widehat{u}^{n+1}))}(u^{n+1}_p - u^n_p) - \frac{1}{f(N_p(\widehat{u}^{n+1}))}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}(\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_p) = \frac{r^{n+1}_p}{\tau\ g(N_p(\widehat{u}^{n+1}))},\\
&\widehat{u}^{n+1}_p = \alpha u^{n+1}_p + (1-\alpha)u^n_p,\\
&\forall p \in \mathcal{M},\ \forall n \in \mathbb{N},
\end{aligned} \tag{41}$$

and

$$\frac{\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_p}{f(N_p(\widehat{u}^{n+1}))\ d_{p\sigma}} + \frac{\widehat{u}^{n+1}_\sigma - \widehat{u}^{n+1}_q}{f(N_q(\widehat{u}^{n+1}))\ d_{q\sigma}} = 0,\ \forall \sigma \in \mathcal{E}_{\rm int} \text{ with } \mathcal{M}_\sigma = \{p, q\},\ \forall n \in \mathbb{N}, \tag{42}$$

we have to replace the fact that the function $s \mapsto s/f(s)$ is increasing by

$$\forall c, d \in \mathbb{R}_+, \int_c^d \frac{z{\rm d}z}{f(z)} \le \frac{\alpha d + (1-\alpha)c}{f(\alpha d + (1-\alpha)c)}(d-c),$$

which is not satisfied for all $\alpha \in [\frac{1}{2}, 1]$ by the example given in (6) (indeed, for $\alpha = \frac{1}{2}$, it implies that the function $s \mapsto s/f(s)$ is concave).

4 Convergence

Thanks to the estimates proved in the above section, we are now in position for proving the convergence of the scheme, using the monotonicity properties of the operators. We first present a few properties which are useful in the convergence study. In this paper, we use the notations "$\rightharpoonup$ weakly" for denoting weak convergence and $\to$ for strong convergence.

Lemma 4.1 *Let Ω be a bounded connected open subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ and let $T > 0$. Let $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be a sequence of space-time discretisations of Ω in the sense of Definition 2.2 such that $h_{\mathcal{D}_m}$ tends to 0 as $m \longrightarrow \infty$. Let $(u_m)_{m\in\mathbb{N}}$ be such that $u_m \in H_{\mathcal{D}_m,\tau_m}$, such that $\|u_m\|_{1,\mathcal{D}_m,\tau_m} \le C$ for all $m \in \mathbb{N}$ and such that there exists $\bar{u} \in L^2(\Omega \times (0,T))$ such that the sequence of functions $u_{\mathcal{D}_m,\tau_m}$ defined, for $u = u_m$, $\mathcal{D} = \mathcal{D}_m$ and $\tau = \tau_m$, by*

$$u_{\mathcal{D},\tau}(x,t) = u_p^{n+1}, \text{ for a.e. } x \in p, t \in]n\tau,(n+1)\tau],\ \forall p \in \mathcal{M},\ \forall n = 0,\ldots,N_T,$$

satisfies $u_{\mathcal{D}_m,\tau_m} \longrightarrow \bar{u}$ in $L^2(\Omega \times (0,T))$ as $m \longrightarrow \infty$.
Then $\bar{u} \in L^2(0,T;H_0^1(\Omega))$. Moreover, defining $G_m \in L^\infty(0,T;L^2(\Omega))$ by

$$G_m(x,t) = d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma}$$

for a.e. $x \in D_{p\sigma}$, and a.e. $t \in]n\tau,(n+1)\tau[$, then $G_m \rightharpoonup \nabla\bar{u}$ weakly in $L^2(\Omega \times (0,T))^d$ as $m \longrightarrow \infty$.

Proof. We first notice that

$$\|G_m\|^2_{L^2(\Omega\times(0,T))^d} = \sum_{n=0}^{N_T}\tau\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|\ d_{p\sigma}}{d}\left|d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma}\right|^2,$$

which provides

$$\|G_m\|^2_{L^2(\Omega\times(0,T))^d} = d\left(\|u_m\|_{1,\mathcal{D}_m,\tau_m}\right)^2.$$

Prolonging G_m (and u_m) by 0 outside $\Omega \times (0,T)$, we get that there exists $\bar{G} \in L^2(\mathbb{R}^d \times (0,T))^d$ such that $G_m \rightharpoonup \bar{G}$ (weakly, up to a subsequence) in $L^2(\mathbb{R}^d \times (0,T))^d$ as $m \longrightarrow \infty$. Moreover, $\bar{G}(x,t) = 0$ for a.e. $x \notin \Omega \times (0,T)$. Let $\psi \in C_c^1(\mathbb{R}^d \times (0,T))^d$. Let us define, for $\mathcal{D} = \mathcal{D}_m$ and $\tau = \tau_m$,

$$\psi_\sigma^{n+1} = \frac{1}{|\sigma|\tau}\int_{n\tau}^{(n+1)\tau}\int_\sigma \psi(x,t)\mathrm{d}s(x)\mathrm{d}t,\ \sigma \in \mathcal{E},\ n = 0,\ldots,N_T,$$

and ψ_m by

$$\psi_m(x,t) = \psi_\sigma^{n+1},$$

for a.e. $x \in D_{p,\sigma}$, all $p \in \mathcal{M}$, $\sigma \in \mathcal{E}_p$, a.e. $t \in]n\tau,(n+1)\tau[$ and all $n = 0,\ldots,N_T$. Thanks to the regularity properties of ψ, we get that $\psi_m \to \psi$ in $L^\infty(\mathbb{R}^d \times (0,T))^d$ as $m \longrightarrow \infty$. Moreover, we have

$$\int_0^T\int_{\mathbb{R}^d} G_m(x,t)\cdot\psi_m(x,t)\mathrm{d}x\mathrm{d}t = \sum_{n=0}^{N_T}\tau\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|\ d_{p\sigma}}{d}d\frac{u_\sigma^{n+1} - u_p^{n+1}}{d_{p\sigma}}\mathbf{n}_{p\sigma}\cdot\psi_\sigma^{n+1},$$

which gives, thanks to the fact that the terms u_σ^{n+1} are multiplied by 0 for all $\sigma \in \mathcal{E}_{\text{int}}$ and using $\int_{n\tau}^{(n+1)\tau}\int_p \mathrm{div}\ \psi(x,t)\mathrm{d}x\mathrm{d}t = \tau\sum_{\sigma\in\mathcal{E}_p}|\sigma|\psi_\sigma^{n+1}\cdot\mathbf{n}_{p\sigma}$,

$$\int_0^T\int_{\mathbb{R}^d} G_m(x,t)\cdot\psi_m(x,t)\mathrm{d}x\mathrm{d}t = -\int_0^T\int_{\mathbb{R}^d} u_m(x,t)\mathrm{div}\ \psi(x,t)\mathrm{d}x\mathrm{d}t.$$

Passing to the limit in the above expression, we get, using weak/strong convergence for the left hand side,

$$\int_0^T \int_{\mathbb{R}^d} \bar{G}(x,t) \cdot \psi(x,t) \mathrm{d}x\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^d} \bar{u}(x,t) \mathrm{div}\ \psi(x,t) \mathrm{d}x\mathrm{d}t.$$

This proves that $\nabla \bar{u} \in L^2(\mathbb{R}^d \times (0,T))^d$ and that $\nabla \bar{u} = \bar{G}$ for a.e. $(x,t) \in \mathbb{R}^d \times (0,T)$. Since $\bar{G}(x,t) = 0$ for a.e. $x \notin \Omega \times (0,T)$, we get that $\bar{u} \in L^2(0,T; H_0^1(\Omega))$. Since $\nabla \bar{u} \in L^2(\mathbb{R}^d \times (0,T))^d$ is uniquely defined, we get that the whole sequence $G_m \rightharpoonup \nabla \bar{u}$ weakly in $L^2(\mathbb{R}^d \times (0,T))^d$, which concludes the proof. □

Lemma 4.2 Strong convergence of the approximate gradient norm of regular function interpolation.
Let Ω be a bounded connected open subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$ and let $T > 0$. Let $(\mathcal{D}, \tau)$ be a space-time discretisation of $\Omega \times (0,T)$ in the sense of Definition 2.2. For any $\varphi \in C_c^\infty(\Omega \times (0,T))$, we define the discrete interpolation of φ, denoted $v \in H_{\mathcal{D},\tau}$, by $v_p^n = \varphi(x_p, n\tau)$ and $v_\sigma^n = \varphi(x_\sigma, n\tau)$, and we define $\mathcal{N}_{\mathcal{D},\tau}$ by

$$\mathcal{N}_{\mathcal{D},\tau}(x,t) = N_p(v^{n+1}),\ \text{for a.e. } x \in p,\ t \in]n\tau, (n+1)\tau],\ \forall p \in \mathcal{M},\ \forall n = 0, \ldots, N_T, \tag{43}$$

Then $\mathcal{N}_{\mathcal{D},\tau} \to |\nabla \varphi|$ in $L^\infty(\Omega \times (0,T))$ as $h_{\mathcal{D}}$ and τ tend to 0.

Proof. We have, for any vector $\mathbf{w} \in \mathbb{R}^d$,

$$|p|\ \mathbf{w} = \sum_{\sigma \in \mathcal{E}_p} |\sigma| \mathbf{w} \cdot (x_\sigma - x_p) \mathbf{n}_{p,\sigma}. \tag{44}$$

Hence we get that

$$|p|\ |\mathbf{w}|^2 = \sum_{\sigma \in \mathcal{E}_p} |\sigma| \mathbf{w} \cdot (x_\sigma - x_p) \mathbf{n}_{p,\sigma} \cdot \mathbf{w} = \sum_{\sigma \in \mathcal{E}_p} |\sigma| d_{p\sigma} (\mathbf{n}_{p,\sigma} \cdot \mathbf{w})^2,$$

thanks to condition (11). This provides that

$$|p|\ |\nabla \varphi(x_p, t)|^2 = \sum_{\sigma \in \mathcal{E}_p} |\sigma| d_{p\sigma} (\mathbf{n}_{p,\sigma} \cdot \nabla \varphi(x_p, t))^2.$$

Writing that

$$\mathbf{n}_{p,\sigma} \cdot \nabla \varphi(x_p, t) = \frac{\varphi(x_\sigma, t) - \varphi(x_p, t)}{d_{p\sigma}} + C_p(t) h_{\mathcal{D}}, \tag{45}$$

with $C_p(t)$ bounded independently of the discretisation, we conclude the proof of the lemma. □

Lemma 4.3 (Strong approximate of the gradient of φ) *For all $\varphi \in C_c^\infty(\Omega \times (0,T))$, we denote by $v_p^n = \varphi(x_p, n\tau)$ and $v_\sigma^n = \varphi(x_\sigma, n\tau)$. We introduce the approximation*

$$\nabla_{p\sigma}^{n+1} \varphi = \frac{v_\sigma^{n+1} - v_p^{n+1}}{d_{p\sigma}} \mathbf{n}_{p\sigma} + \nabla \varphi(x_p, (n+1)\tau) - (\nabla \varphi(x_p, (n+1)\tau) \cdot \mathbf{n}_{p\sigma}) \mathbf{n}_{p\sigma}, \tag{46}$$

and $\nabla_{\mathcal{D},\tau} \varphi(x,t) = \nabla_{p\sigma}^{n+1} \varphi$ for $x \in D_{p\sigma}$, $t \in [n\tau, (n+1)\tau]$.

Then $\nabla_{\mathcal{D},\tau} \varphi \to \nabla \varphi$ in $L^\infty(\Omega \times (0,T))$ as $h_{\mathcal{D}}$ and τ tend to 0.

Proof. The proof relies on (45). □

Let us denote by (HC) the following hypotheses:

- Hypotheses (H) are fulfilled.
- The sequence $(\mathcal{D}_m, \tau_m)_{m \in \mathbb{N}}$ denotes a sequence of space-time discretisations of $\Omega \times (0,T)$ in the sense of Definition 2.2 such that $h_{\mathcal{D}_m}$ and $\tau_m > 0$ tends to 0 as $m \longrightarrow \infty$.

- There exists some $\theta > 0$ with $\theta < \theta_{\mathcal{D}_m}$ for all $m \in \mathbb{N}$, where $\theta_{\mathcal{D}}$ is defined by (12).
- For all $m \in \mathbb{N}$, the family $(u_p^n)_{p\in\mathcal{M},n\in\mathbb{N}}$ is such that (16), (17), (20) and (21), (22), (23) hold and the function $u_{\mathcal{D}_m,\tau_m}$ is defined by (24).

We can now state the following Lemma, using the compactness properties issued from the above estimates.

Lemma 4.4 (Convergence properties) *Let Hypotheses* (HC) *be fulfilled.*
Then there exists a subsequence of $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, there exists a function $\bar{u} \in L^\infty(0, T; H_0^1(\Omega)) \cap C^0(0, T; L^2(\Omega))$, such that $\bar{u}_t \in L^2(\Omega \times (0, T))$, $u(.,0) = u_0$, and $u_{\mathcal{D}_m,\tau_m}$ tends to $\bar{u} \in L^\infty(0, T; H_0^1(\Omega))$ in $L^\infty(0, T; L^2(\Omega))$, and there exists functions $\bar{H} \in L^2(\Omega \times (0, T))^d$, $\bar{w} \in L^2(\Omega \times (0, T))$ such that $H_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ and $\widetilde{H}_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ weakly in $L^2(\Omega \times (0, T))^d$ (see definition (26)), and such that $w_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{w}$ and $z_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{u}_t$ weakly in $L^2(\Omega \times (0, T))$ as $m \to \infty$. Moreover, $G_{\mathcal{D}_m,\tau_m} \rightharpoonup \nabla\bar{u}$ weakly in $L^2(\Omega \times (0, T))^d$ (see definition (26)), $N_{\mathcal{D}_m,\tau_m} - \widetilde{N}_{\mathcal{D}_m,\tau_m} \to 0$ in $L^2(\Omega \times (0, T))$ (see definition (25)) and the following relation holds:

$$\lim_{m\to\infty} \int_0^T \int_\Omega \frac{N_{\mathcal{D}_m,\tau_m}(x,t)^2}{f(N_{\mathcal{D}_m,\tau_m}(x,t))} \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{H}(x,t) \cdot \nabla\bar{u}(x,t)\mathrm{d}x\mathrm{d}t. \tag{47}$$

Proof. From the definition of F and Hypotheses (H) (which imply $F(s) \geq s^2/2b$), $u_{\mathcal{D}_m,\tau_m}(\cdot, t)$ is uniformly bounded in $H_{\mathcal{D}}$ for all $t \in [0, T]$. Hence we can apply Theorem 6.1, which is a generalisation of Ascoli's theorem and shows that the convergence property $u_{\mathcal{D}_m,\tau_m}(\cdot, t) \to \bar{u} \in C^0(0, T; L^2(\Omega))$ holds in $L^\infty(0, T; L^2(\Omega))$. Thanks to (16), we have $\bar{u}(\cdot, 0) = u_0$. We also get, thanks to Lemma 4.1, that $\bar{u} \in L^\infty(0, T; H_0^1(\Omega))$ and that $G_{\mathcal{D},\tau} \rightharpoonup \nabla\bar{u}$ weakly in $L^2(\Omega \times (0, T))^d$.
From Lemma 3.2 we get that $w_{\mathcal{D},\tau}$ remains bounded in $L^2(\Omega \times (0, T))$ for all $m \in \mathbb{N}$. Therefore there exists a function $\bar{w} \in L^2((\Omega \times (0, T))$ such that, up to a subsequence of the preceding one, $w_m \rightharpoonup \bar{w}$ weakly in $L^2(\Omega \times (0, T))$. Similarly, we have $z_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{u}_t$ weakly in $L^2(\Omega \times (0, T))$, which shows that $\bar{u}_t \in L^2(\Omega \times (0, T))$. Similarly, $\widetilde{H}_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ weakly in $L^2(\Omega \times (0, T))^d$, up to a subsequence of the preceding one. Note that in the proof below, we drop some indices m for the simplicity of notation.

Let us first focus on the difference between $N_{\mathcal{D}_m,\tau_m}$ and $\widetilde{N}_{\mathcal{D}_m,\tau_m}$ on one hand, and that between $\widetilde{H}_{\mathcal{D}_m,\tau_m}$ and $H_{\mathcal{D}_m,\tau_m}$ on the other hand. We have, for $x \in p$ and $t \in]n\tau, (n + 1)\tau[$,

$$N_{\mathcal{D},\tau}(x, t) - \widetilde{N}_{\mathcal{D},\tau}(x, t) = N_p(u^{n+1}) - N_p(u^n).$$

Using (35), we get the existence of $C > 0$ independent of m such that

$$\|N_{\mathcal{D}_m,\tau_m} - \widetilde{N}_{\mathcal{D}_m,\tau_m}\|^2_{L^2(\Omega\times(0,T))} \leq C\tau_m,$$

which provides

$$\lim_{m\to\infty} \|N_{\mathcal{D}_m,\tau_m} - \widetilde{N}_{\mathcal{D}_m,\tau_m}\|_{L^2(\Omega\times(0,T))} = 0. \tag{48}$$

Using the Cauchy-Schwarz inequality, we have

$$\int_0^T \int_\Omega |\widetilde{H}_{\mathcal{D}_m,\tau_m}(x,t) - H_{\mathcal{D}_m,\tau_m}(x,t)|\mathrm{d}x\mathrm{d}t \leq \|G_{\mathcal{D}}\|_{L^2(\Omega\times(0,T))^d} \left\| \frac{1}{f(\widetilde{N}_{\mathcal{D}_m,\tau_m})} - \frac{1}{f(N_{\mathcal{D}_m,\tau_m})} \right\|_{L^2(\Omega\times(0,T))^d},$$

which proves that $\widetilde{H}_{\mathcal{D}_m,\tau_m} - H_{\mathcal{D}_m,\tau_m} \to 0$ in $L^1(\Omega \times (0, T))^d$ thanks to (48). Note that this shows that $H_{\mathcal{D}_m,\tau_m} \rightharpoonup \bar{H}$ weakly in $L^2(\Omega \times (0, T))$. One of the difficulties is now to identify $\bar{H}$ with $\nabla u/f(|\nabla u|)$. This will be done in further lemmas, thanks to the property (47) stated in the present lemma, that we have now to prove.

Let $\varphi \in C_c^\infty(\Omega \times (0, T))$ be given. We denote by $v_p^n = \varphi(x_p, n\tau)$ and $v_\sigma^n = \varphi(x_\sigma, n\tau)$. Multiplying (27) by τv_p^{n+1}, summing on n and p, we get $T_{1m} = T_{2m}$ with

$$T_{1m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^n))} (v_\sigma^{n+1} - v_p^{n+1}),$$

and

$$T_{2m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} |p| \; w_p^{n+1} v_p^{n+1}.$$

Using the approximation $\nabla_{\mathcal{D},\tau}\varphi$ of $\nabla\varphi$ provided in Lemma 4.3, we can write that

$$T_{1m} = \int_0^T \int_\Omega \widetilde{H}_{\mathcal{D},\tau} \cdot \nabla_{\mathcal{D},\tau}\varphi \mathrm{d}x\mathrm{d}t.$$

Hence, by weak/strong convergence,

$$\lim_{m\to\infty} T_{1m} = \int_0^T \int_\Omega \bar{H} \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t.$$

We have on the other hand

$$\lim_{m\to\infty} T_{2m} = \int_0^T \int_\Omega \bar{w}\varphi \mathrm{d}x\mathrm{d}t.$$

Hence

$$\int_0^T \int_\Omega \bar{H} \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{w}\varphi \mathrm{d}x\mathrm{d}t.$$

Since the above equality holds for all $\varphi \in C_c^\infty(\Omega\times(0,T))$, it also holds by density for all $v \in L^2(0,T;H_0^1(\Omega))$. Hence we get

$$\int_0^T \int_\Omega \bar{H} \cdot \nabla\bar{u} \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{w}\bar{u} \mathrm{d}x\mathrm{d}t. \tag{49}$$

We now multiply (27) by τu_p^{n+1}, sum on n and p. We get $\widetilde{T}_{3m} = T_{4m}$ with $\widetilde{T}_{3m}$ defined by

$$\widetilde{T}_{3m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \frac{(u_\sigma^{n+1} - u_p^{n+1})^2}{f(N_p(u^n))} = \int_0^T \int_\Omega \frac{N_{\mathcal{D},\tau}(x,t)^2}{f(\widetilde{N}_{\mathcal{D},\tau}(x,t))} \mathrm{d}x\mathrm{d}t, \tag{50}$$

and

$$T_{4m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} |p| \; w_p^{n+1} u_p^{n+1}.$$

We have, by weak/strong convergence,

$$\lim_{m\to\infty} T_{4m} = \int_0^T \int_\Omega \bar{w}\bar{u} \mathrm{d}x\mathrm{d}t,$$

which leads, using (49), to

$$\lim_{m\to\infty} T_{3m} = \int_0^T \int_\Omega \bar{w}\bar{u} \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{H} \cdot \nabla\bar{u} \mathrm{d}x\mathrm{d}t,$$

We now define

$$T_{3m} = \int_0^T \int_\Omega \frac{N_{\mathcal{D}_m,\tau_m}(x,t)^2}{f(N_{\mathcal{D}_m,\tau_m}(x,t))} \mathrm{d}x\mathrm{d}t = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \frac{(u_\sigma^{n+1} - u_p^{n+1})^2}{f(N_p(u^{n+1}))},$$

again dropping some indices m for the simplicity of notation. Let us now prove that $\widetilde{T}_{3m}$ and T_{3m} have the same limit. Writing

$$\widetilde{T}_{3m} - T_{3m} = -\tau \sum_{p\in\mathcal{M}} |p| \; \frac{N_p(u^{N_T+1})^2}{f(N_p(u^{N_T+1}))} + \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} |p| \; \frac{N_p(u^{n+1})^2 - N_p(u^n)^2}{f(N_p(u^n))} + \tau \sum_{p\in\mathcal{M}} |p| \; \frac{N_p(u^0)^2}{f(N_p(u^0))},$$

we get, using (35) for the study of the first term in the right hand side of the above equation, (48) for the study of the second term and (38) for the third one, that

$$\lim_{m\to\infty}(\widetilde{T}_{3m} - T_{3m}) = 0,$$

Hence we also get that

$$\lim_{m\to\infty} T_{3m} = \int_0^T \int_\Omega \bar{w}\bar{u}\mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \bar{H}\cdot\nabla\bar{u}\mathrm{d}x\mathrm{d}t,$$

which completes the proof of (47). □
The problem is now to show the strong convergence in $L^2(\Omega\times(0,T))$ of $N_{\mathcal{D}}(u_{\mathcal{D},\tau})$ to $|\nabla\bar{u}|$. This will result from property (47), from Minty trick and from Leray-Lions trick. Let us start with the following property:

Lemma 4.5 *For all $u, v \in H_{\mathcal{D}}$,*

$$\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))})\right)(u_\sigma - u_p - v_\sigma + v_p) \geq 0.$$

Proof. We have that

$$\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))})\right)(u_\sigma - u_p - v_\sigma + v_p)$$
$$= \sum_{\sigma\in\mathcal{E}_p}\left(\frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)^2}{f(N_p(u))} + \frac{|\sigma|}{d_{p\sigma}}\frac{(v_\sigma - v_p)^2}{f(N_p(v))} - \frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)(v_\sigma - v_p)}{f(N_p(u))} - \frac{|\sigma|}{d_{p\sigma}}\frac{(u_\sigma - u_p)(v_\sigma - v_p)}{f(N_p(v))}\right).$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))})\right)(u_\sigma - u_p - v_\sigma + v_p) \geq \frac{|p|\ N_p(u)^2}{f(N_p(u))} + \frac{|p|\ N_p(v)^2}{f(N_p(v))}$$
$$-\frac{|p|\ N_p(u)N_p(v)}{f(N_p(u))} - \frac{|p|\ N_p(u)N_p(v)}{f(N_p(v))},$$

which gives

$$\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma - u_p}{f(N_p(u))} - \frac{v_\sigma - v_p}{f(N_p(v))})\right)(u_\sigma - u_p - v_\sigma + v_p) \geq$$
$$|p|\ \left(\frac{N_p(u)}{f(N_p(u))} - \frac{N_p(v)}{f(N_p(v))}\right)(N_p(u) - N_p(v)).$$

This last expression is non negative thanks to the hypotheses (H). □
We now continue with the use of Minty trick.

Lemma 4.6 *Let Hypotheses (HC) be fulfilled. We assume that the sequence $(\mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ denotes an extracted sub-sequence, the existence of which is provided by Lemma 4.4. Let $\varphi \in C_c^\infty(\Omega\times(0,T))$ be given. We denote by $v_p^n = \varphi(x_p, n\tau)$ and $v_\sigma^n = \varphi(x_\sigma, n\tau)$ and by*

$$T_m = \sum_{n=0}^{N_T}\tau\sum_{p\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_p}\frac{|\sigma|}{d_{p\sigma}}\left(\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^{n+1}))} - \frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(v^{n+1}))})\right)(u_\sigma^{n+1} - u_p^{n+1} - v_\sigma^{n+1} + v_p^{n+1}). \quad (51)$$

Then the following holds

$$\lim_{m\to\infty} T_m = \int_0^T \int_\Omega (\bar{H} - \frac{\nabla\varphi}{f(|\nabla\varphi|)})(\nabla\bar{u} - \nabla\varphi)\mathrm{d}x\mathrm{d}t, \quad (52)$$

and

$$\int_0^T \int_\Omega \bar{H} \cdot \nabla v \mathrm{d}x\mathrm{d}t = \int_0^T \int_\Omega \frac{\nabla \bar{u}}{f(|\nabla \bar{u}|)} \cdot \nabla v \mathrm{d}x\mathrm{d}t,\ \forall v \in L^2(0,T;H_0^1(\Omega)). \tag{53}$$

PROOF. We remark that $T_m = T_{3m} - T_{5m} - T_{6m} + T_{7m}$, with T_{3m} defined by (50) and

$$T_{5m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \left(\frac{u_\sigma^{n+1} - u_p^{n+1}}{f(N_p(u^{n+1}))}) \right) (v_\sigma^{n+1} - v_p^{n+1}),$$

$$T_{6m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \left(\frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(v^{n+1}))}) \right) (u_\sigma^{n+1} - u_p^{n+1}),$$

$$T_{7m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} \left(\frac{v_\sigma^{n+1} - v_p^{n+1}}{f(N_p(v^{n+1}))}) \right) (v_\sigma^{n+1} - v_p^{n+1}).$$

We have that $N_{\mathcal{D}}(v_{\mathcal{D},\tau}) \to |\nabla\varphi|$ in $L^\infty(\Omega\times(0,T))$, which leads to

$$\lim_{m\to\infty} T_{5m} = \int_0^T \int_\Omega \bar{H} \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t$$

$$\lim_{m\to\infty} T_{6m} = \int_0^T \int_\Omega \frac{\nabla\varphi}{f(|\nabla\varphi|)} \cdot \nabla\bar{u} \mathrm{d}x\mathrm{d}t,$$

$$\lim_{m\to\infty} T_{7m} = \int_0^T \int_\Omega \frac{\nabla\varphi}{f(|\nabla\varphi|)} \cdot \nabla\varphi \mathrm{d}x\mathrm{d}t.$$

Hence, gathering the above results, we get (52). We have that $T_m \geq 0$ thanks to Lemma 4.5. Hence (52) provides

$$\int_0^T \int_\Omega (\bar{H} - \frac{\nabla\varphi}{f(|\nabla\varphi|)})(\nabla\bar{u} - \nabla\varphi) \mathrm{d}x\mathrm{d}t \geq 0,$$

and therefore we get by density

$$\forall v \in L^2(0,T;H_0^1(\Omega)),\ \int_0^T \int_\Omega (\bar{H} - \frac{\nabla v}{f(|\nabla v|)})(\nabla\bar{u} - \nabla v) \mathrm{d}x\mathrm{d}t \geq 0. \tag{54}$$

We can now apply Minty trick, taking in (54) $v = \bar{u} - \lambda\psi$, with $\lambda > 0$ and $\psi \in C_c^\infty(\Omega\times(0,T))$. We get, dividing by λ,

$$\int_0^T \int_\Omega (\bar{H} - \frac{\nabla(\bar{u} - \lambda\psi)}{f(|\nabla(\bar{u} - \lambda\psi)|)}) \nabla\psi \mathrm{d}x\mathrm{d}t \geq 0.$$

We can let $\lambda \longrightarrow 0$ in the above inequality, using Lebesgue's dominated convergence theorem. We then get

$$\int_0^T \int_\Omega (\bar{H} - \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)}) \nabla\psi \mathrm{d}x\mathrm{d}t \geq 0.$$

Since this also holds for $-\psi$, we get

$$\int_0^T \int_\Omega (\bar{H} - \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)}) \nabla\psi \mathrm{d}x\mathrm{d}t = 0.$$

The above equality can again be extended to all $v \in L^2(0,T;H_0^1(\Omega))$, which achieves the proof of (53). □

We have now the following lemma, which uses Leray-Lions trick.

Lemma 4.7 *Under the same hypotheses as Lemma 4.6, $N_{\mathcal{D}_m,\tau_m} \to |\nabla \bar{u}|$ in $L^2(\Omega \times (0,T))$ as m tends to ∞.*

PROOF. For a given $m \in \mathbb{N}$, we drop the indices m in $\mathcal{D}$, τ in order to lighten the notation. Let $\varphi \in C_c^\infty(\Omega \times (0,T))$, we denote by $v_p^n = \varphi(x_p, n\tau)$ and $v_\sigma^n = \varphi(x_\sigma, n\tau)$. Let us denote

$$T_{8m} = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} |p| \left(\frac{N_p(u^{n+1})}{f(N_p(u^{n+1}))} - \frac{N_p(v^{n+1})}{f(N_p(v^{n+1}))} \right) \left(N_p(u^{n+1}) - N_p(v^{n+1}) \right).$$

We have

$$T_{8m} = \int_0^T \int_\Omega \left(\frac{N_\mathcal{D}(u_{\mathcal{D},\tau})}{f(N_\mathcal{D}(u_{\mathcal{D},\tau}))} - \frac{N_\mathcal{D}(v_{\mathcal{D},\tau})}{f(N_\mathcal{D}(v_{\mathcal{D},\tau}))} \right) (N_\mathcal{D}(u_{\mathcal{D},\tau}) - N_\mathcal{D}(v_{\mathcal{D},\tau}))\,\mathrm{d}x\mathrm{d}t.$$

From the proof of lemma 4.5, we have

$$0 \le T_{8m} \le T_m.$$

We write $T_{8m} = T_{9m} - T_{10m} - T_{11m}$, with

$$T_{9m} = \int_0^T \int_\Omega \left(\frac{N_\mathcal{D}(u_{\mathcal{D},\tau})}{f(N_\mathcal{D}(u_{\mathcal{D},\tau}))} - \frac{|\nabla \bar{u}|}{f(|\nabla \bar{u}|)} \right) (N_\mathcal{D}(u_{\mathcal{D},\tau}) - |\nabla \bar{u}|)\,\mathrm{d}x\mathrm{d}t,$$

$$\begin{aligned} T_{10m} = & -\int_0^T \int_\Omega \left(\frac{|\nabla\varphi|^2}{f(|\nabla\varphi|)} - \frac{|\nabla \bar{u}|^2}{f(|\nabla \bar{u}|)} \right) \mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega \frac{N_\mathcal{D}(u_{\mathcal{D},\tau})}{f(N_\mathcal{D}(u_{\mathcal{D},\tau}))} (|\nabla\varphi| - |\nabla \bar{u}|)\,\mathrm{d}x\mathrm{d}t \\ & + \int_0^T \int_\Omega N_\mathcal{D}(u_{\mathcal{D},\tau}) \left(\frac{|\nabla\varphi|}{f(|\nabla\varphi|)} - \frac{|\nabla \bar{u}|}{f(|\nabla \bar{u}|)} \right) \mathrm{d}x\mathrm{d}t, \end{aligned}$$

$$\begin{aligned} T_{11m} = & \int_0^T \int_\Omega \left(\frac{|\nabla\varphi|^2}{f(|\nabla\varphi|)} - \frac{N_\mathcal{D}(v_{\mathcal{D},\tau})^2}{f(N_\mathcal{D}(v_{\mathcal{D},\tau}))} \right) \mathrm{d}x\mathrm{d}t \\ & - \int_0^T \int_\Omega \frac{N_\mathcal{D}(u_{\mathcal{D},\tau})}{f(N_\mathcal{D}(u_{\mathcal{D},\tau}))} (|\nabla\varphi| - N_\mathcal{D}(v_{\mathcal{D},\tau}))\,\mathrm{d}x\mathrm{d}t \\ & - \int_0^T \int_\Omega N_\mathcal{D}(u_{\mathcal{D},\tau}) \left(\frac{|\nabla\varphi|}{f(|\nabla\varphi|)} - \frac{N_\mathcal{D}(v_{\mathcal{D},\tau})}{f(N_\mathcal{D}(v_{\mathcal{D},\tau}))} \right) \mathrm{d}x\mathrm{d}t, \end{aligned}$$

We then deduce, using Cauchy-Schwarz inequalities and estimates on the scheme,

$$0 \le T_{9m} \le T_m + C\| |\nabla\varphi| - |\nabla \bar{u}| \|_{L^2(\Omega\times(0,T))} + C\| |\nabla\varphi| - N_\mathcal{D}(v_{\mathcal{D},\tau}) \|_{L^2(\Omega\times(0,T))}.$$

Hence, passing to the limit $m \to \infty$, since $N_\mathcal{D}(v_{\mathcal{D},\tau}) \to |\nabla\varphi|$ in $L^\infty(\Omega \times (0,T))$, we get

$$0 \le \limsup_{m\to\infty} T_{9m} \le \int_0^T \int_\Omega (\frac{\nabla \bar{u}}{f(|\nabla \bar{u}|)} - \frac{\nabla\varphi}{f(|\nabla\varphi|)})(\nabla \bar{u} - \nabla\varphi)\mathrm{d}x\mathrm{d}t + C\| |\nabla\varphi| - |\nabla \bar{u}| \|_{L^2(\Omega\times(0,T))}.$$

Since this holds for any $\varphi \in C_c^\infty(\Omega \times (0,T))$, we can let $\varphi \to \bar{u}$ in $L^2(0,T;H_0^1(\Omega))$. Then the right hand side of the above inequality tends to 0, and we get

$$\lim_{m\to\infty} \int_0^T \int_\Omega \left(\frac{N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})}{f(N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m}))} - \frac{|\nabla \bar{u}|}{f(|\nabla \bar{u}|)} \right) (N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m}) - |\nabla \bar{u}|)\,\mathrm{d}x\mathrm{d}t = 0.$$

Simple conclusion of the proof in the case where $\left(\frac{c}{f(c)} - \frac{d}{f(d)}\right)(c-d) \ge \alpha(c-d)^2$ (this holds if the function $x \mapsto x/f(x)$ has its derivative greater or equal to $\alpha > 0$; this is satisfied by the example provided in (6)). We immediately get the conclusion of the lemma.

More complex conclusion of the proof in the general case. Let us now apply lemma (6.1). We get that $N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})$ converges a.e. to $|\nabla\bar{u}|$. We then remark that, thanks to (47) and (53), we have

$$\lim_{m\to\infty}\int_0^T\int_\Omega \frac{N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})^2}{f(N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m}))}\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \frac{|\nabla\bar{u}|^2}{f(|\nabla\bar{u}|)}\mathrm{d}x\mathrm{d}t.$$

We now apply lemma 6.2, which shows that $\frac{N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})^2}{f(N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m}))} \to \frac{|\nabla\bar{u}|^2}{f(|\nabla\bar{u}|)}$ in $L^1(\Omega\times(0,T))$. This L^1-convergence gives the equi-integrability of the family of functions $\frac{N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})^2}{f(N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m}))}$, which, in turn, gives that the family of functions $N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})^2$ is equi-integrable. Finally, we obtain (using Vitali's theorem) the convergence of $N_{\mathcal{D}_m}(u_{\mathcal{D}_m,\tau_m})$ to $|\nabla\bar{u}|$ in $L^2(\Omega)$, as $m\to\infty$. This completes the proof. □

We can now conclude the convergence of the scheme. We introduce the following strongly convergent approximation for the gradient of the unknown:

$$\begin{aligned}&\widehat{G}_{\mathcal{D},\tau}(x,t) = \frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}|\sigma|(u_\sigma^{n+1}-u_p^{n+1})\mathbf{n}_{p\sigma},\\ &\text{for a.e. } x\in p, \quad \text{for a.e. } t\in]n\tau,(n+1)\tau[,\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}.\end{aligned} \tag{55}$$

(recall that $G_{\mathcal{D},\tau}(x,t)$ is only weakly convergent).

Theorem 4.1 *Let Hypotheses* (HC) *be fulfilled. We assume that the sequence* $(\mathcal{D}_m,\tau_m)_{m\in\mathbb{N}}$ *denotes an extracted sub-sequence, the existence of which is provided by Lemma 4.4.*
Then the function $\bar{u}\in L^\infty(0,T;H_0^1(\Omega))$*, such that* $u_{\mathcal{D}_m,\tau_m}\to\bar{u}$ *in* $L^\infty(0,T;L^2(\Omega))$*, is a weak solution of (1)-(2)-(3) in the sense of Definition 1.1. Moreover,* $\widehat{G}_{\mathcal{D}_m,\tau_m}\to\nabla\bar{u}$ *in* $L^2(\Omega\times(0,T))^d$ *(see (55)) and* $N_{\mathcal{D}_m,\tau_m}\to|\nabla\bar{u}|$*,* $\widetilde{N}_{\mathcal{D}_m,\tau_m}\to|\nabla\bar{u}|$ *in* $L^2(\Omega\times(0,T))$.

Proof.

Using Lemma (4.6), we get that

$$\int_0^T\int_\Omega \frac{\nabla\bar{u}}{f(|\nabla\bar{u}|)}\cdot\nabla v\mathrm{d}x\mathrm{d}t = \int_0^T\int_\Omega \bar{w}v\mathrm{d}x\mathrm{d}t,\ \forall v\in L^2(0,T;H_0^1(\Omega)), \tag{56}$$

Thanks to Lemma 4.7, we get that $\bar{w}=(r-u_t)/g(|\nabla u|)$, and the proof that $\bar{u}$ is a weak solution of (1)-(2)-(3) in the sense of Definition 1.1 is complete.

Let us turn to the proof of the strong convergence of $\widehat{G}_{\mathcal{D}_m,\tau_m}$. Let us first remark that, thanks to (44), the expression of $\widehat{G}_{\mathcal{D},\tau}$, applied to the interpolation of some regular function φ, is strongly consistent with $\nabla\varphi$.
We can then follow the reasoning of [20] in order to prove the strong convergence of $\widehat{G}_{\mathcal{D}_m,\tau_m}$ to $\nabla\bar{u}$. Indeed, let $\varphi\in C_c^\infty(\Omega\times(0,T))$ be given (this function is devoted to approximate $\bar{u}$ in $L^2(0,T;H_0^1(\Omega))$). We define, for $m\in\mathbb{N}$, $p\in\mathcal{M}_m$ and $\sigma\in\mathcal{E}_m$, the values $v_p^n=\varphi(x_p,n\tau)$ and $v_\sigma^n=\varphi(x_\sigma,n\tau)$, which are used in the definition of

$$\begin{aligned}&\widehat{\nabla}_{\mathcal{D}_m,\tau_m}\varphi(x,t) = \frac{1}{|p|}\sum_{\sigma\in\mathcal{E}_p}|\sigma|(v_\sigma^{n+1}-v_p^{n+1})\mathbf{n}_{p\sigma},\\ &\text{for a.e. } x\in p, \quad \text{for a.e. } t\in]n\tau,(n+1)\tau[,\ \forall p\in\mathcal{M},\ \forall n\in\mathbb{N}.\end{aligned} \tag{57}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$\int_0^T\int_\Omega |\widehat{G}_{\mathcal{D}_m,\tau_m}(x,t)-\nabla\bar{u}(x,t)|^2\mathrm{d}x\mathrm{d}t \le 3\ (T_{12m}+T_{13m}+T_{14m}),$$

with $T_{12m} = \int_0^T \int_\Omega |\widehat{G}_{\mathcal{D}_m,\tau_m}(x,t) - \widehat{\nabla}_{\mathcal{D}_m,\tau_m}\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t$, $T_{13m} = \int_0^T \int_\Omega |\widehat{\nabla}_{\mathcal{D}_m,\tau_m}\varphi(x,t) - \nabla\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t$, and $T_{14m} = \int_0^T \int_\Omega |\nabla\varphi(x,t) - \nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t$. In a similar way as in Lemma 4.2, we have $\lim_{m\to\infty} T_{13m} = 0$. Using the Cauchy-Schwarz inequality, we have

$$T_{12m} \le d\langle u_{\mathcal{D}_m,\tau_m} - v_{\mathcal{D}_m,\tau_m}, u_{\mathcal{D}_m,\tau_m} - v_{\mathcal{D}_m,\tau_m}\rangle, \tag{58}$$

defining $\langle\cdot,\cdot\rangle$ by

$$\langle u, v\rangle = \sum_{n=0}^{N_T} \tau \sum_{p\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_p} \frac{|\sigma|}{d_{p\sigma}} (u_\sigma^{n+1} - u_p^{n+1})(v_\sigma^{n+1} - v_p^{n+1}).$$

Developing equation (58), we obtain

$$T_{12m} \le d\ (\langle u_{\mathcal{D}_m,\tau_m}, u_{\mathcal{D}_m,\tau_m}\rangle - 2\langle u_{\mathcal{D}_m,\tau_m}, v_{\mathcal{D}_m,\tau_m}\rangle + \langle v_{\mathcal{D}_m,\tau_m}, v_{\mathcal{D}_m,\tau_m}\rangle).$$

Since, from the definitions (26) and (46), we have

$$\langle u_{\mathcal{D}_m,\tau_m}, v_{\mathcal{D}_m,\tau_m}\rangle = \int_0^T \int_\Omega G_{\mathcal{D}_m,\tau_m}(x,t) \cdot \nabla_{\mathcal{D}_m,\tau_m}\varphi(x,t) \mathrm{d}x\mathrm{d}t,$$

using the convergence properties of $G_{\mathcal{D}_m,\tau_m}$ and $\nabla_{\mathcal{D}_m,\tau_m}\varphi$, we obtain

$$\lim_{m\to\infty} \langle u_{\mathcal{D}_m,\tau_m}, v_{\mathcal{D}_m,\tau_m}\rangle = \int_0^T \int_\Omega \nabla\bar{u}(x,t) \cdot \nabla\varphi(x,t) \mathrm{d}x\mathrm{d}t.$$

Since $\langle v_{\mathcal{D},\tau}, v_{\mathcal{D},\tau}\rangle = \int_0^T \int_\Omega \mathcal{N}_{\mathcal{D},\tau}(x,t)^2 \mathrm{d}x\mathrm{d}t$, Lemma 4.2 states that

$$\lim_{m\to\infty} \langle v_{\mathcal{D}_m,\tau_m}, v_{\mathcal{D}_m,\tau_m}\rangle = \int_0^T \int_\Omega |\nabla\varphi(x,t)|^2 \mathrm{d}x\mathrm{d}t.$$

Since $\langle u_{\mathcal{D},\tau}, u_{\mathcal{D},\tau}\rangle = \int_0^T \int_\Omega N_{\mathcal{D},\tau}(x,t)^2 \mathrm{d}x\mathrm{d}t$, we get from Lemma 4.7 that

$$\lim_{m\to\infty} \langle u_{\mathcal{D}_m,\tau_m}, u_{\mathcal{D}_m,\tau_m}\rangle = \int_0^T \int_\Omega |\nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t.$$

Gathering the above results, we get that

$$\lim_{m\to\infty} \langle u_{\mathcal{D}_m,\tau_m} - v_{\mathcal{D}_m,\tau_m}, u_{\mathcal{D}_m,\tau_m} - v_{\mathcal{D}_m,\tau_m}\rangle = \int_0^T \int_\Omega |\nabla\bar{u} - \nabla\varphi|^2 \mathrm{d}x\mathrm{d}t,$$

which yields

$$\limsup_{m\to\infty} T_{12m} \le d \int_0^T \int_\Omega |\nabla\bar{u} - \nabla\varphi|^2 \mathrm{d}x\mathrm{d}t.$$

From the above results, we obtain that

$$\int_0^T \int_\Omega |\widehat{G}_{\mathcal{D}_m,\tau_m}(x,t) - \nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t \le 3(d+1) \int_0^T \int_\Omega |\nabla\varphi(x,t) - \nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t + T_{15m},$$

with (noting that φ is fixed) $\lim_{m\to\infty} T_{15m} = 0$. Let $\varepsilon > 0$; we may choose φ such that $\int_0^T \int_\Omega |\nabla\varphi(x,t) - \nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t \le \varepsilon$, and we may then choose m large enough so that $T_{15m} \le \varepsilon$. This completes the proof that

$$\lim_{m\to\infty} \int_0^T \int_\Omega |\widehat{G}_{\mathcal{D}_m,\tau_m}(x,t) - \nabla\bar{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t = 0. \tag{59}$$

□

Remark 4.1 *The above convergence theorem also holds for the fully implicit scheme, under almost the same hypotheses (the hypothesis that f is non-decreasing is not necessary).*

5 Numerical experiments

In this section we present several examples to illustrate the numerical properties of the proposed finite volume (FV) schemes. In particular, we investigate the comparison of these properties with those obtained using the classical finite difference (FD) approach, cf. [33], Chapter 6. We focus on experimental convergence orders in Section 5.1, and on examples of image processing in Section 5.2.

5.1 Comparison of experimental convergence orders

In the examples handled in this section, we consider the square domain $\Omega = [-1.25, 1.25] \times [-1.25, 1.25]$ and we approximate Problem (1)-(2)-(3), where the functions f and g are given by (6). The parameter b is then chosen large enough, such that the second regularisation does not play any role in the finite volume schemes (this is possible since the discrete gradient cannot exceed C/h where h is the space step of the discretisation). Therefore the functions f and g are in fact given by (5). We study below the sensitivity of the methods with respect to the first regularisation parameter a.
The number of finite volumes along each boundary side is denoted by n, which means that n^2 is the total number of finite volumes, and correspondingly the total number of grid points in the finite difference scheme given in [33], used for the purpose of comparison. Then $h = 2.5/n$ is the length of the side of each square finite volume.
Since the solution of the finite volume schemes is obtained through the resolution of linear algebraic system (once in semi-implicit and several times in fully-implicit case) at every discrete time step, we used the Successive Over Relaxation (SOR) iterative solver. The numerical convergence of the SOR solver as well as that of the nonlinear iterations is measured through the square of relative residual drops. Typically, about 20 SOR iterations inside semi-implicit scheme in each time step, and additionally about 5 nonlinear iterations in every time step in case of fully implicit finite volume scheme, are needed for obtaining the results presented in this section,
In the tables below we present the errors committed by the numerical schemes on various examples, the experimental order of convergence (EOC) in several functional spaces and CPU times (in seconds) for the methods. The considered errors are $E_2 = \|u_{\mathcal{D},\tau} - \bar{u}\|_{L^2(\Omega\times(0,T))}$, $E_\infty = \|u_{\mathcal{D},\tau} - \bar{u}\|_{L^\infty(0,T;L^2(\Omega))}$, $EG_2 = \|\widehat{G}_{\mathcal{D},\tau} - \nabla\bar{u}\|_{L^2(\Omega\times(0,T))^2}$ and $EG_\infty = \|\widehat{G}_{\mathcal{D},\tau} - \nabla\bar{u}\|_{L^\infty(0,T;L^2(\Omega))^2}$.

Example 1. In this example, the exact solution is a paraboloid moving up in time, given by $u(x, y, t) = \frac{1}{2}(x^2 + y^2) + t$, which is the solution to (1)-(2)-(5) with $a^2 = 1/2$, $r(x, y, t) = -\frac{1}{2}(x^2 + y^2 + \frac{1}{2})^{-\frac{3}{2}}$, $u_0(x, y) = \frac{1}{2}(x^2 + y^2)$ and (3) is replaced by the exact non-homogeneous Dirichlet boundary conditions provided by the solution. We consider the time interval $[0, T] = [0, 0.3125]$.
For this simple quadratic polynomial example, the finite difference scheme happens to be exact. So only the results for the semi-implicit and fully implicit finite volume schemes are summarised in Table 1.
We can observe that the fully implicit scheme is about three times more precise than the semi-implicit one using the same time step size. On the other hand the semi-implicit scheme is about three times faster.
We also tested the Crank-Nicolson α-schemes (39)-(40) for the semi-implicit scheme and (41)-(42) for the fully implicit scheme.
With $\alpha = 0.6$, the α scheme slightly improves the accuracy of the semi-implicit one and does not significantly modify that of the fully implicit scheme. Since the results are very similar to the previous ones (all schemes have the same experimental order of convergence which is, for the coupling $\tau = h^2$, equal to 2 for the solution error and equal to 1 for the gradient error) we do not include the tables reporting them.
We also tested the α-schemes with $\alpha = 0.5$ instead of 0.6 and $\tau = h$. In such case we got a very poor order of convergence close to 0.8 for the semi-implicit scheme and a divergent behaviour for the fully implicit one, that is compatible with the expectations of the end of Section 3.

Example 2. In this example, the truncated paraboloid function shrinking in time, given by $u(x, y, t) = \min\{\frac{1}{2}(x^2 + y^2 - 1) + t, 0\}$ [27] is the exact viscosity solution to (1)-(2)-(3)-(4) with $r = 0$ and $u_0(x, y) =$

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	4.84e-02	-	1.12e-01	-	4.21e-01	-	8.82e-01	-	-
20	1.5625e-02	1.71e-02	1.504	4.03e-02	1.478	2.03e-01	1.055	4.04e-01	1.126	-
40	3.90625e-03	5.14e-03	1.731	1.18e-02	1.770	8.88e-02	1.192	1.67e-01	1.278	1.0e-01
80	9.76563e-04	1.41e-03	1.867	3.17e-03	1.899	3.81e-02	1.220	6.94e-02	1.265	1.6e-00
160	2.44141e-04	3.68e-04	1.936	8.19e-04	1.953	1.68e-02	1.180	3.03e-02	1.198	2.4e+01
320	6.10352e-05	9.45e-05	1.963	2.09e-04	1.967	7.73e-03	1.121	1.39e-02	1.128	4.1e+02

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	3.89e-02	-	7.25e-02	-	3.40e-01	-	6.62e-01	-	-
20	1.5625e-02	1.01e-02	1.944	1.83e-02	1.984	1.51e-01	1.171	2.77e-01	1.255	7.0e-02
40	3.90625e-03	2.53e-03	1.996	4.55e-03	2.011	6.69e-02	1.176	1.20e-01	1.206	9.4e-01
80	9.76563e-04	6.31e-04	2.004	1.13e-03	2.007	3.08e-02	1.120	5.51e-02	1.126	1.0e+01
160	2.44141e-04	1.57e-04	2.004	2.82e-04	2.005	1.47e-02	1.071	2.62e-02	1.071	1.6e+02
320	6.10352e-05	3.93e-05	2.002	7.04e-05	2.002	7.14e-03	1.038	1.28e-02	1.038	1.9e+03

Table 1: Example 1, error reports, EOCs and CPU times for the semi-implicit FV scheme (top) and the fully implicit FV scheme (bottom).

$\min\{\frac{1}{2}(x^2+y^2-1), 0\}$ (which is the non-regularised mean curvature flow level set equation), again during the time interval $[0, T] = [0, 0.3125]$. The initial condition and exact solution at time $T = 0.3125$ are plotted in Figure 1. Numerical results, for $n = 160$, $T = 0.3125$, obtained by the finite difference and finite volume methods are plotted in Figure 2. Since the gradient of the solution is discontinuous along a shrinking circle, a second order accuracy cannot be expected in this case. For the FV schemes, we set $\tau = h^2$ and for the finite difference scheme, due to stability reasons, we have to set $\tau = h^2/4$.

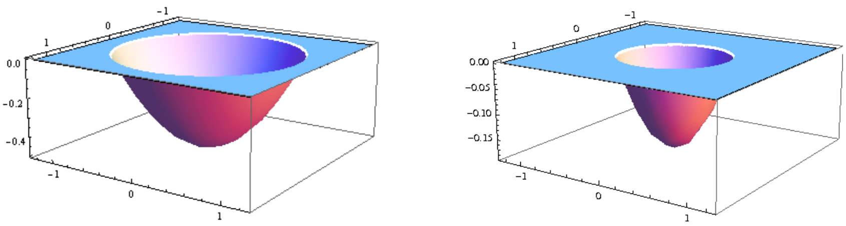

Figure 1: Example 2, the initial condition (left) and exact solution at time T=0.3125

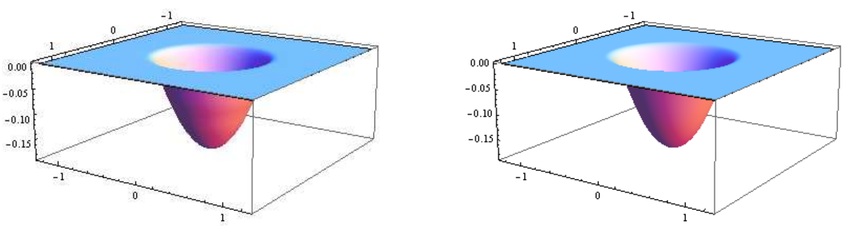

Figure 2: Example 2, numerical solutions by semi-implicit FV (left) and FD (right) schemes, $n = 160$, $T = 0.3125$.

For the numerical implementation, the Evans-Spruck type regularisation (5) has been used (this is needed, since the solution contains flat regions). The results obtained, using the fixed regularisation parameter $a^2 = 10^{-6}$, with the semi-implicit and fully implicit finite volume schemes and with the classical finite difference method for the mean curvature flow level set equation are summarised in Table 2.
The results obtained, using the coupling $a = h^2$ with the space step, with the semi-implicit and fully

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	5.11e-02	-	1.16e-01	-	2.65e-01	-	5.39e-01	-	-
20	1.5625e-02	2.97e-02	0.785	7.05e-02	0.713	2.17e-01	0.287	4.52e-01	0.254	1.0e-02
40	3.90625e-03	1.60e-02	0.894	3.84e-02	0.879	1.73e-01	0.328	3.47e-01	0.384	1.4e-01
80	9.76563e-04	8.39e-03	0.928	2.00e-02	0.938	1.38e-01	0.328	2.71e-01	0.355	2.3e+00
160	2.44141e-04	4.35e-03	0.949	1.04e-02	0.949	1.10e-01	0.331	2.14e-01	0.342	3.2e+01
320	6.10352e-05	2.27e-03	0.935	5.44e-03	0.932	8.72e-02	0.332	1.69e-01	0.337	5.2e+02

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	1.61e-02	-	3.36e-02	-	1.69e-01	-	3.53e-01	-	0.0e-00
20	1.5625e-02	8.31e-03	0.952	1.95e-02	0.784	1.53e-01	0.140	3.56e-01	-0.01	1.0e-01
40	3.90625e-03	4.35e-03	0.934	9.93e-03	0.973	1.30e-01	0.239	2.77e-01	0.362	1.6e+00
80	9.76563e-04	2.35e-03	0.890	5.28e-03	0.912	1.10e-01	0.244	2.29e-01	0.277	1.9e+01
160	2.44141e-04	1.30e-03	0.850	3.01e-03	0.809	9.06e-02	0.273	1.83e-01	0.321	3.0e+02
320	6.10352e-05	7.98e-04	0.707	1.94e-03	0.637	7.51e-02	0.271	1.48e-01	0.306	3.2e+03

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	1.5625e-2	2.11e-02	-	4.80e-02	-	2.23e-01	-	4.78e-01	-	-
20	3.90625e-03	1.22e-02	0.793	2.84e-02	0.760	1.83e-01	0.287	3.77e-01	0.344	4.8e-270
40	9.76563e-04	7.15e-03	0.768	1.60e-02	0.830	1.50e-01	0.284	2.98e-01	0.338	3.0e-01
80	2.44141e-04	4.30e-03	0.733	9.42e-03	0.762	1.25e-01	0.266	2.47e-01	0.272	2.0e-00
160	6.10352e-05	2.60e-03	0.727	5.69e-03	0.727	1.05e-01	0.258	2.05e-01	0.272	1.6e+01
320	1.52584e-05	1.59e-03	0.709	3.52e-03	0.695	8.75e-02	0.257	1.71e-01	0.260	2.4e+02

Table 2: Example 2, errors, EOC and CPU times with $a^2 = 10^{-6}$, for semi-implicit FV scheme (top), fully implicit FV scheme (middle), explicit FD scheme (bottom).

implicit finite volume schemes and with the classical finite difference method for the mean curvature flow level set equation are summarised in Table 3. For the finite volume schemes such coupling seems optimal with respect to obtained precision and experimental order of convergence, in the finite difference scheme it does not play and important role.
These results seem to indicate that the numerical schemes experimentally converge also in this singular example. For the fine grids the precision of the finite difference and semi-implicit finite volume scheme is comparable, which is also observable in Figure 2, where the results show similar smoothing of the gradient jump. Note that the fully implicit FV scheme is about three times more precise than the other ones.

Example 3. In this example, the exact viscosity solution of (1)-(2)-(3)-(4) is, at any time $t \in [0, \frac{1}{2}]$, equal to the characteristic function of $R_t = \{(x, y) \in \mathbb{R}^2, x^2 + y^2 + 2t \leq 1\}$ (the inside of the circle with centre $(0, 0)$ and radius $r(t) = \sqrt{1 - 2t}$). We consider the numerical approximation of this solution during the time interval $[0, T] = [0, 0.25]$.
The circle with radius $r(t)$ is plotted by red dashed line in Figure 3 in order to compare the diffusive behaviours of the fully implicit FV scheme (setting $\tau = h^2$) and classical finite difference scheme (setting $\tau = h^2/4$) at times $t = 0$, $t = 0.25$, using $n = 50$ and $n = 250$. This figure shows that that FV scheme is significantly less diffusive on both coarse and fine grids. Moreover, for the finite difference method the minimum-maximum principle is not fulfilled in this discontinuous case: there are always about 2 percents overshoot and undershoot in the solution which remain stable, even with smaller time steps. In Table 4 we report the errors $E_1 = \|u_{\mathcal{D},\tau} - \bar{u}\|_{L^1(\Omega \times (0,T))}$, $E_\infty = \|u_{\mathcal{D},\tau} - \bar{u}\|_{L^\infty(0,T;L^1(\Omega))}$ for different grid sizes, selecting for each method the optimal regularisation a in (5). Table 4 shows that the finite volume scheme is more precise, especially for the $L^\infty(0, T; L^1(\Omega))$ norm. We do not report in details the results of the semi-implicit finite volume scheme, which behaves in a similar way to the fully implicit scheme, provided that $\tau = h^2/16$. This restriction on the time step makes the CPU time of the semi-implicit FV scheme similar to that of the fully implicit scheme.

n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	5.72e-02	-	1.29e-01	-	2.60e-01	-	5.25e-01	-	-
20	1.5625e-02	3.16e-02	0.856	7.51e-02	0.778	2.14e-01	0.280	4.44e-01	0.244	1.0e-03
40	3.90625e-03	1.63e-02	0.951	3.93e-02	0.935	1.72e-01	0.315	3.45e-01	0.365	1.4e-01
80	9.76563e-04	8.39e-03	0.963	2.00e-02	0.973	1.38e-01	0.319	2.71e-01	0.346	2.3e-00
160	2.44141e-04	4.25e-03	0.979	1.01e-02	0.982	1.10e-01	0.327	2.15e-01	0.337	3.5e+01
320	6.10352e-05	2.15e-03	0.984	5.10e-03	0.990	8.79e-02	0.325	1.71e-01	0.330	5.2e+02
n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	6.25e-02	3.45e-02	-	8.10e-02	-	1.96e-01	-	3.90e-01	-	-
20	1.5625e-2	1.20e-02	1.528	2.95e-02	1.457	1.60e-01	0.289	3.80e-01	0.038	1.0e-01
40	3.90625e-03	4.96e-03	1.270	1.17e-02	1.340	1.32e-01	0.281	2.84e-01	0.421	1.6e-00
80	9.76563e-04	2.34e-03	1.082	5.26e-03	1.146	1.10e-01	0.269	2.29e-01	0.312	2.0e+01
160	2.44141e-04	1.17e-03	1.007	2.61e-03	1.012	8.94e-02	0.293	1.80e-01	0.345	3.1e+02
320	6.10352e-05	5.87e-04	0.990	1.32e-03	0.982	7.22e-02	0.308	1.42e-01	0.341	3.6e+03
n	τ	E_2	EOC	E_∞	EOC	EG_2	EOC	EG_∞	EOC	CPU
10	1.5625e-2	3.14e-02	-	7.79e-02	-	2.33e-01	-	4.98e-01	-	-
20	3.90625e-03	1.37e-02	1.196	3.35e-02	1.220	1.83e-01	0.351	3.81e-01	0.384	1.0e-04
40	9.76563e-04	7.31e-03	0.910	1.65e-02	1.020	1.50e-01	0.283	2.98e-01	0.355	1.0e-02
80	2.44141e-04	4.30e-03	0.766	9.41e-03	0.809	1.25e-01	0.263	2.47e-01	0.272	2.0e+00
160	6.10352e-05	2.56e-03	0.746	5.58e-03	0.755	1.05e-02	0.257	2.04e-01	0.273	1.6e+01
320	1.52588e-05	1.53e-03	0.743	3.32e-03	0.748	8.79e-02	0.253	1.71e-01	0.255	2.4e+02

Table 3: Example 2, errors, EOC and CPU times with $a = h^2$ for semi-implicit FV scheme (top), fully implicit FV scheme (middle), explicit FD scheme (bottom).

n	τ	E_1	EOC	E_∞	EOC	CPU
50	6.25e-04	1.000e-01	-	4.341e-01	-	5.000e-02
100	1.5625e-4	6.596e-02	0.600	3.002e-01	0.532	8.300e-01
200	3.90625e-05	4.496e-02	0.553	2.090e-01	0.522	1.317e+01
n	τ	E_1	EOC	E_∞	EOC	CPU
50	2.50e-03	4.750e-02	-	2.081e-01	-	1.773e+01
100	6.25e-04	2.859e-02	0.732	1.315e-01	0.662	2.632e+02
200	1.5625e-04	1.757e-02	0.703	8.278e-02	0.668	5.353e+03

Table 4: Example 3, errors, EOC and CPU times for the explicit FD scheme with $a^2 = 10^{-14}$ and $\tau = h^2/4$ (top), and for the fully implicit FV scheme with $a = h^2$ and $\tau = h^2$ (bottom).

5.2 Image processing examples

We now turn to the assessment of the behaviour of the schemes in the framework of image processing applications. More particularly, one of the best applications of filtering by mean curvature flow models happens to be the filtering of salt-and-pepper noise. Indeed, the small level sets of image intensity disappear much faster than larger objects (e.g. unit circle given by its characteristic function). In fact due to curvature blow-up phenomenon in the mean curvature flow models, there is an infinite speed of shrinking before the extinction of an object. The following examples show a comparison of finite volume and finite difference schemes used for this denoising task.

Example 4. In this example, we consider the same data as Example 3, adding 20 percent salt-and-pepper noise to the initial data and setting $n = 200$. Figure 4 shows the filtering effect of the fully implicit finite volume scheme after one and three time steps with $\tau = h^2$. In accordance with analytical mean curvature flow theory, the small separated spots are disappearing fast and three steps are sufficient to get filtering result with no noise and no unit circle deterioration. Due to stability reasons, the condition $\tau = h^2/4$ must hold for the finite difference scheme, and a greater number of time steps (corresponding to approximately three times longer time) is needed for smoothing out the noise. This fact together with

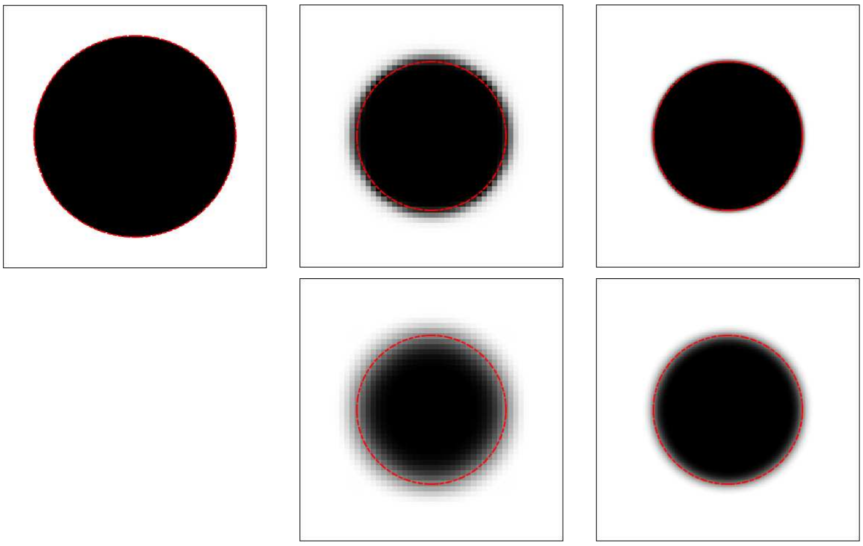

Figure 3: Example 3, initial condition in the form of characteristic function of the unit circle (top left), fully implicit FV with $n = 50$ (top middle), fully implicit FV with $n = 250$ (top right), explicit FD with $n = 50$ (bottom middle), explicit FD with $n = 250$ (bottom right) at time 0.25.

more diffusive scheme behaviour causes a blurring of the edge.

Example 5. In this example we add 20 percent salt-and-pepper noise to the characteristic function of a quatrefoil, and we set again $n = 200$. Figure 5 shows almost perfect quatrefoil reconstruction using the fully implicit finite volume scheme, whereas a much more diffused reconstruction is obtained using the finite difference scheme, especially in the central part.

Example 6. In the last example we add 50 percent salt-and-pepper noise to the characteristic function of the quatrefoil. Figure 6 shows again very good quatrefoil reconstruction using the fully implicit finite volume scheme. The slightly diffused central part in the filtering result corresponds to "blurred" part visible also in noisy image due to high level of noise. The finite difference scheme cannot get rid of the noise without adding too much diffusion.

6 Conclusions

The mathematical properties proved in this paper show that the fully implicit and the semi-implicit schemes have interesting mathematical and numerical properties. It remains the difficult task of extending these convergence properties to the non-regularised mean curvature motion model.

From the point of view of numerical applications, the numerical examples presented here seem to show that the precision of the finite difference scheme is between that of the semi-implicit and fully implicit finite volume schemes. The finite difference scheme is the fastest due to its simplicity, although it requires smaller time steps in these examples (on some examples, the CPU ratio can be about only two). On the other hand, there is no guarantee of stability for the finite difference scheme which is, in opposite, the advantage of both semi-implicit and fully implicit finite volume schemes (this is checked on some discontinuous problems).

Finally, considering image processing purposes, the examples proposed here are showing that the finite volume schemes are substantially less diffusive during salt-and-pepper noise filtering than the finite difference method proposed in [33]. Thus finite volume schemes significantly improve the capability to filter noisy images without blurring edges, using nonlinear diffusion methods based on mean curvature flow.

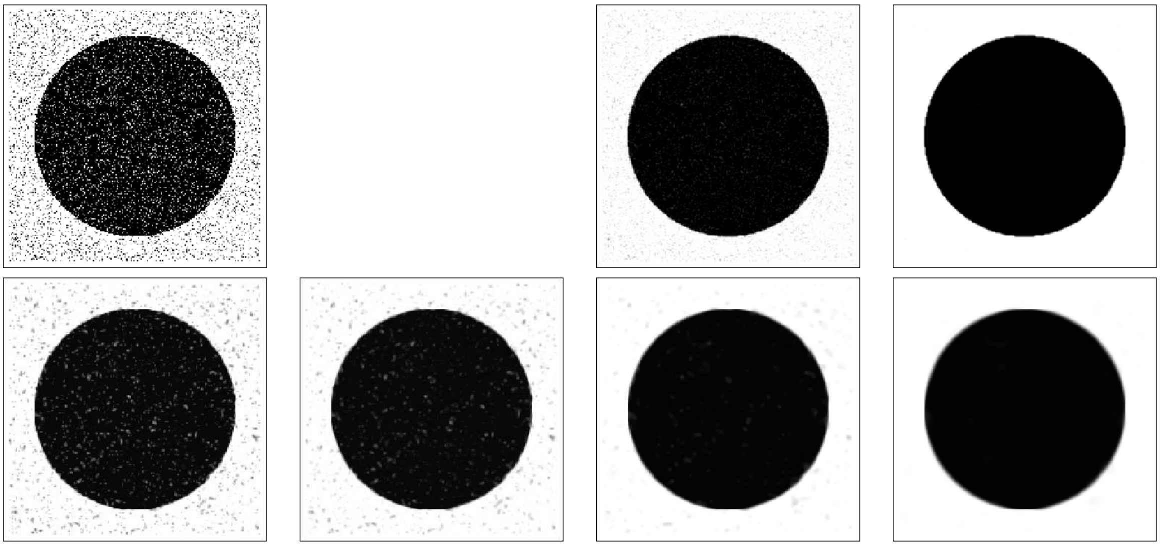

Figure 4: Example 4, initial noisy image with 20 percent salt-and-pepper noise (top extreme left), filtering result by the fully implicit FV scheme with $\tau = h^2$ after one (top middle right) and three (top extreme right) time steps, filtering result by the explicit FD scheme after 2 (bottom extreme left), 4 (bottom middle left), 12 (bottom middle right) and 40 (bottom extreme right) time steps, $\tau = h^2/4$.

Acknowledgements. The authors thank Professor Guy Barles for interesting and fruitful discussions. The work of second and third author was supported by grants APVV-0351-07 and VEGA 1/0269/09.

Appendix

Lemma 6.1 *Let b be a continuous strictly increasing function from $\mathbb{R}$ to $\mathbb{R}$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $\beta \in R$ such that $(b(\beta_n) - b(\beta))(\beta_n - \beta) \longrightarrow 0$ as $n \longrightarrow \infty$. Then, $\beta_n \longrightarrow \beta$ as $n \longrightarrow \infty$.*

Proof. We first remark that the mapping $s \mapsto (b(\gamma + s\delta) - b(\gamma))\delta$ is increasing, for all $\delta, \gamma \in \mathbb{R}$. This will be used to prove that the sequence $(\beta_n)_{n\in\mathbb{N}}$ is bounded. Indeed, if the sequence $(\beta_n)_{n\in\mathbb{N}}$ is unbounded, we can assume, up to a subsequence, that $|\beta_n| \longrightarrow \infty$ as $n \longrightarrow \infty$ and then, once again up to a subsequence, that $|\beta_n - \beta| \geq 1$ for all $n \in \mathbb{N}$ and $\frac{\beta_n - \beta}{|\beta_n - \beta|} \longrightarrow \gamma$ as $n \longrightarrow \infty$ (for some $\gamma \in \mathbb{R}$ with $|\gamma| = 1$). Therefore, one has:

$$(b(\beta_n) - b(\beta))(\beta_n - \beta) \geq \left(b(\beta + \frac{\beta_n - \beta}{|\beta_n - \beta|}) - b(\beta) \right) \frac{\beta_n - \beta}{|\beta_n - \beta|}.$$

Then, passing to the limit as $n \longrightarrow \infty$,

$$0 = \lim_{n\longrightarrow\infty} (b(\beta_n) - b(\beta))(\beta_n - \beta) \geq (b(\beta + \gamma) - b(\beta)) \cdot \gamma > 0.$$

which is impossible.
Since the sequence $(\beta_n)_{n\in\mathbb{N}}$ is bounded, we can assume, up to a subsequence, that $\beta_n \longrightarrow \gamma$, as $n \longrightarrow \infty$, for some $\gamma \in \mathbb{R}$. Then, since $(b(\beta_n) - b(\beta))(\beta_n - \beta) \longrightarrow 0$, one has $(b(\gamma) - b(\beta))(\gamma - \beta) = 0$, which gives $\gamma = \beta$ and the convergence of the whole sequence $(\beta_n)_{n\in\mathbb{N}}$ to β follows.
□

Lemma 6.2 *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence non-negative functions in $L^1(\Omega)$. Let $F \in L^1(\Omega)$ be such that $F_n \longrightarrow F$ a.e. in Ω and $\int_\Omega F_n(x)\mathrm{d}x \longrightarrow \int_\Omega F(x)\mathrm{d}x$, as $n \longrightarrow \infty$. Then, $F_n \longrightarrow F$ in $L^1(\Omega)$ as $n \longrightarrow \infty$.*

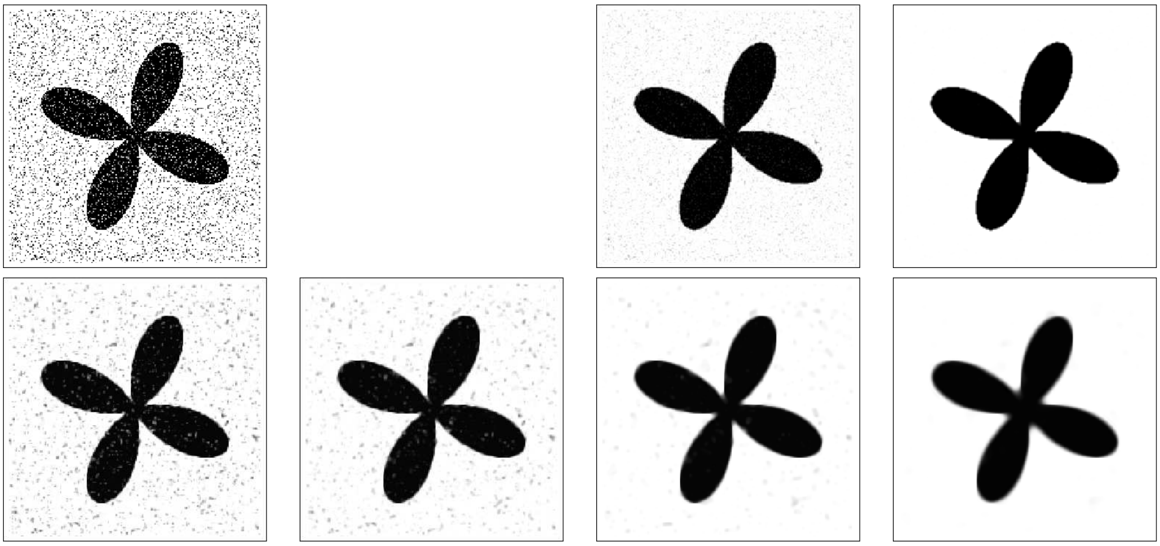

Figure 5: Example 5, initial noisy image with 20 percent salt-and-pepper noise (top extreme left), filtering result by the fully implicit FV scheme with $\tau = h^2$ after one (top middle right) and three (top extreme right) time steps, filtering result by the explicit FD scheme after 2 (bottom extreme left), 4 (bottom middle left), 12 (bottom middle right) and 40 (bottom extreme right) time steps, $\tau = h^2/4$.

PROOF. The proof of this lemma is very classical. Applying the Dominated Convergence Theorem to the sequence $(F - F_n)^+$ leads to $\int_\Omega (F(x) - F_n(x))^+ \mathrm{d}x \longrightarrow 0$ as $n \longrightarrow \infty$. Then, since $|F - F_n| = 2(F - F_n)^+ - (F - F_n)$, we conclude that $F_n \longrightarrow F$ in $L^1(\Omega)$ as $n \longrightarrow \infty$.
□

Theorem 6.1 (A variant of Ascoli's theorem) *Let Ω be a polyhedral open bounded connected subset of $\mathbb{R}^d$, with $d \in \mathbb{N} \setminus \{0\}$ and $T > 0$. Let $u_0 \in H_0^1(\Omega)$ be given. Let $(u_m, \mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ be a sequence such that, for all $m \in \mathbb{N}$, $(\mathcal{D}_m, \tau_m)$ is a space-time discretisation of $\Omega \times (0, T)$ in the sense of Definition 2.2, $u_m \in H_{\mathcal{D}_m,\tau_m}$ and $h_{\mathcal{D}_m}$ and τ_m tend to 0 as $m \longrightarrow \infty$. For all $m \in \mathbb{N}$, setting $\mathcal{D} = \mathcal{D}_m$ and $\tau = \tau_m$, we define the functions $u_{\mathcal{D},\tau}(x, t)$, for all $t \in [0, T]$ and a.e. $x \in \Omega$ by*

$$\begin{aligned} &u_{\mathcal{D},\tau}(x, 0) = u_p^0 = \frac{1}{|p|}\int_p u_0(x)\mathrm{d}x, \\ &u_{\mathcal{D},\tau}(x, t) = u_p^{n+1}, \text{ for a.e. } x \in p,\ \forall t \in]n\tau, (n + 1)\tau],\ \forall p \in \mathcal{M},\ \forall n = 0, \dots, N_T, \end{aligned} \tag{60}$$

and the function $z_{\mathcal{D},\tau}$ by

$$z_{\mathcal{D},\tau}(x, t) = \frac{u_p^{n+1} - u_p^n}{\tau}, \text{ for a.e. } x \in p, \quad \text{for a.e. } t \in]n\tau, (n + 1)\tau[,\ \forall p \in \mathcal{M},\ \forall n \in \mathbb{N}, \tag{61}$$

with u_p^0 defined by (60). We assume that there exists $C > 0$ (hence independent of m) such that $\|u_m^{n+1}\|_{1,\mathcal{D}_m} \le C$ for all $n = 0, \dots, N_{Tm}$ and $\|z_{\mathcal{D}_m,\tau_m}\|^2_{L^2(\Omega\times(0,T))} \le C$.
Then there exists $\bar{u} \in C^0(0, T; L^2(\Omega))$ with $\bar{u}(\cdot, 0) = u_0$ and a subsequence of $(u_m, \mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, again denoted $(u_m, \mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \|u_{\mathcal{D}_m,\tau_m}(t) - u(t)\|_{L^2(\Omega)} = 0. \tag{62}$$

PROOF.

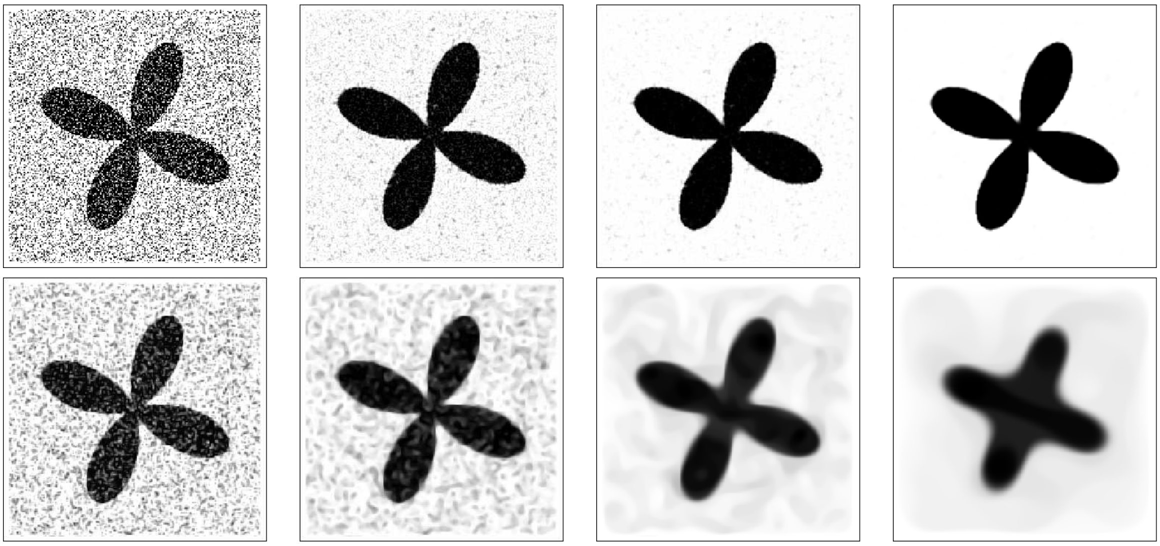

Figure 6: Example 6, initial noisy image with 50 percent salt-and-pepper noise (top extreme left), filtering result by the fully implicit FV scheme after 1 (top middle left), 2 (top middle right) and 6 (top extreme right) time steps, $\tau = h^2$, filtering result by the explicit FD scheme after 2 (bottom extreme left), 10 (bottom middle left), 100 (bottom middle right) and 500 (bottom extreme right) time steps, $\tau = h^2/4$.

We first remark that, for $m \in \mathbb{N}$, denoting $\mathcal{D} = \mathcal{D}_m$ and $\tau = \tau_m$ and for all $t_1 \in](n_1-1)\tau, n_1\tau]$ and $t_2 \in](n_2-1)\tau, n_2\tau]$, for $n_1, n_2 = 0, \ldots, N_T$ with $n_2 > n_1$ and a.e. $x \in p$, we have

$$(u_{\mathcal{D},\tau}(x,t_1) - u_{\mathcal{D},\tau}(x,t_2))^2 \le \left(\sum_{n=n_1}^{n_2-1} |u_p^{n+1} - u_p^n|\right)^2 \le (n_2-n_1)\tau \sum_{n=n_1}^{n_2-1} \frac{(u_p^{n+1}-u_p^n)^2}{\tau}.$$

Hence we get that

$$\int_\Omega (u_{\mathcal{D}_m,\tau_m}(x,t_1) - u_{\mathcal{D}_m,\tau_m}(x,t_2))^2 \mathrm{d}x \le C\ (|t_2-t_1| + \tau_m),\ \forall t_1, t_2 \in [0,T],\ \forall m \in \mathbb{N}. \tag{63}$$

The proof can now follow that of Ascoli's theorem. Let $(t_k)_{k\in\mathbb{N}}$ be a dense sequence in $[0,T]$. For $t = t_0$, if $t_O > 0$, we can use the result given in [17] since $\|u_m^{n+1}\|_{1,\mathcal{D}_m} \le C$ for all $n = 0, \ldots, N_T$:

$$\|u_{\mathcal{D}_m,\tau_m}(\cdot+\xi,t_0) - u_{\mathcal{D}_m,\tau_m}(\cdot,t_0)\|^2_{L^2(\Omega\times(0,T))} \le C|\xi|(|\xi| + 4h_{\mathcal{D}_m}),\ \forall \xi \in \mathbb{R}^d.$$

Hence we can extract, thanks to Kolmogorov's theorem, a subsequence of $(u_m, \mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$, denoted by $(u_{\psi_0(m)}, \mathcal{D}_{\psi_0(m)}, \tau_{\psi_0(m)})_{m\in\mathbb{N}}$, where ψ_0 is an increasing injection from $\mathbb{N}$ to $\mathbb{N}$, such that $u_{\mathcal{D}_{\psi_0(m)},\tau_{\psi_0(m)}}(\cdot,t_0)$ converges in $L^2(\Omega)$ to some function. Note also that, thanks to (60), $u_{\mathcal{D}_m,\tau_m}(\cdot,0)$ converges in $L^2(\Omega)$ to u_0. Similarly, for $t = t_1$, one extracts, again thanks to Kolmogorov's theorem, a subsequence of $(u_{\psi_0(m)}, \mathcal{D}_{\psi_0(m)}, \tau_{\psi_0(m)})_{m\in\mathbb{N}}$, denoted $(u_{\psi_1(m)}, \mathcal{D}_{\psi_1(m)}, \tau_{\psi_1(m)})_{m\in\mathbb{N}}$ such that $u_{\mathcal{D}_{\psi_1(m)},\tau_{\psi_1(m)}}(\cdot,t_1)$ converges in $L^2(\Omega)$ to some function. We reproduce this mechanism by induction for all $k \in \mathbb{N}$, allowing to consider the diagonal sequence $(u_{\psi_m(m)}, \mathcal{D}_{\psi_m(m)})_{m\in\mathbb{N}}$, which is then such that $u_{\mathcal{D}_{\psi_m(m)},\tau_{\psi_m(m)}}(\cdot,t_k)$ converges in $L^2(\Omega)$ as $m \to \infty$ for all $k \in \mathbb{N}$ (recall that the sequence $(u_{\psi_m(m)}, \mathcal{D}_{\psi_m(m)}, \tau_{\psi_m(m)})_{m\in\mathbb{N}}$ is extracted from $(u_{\psi_k(m)}, \mathcal{D}_{\psi_k(m)}, \tau_{\psi_k(m)})_{m\in\mathbb{N},m\ge k}$). We now denote, for simplicity, $(u_m, \mathcal{D}_m, \tau_m)_{m\in\mathbb{N}}$ instead of $(u_{\psi_m(m)}, \mathcal{D}_{\psi_m(m)}, \tau_{\psi_m(m)})_{m\in\mathbb{N}}$.

Then the property (63) allows to show that, for all $t \in \mathbb{R}_+$, $(u_{\mathcal{D}_m,\tau_m}(t))_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. Indeed, for $\varepsilon \in]0,1[$, one first chooses $k \in \mathbb{N}$ such that $|t - t_k| \le \varepsilon^2$, then $n_0 \in \mathbb{N}$ such that $\tau_m \le \varepsilon^2$ for all $n \ge n_0$, and $\|u_{\mathcal{D}_n,\tau_n}(t_k) - u_{\mathcal{D}_p,\tau_p}(t_k)\|_{L^2(\Omega)} \le \varepsilon$ for all $n, p \ge n_0$. The inequality

$\|u_{\mathcal{D}_n,\tau_n}(t)-u_{\mathcal{D}_p,\tau_p}(t)\|_{L^2(\Omega)} \leq \|u_{\mathcal{D}_n,\tau_n}(t)-u_{\mathcal{D}_n,\tau_n}(t_k)\|_{L^2(\Omega)}+\|u_{\mathcal{D}_n,\tau_n}(t_k)-u_{\mathcal{D}_p,\tau_p}(t_k)\|_{L^2(\Omega)}+\|u_{\mathcal{D}_p,\tau_p}(t_k)-u_{\mathcal{D}_p,\tau_p}(t)\|_{L^2(\Omega)} \leq (2\sqrt{2C}+1)\varepsilon$ for all $n, p \geq n_0$ follows.
One then defines, for all $t \in \mathbb{R}_+$, $\bar{u}(t)$ as the limit of $(u_{\mathcal{D}_m,\tau_m}(t))_{m\in\mathbb{N}}$. Passing to the limit $m \to \infty$ in (63) provides

$$\|\bar{u}(t_2) - \bar{u}(t_1)\|^2_{L^2(\Omega)} \leq C\ |t_2 - t_1|,\ \forall t_1, t_2 \in [0, T],\ \forall T > 0, \tag{64}$$

which shows that $u \in C^0(\mathbb{R}_+; L^2(\Omega))$. Then (62) is again an easy consequence of (63). Indeed, let $T \geq 0$ and $\varepsilon > 0$ be given. Since, for all $k = 0, \ldots, \lfloor T/\varepsilon^2 \rfloor$ (where $\lfloor x \rfloor$ denotes the greater integer lower of equal to x), the sequence $(u_{\mathcal{D}_m,\tau_m}(k\varepsilon^2))_{m\in\mathbb{N}}$ converges to $u(k\varepsilon^2)$, let $n_0 \in \mathbb{N}$ be such that $\|u_{\mathcal{D}_m,\tau_m}(k\varepsilon^2) - u(k\varepsilon^2)\|_{L^2(\Omega)} \leq \varepsilon$ for all $k = 0, \ldots, \lfloor T/\varepsilon^2 \rfloor$ and all $m \geq n_0$, and such that $\tau_m \leq \varepsilon^2$ for all $m \geq n_0$. Then, for all $t \in [0, T]$ and $m \geq n_0$, letting $k = \lfloor t/\varepsilon^2 \rfloor$, we get using (64) and (63), $\|\bar{u}(t) - u_{\mathcal{D}_m,\tau_m}(t)\|_{L^2(\Omega)} \leq \|\bar{u}(t) - \bar{u}(k\varepsilon^2)\|_{L^2(\Omega)} + \|\bar{u}(k\varepsilon^2) - u_{\mathcal{D}_m,\tau_m}(k\varepsilon^2)\|_{L^2(\Omega)} + \|u_{\mathcal{D}_m,\tau_m}(k\varepsilon^2) - u_{\mathcal{D}_m,\tau_m}(t)\|_{L^2(\Omega)} \leq (\sqrt{C} + 1 + \sqrt{2C})\varepsilon$, which concludes the proof of (62). □

References

[1] Luis Alvarez, Frédéric Guichard, Pierre-Louis Lions, and Jean-Michel Morel. Axioms and fundamental equations of image processing. *Arch. Rational Mech. Anal.*, 123(3):199–257, 1993.

[2] Boris Andreianov, Franck Boyer, and Florence Hubert. Discrete duality finite volume schemes for Leray-Lions-type elliptic problems on general 2D meshes. *Numer. Methods Partial Differential Equations*, 23(1):145–195, 2007.

[3] Sigurd Angenent and Morton E. Gurtin. Multiphase thermomechanics with interfacial structure. II. Evolution of an isothermal interface. *Arch. Rational Mech. Anal.*, 108(4):323–391, 1989.

[4] G. Barles and P. E. Souganidis. Convergence of approximation schemes for fully nonlinear second order equations. *Asymptotic Anal.*, 4(3):271–283, 1991.

[5] P. Bourgine, R. Čunderlik, O. Drblíková, K. Mikula, N. Peyriéras, M. Remešíková, A. Sarti, and B. Rizzi. 4d embryogenesis image analysis using pde methods of image processing. *to appear in Kybernetika*, 2010.

[6] Vicent Caselles, Ron Kimmel, and Guillermo Sapiro. Geodesic active contours. *International Journal of Computer Vision*, 22(1):61–79, 1997.

[7] Francine Catté, Pierre-Louis Lions, Jean-Michel Morel, and Tomeu Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM J. Numer. Anal.*, 29(1):182–193, 1992.

[8] Yun-Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differ. Geom.*, 33(3):749–786, 1991.

[9] Yunmei Chen, B.C. Vemuri, and Li Wang. Image denoising and segmentation via nonlinear diffusion. *Comput. Math. Appl.*, 39(5-6):131–149, 2000.

[10] S. Corsaro, K. Mikula, A. Sarti, and F. Sgallari. Semi-implicit covolume method in 3D image segmentation. *SIAM J. Sci. Comput.*, 28(6):2248–2265, 2006.

[11] Klaus Deckelnick and Gerhard Dziuk. Error estimates for a semi-implicit fully discrete element scheme for the mean curvature flow of graphs. *Interfaces Free Bound.*, 2(4):341–359, 2000.

[12] Klaus Deckelnick and Gerhard Dziuk. Convergence of numerical schemes for the approximation of level set solutions to mean curvature flow. In *Numerical methods for viscosity solutions and applications (Heraklion, 1999)*, volume 59 of *Ser. Adv. Math. Appl. Sci.*, pages 77–93. World Sci. Publ., River Edge, NJ, 2001.

[13] Klaus Deckelnick and Gerhard Dziuk. Numerical approximation of mean curvature flow of graphs and level sets. In *Mathematical aspects of evolving interfaces (Funchal, 2000)*, volume 1812 of *Lecture Notes in Math.*, pages 53–87. Springer, Berlin, 2003.

[14] Olga Drblíková and Karol Mikula. Convergence analysis of finite volume scheme for nonlinear tensor anisotropic diffusion in image processing. *SIAM J. Numer. Anal.*, 46(1):37–60, 2008.

[15] Jérôme Droniou. Finite volume schemes for fully nonlinear elliptic equations in divergence form. *ESAIM, Math. Model. Numer. Anal.*, 40(6):1069–1100, 2006.

[16] L.C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differ. Geom.*, 33(3):635–681, 1991.

[17] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. Finite volume methods. In *Handbook of numerical analysis, Vol. VII*, Handb. Numer. Anal., VII, pages 713–1020. North-Holland, Amsterdam, 2000.

[18] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. Finite volume approximation of elliptic problems and convergence of an approximate gradient. *Appl. Numer. Math.*, 37(1-2):31–53, 2001.

[19] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. Cell centred discretisation of non linear elliptic problems on general multidimensional polyhedral grids. *J. Numer. Math.*, 17(3):173–193, 2009.

[20] Robert Eymard, Thierry Gallouët, and Raphaèle Herbin. Discretisation of heterogeneous and anisotropic diffusion problems on general nonconforming meshes. SUSHI: a scheme using stabilization and hybrid interfaces. *to be published in IMAJNA*, 2009.

[21] Angela Handlovičová and Karol Mikula. Stability and consistency of the semi-implicit co-volume scheme for regularized mean curvature flow equation in level set formulation. *Appl. Math., Praha*, 53(2):105–129, 2008.

[22] Angela Handlovičová, Karol Mikula, and Fiorella Sgallari. Semi-implicit complementary volume scheme for solving level set like equations in image processing and curve evolution. *Numer. Math.*, 93(4):675–695, 2003.

[23] Satyanad Kichenassamy, Arun Kumar, Peter Olver, Allen Tannenbaum, and Anthony jun. Yezzi. Conformal curvature flows: from phase transitions to active vision. *Arch. Ration. Mech. Anal.*, 134(3):275–301, 1996.

[24] K. Mikula, N. Peyriéras, M. Remešíková, and A. Sarti. 3D embryogenesis image segmentation by the generalized subjective surface method using the finite volume technique. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 585–592. Wiley, 2008.

[25] Karol Mikula and Narisoa Ramarosy. Semi-implicit finite volume scheme for solving nonlinear diffusion equations in image processing. *Numer. Math.*, 89(3):561–590, 2001.

[26] Karol Mikula, Alessandro Sarti, and Fiorella Sgallari. Co-volume method for Riemannian mean curvature flow in subjective surfaces multiscale segmentation. *Comput. Vis. Sci.*, 9(1):23–31, 2006.

[27] Adam M. Oberman. A convergent monotone difference scheme for motion of level sets by mean curvature. *Numer. Math.*, 99(2):365–379, 2004.

[28] Stanley Osher and Ronald Fedkiw. *Level set methods and dynamic implicit surfaces.* Applied Mathematical Sciences. 153. New York, NY: Springer. xiv, 273 p., 2003.

[29] Stanley Osher and James A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton-Jacobi formulations. *J. Comput. Phys.*, 79(1):12–49, 1988.

[30] Pietro Perona and Jitendra Malik. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions on pattern analysis and machine intelligence*, 12(7):629–639, 1990.

[31] Alessandro Sarti, Ravi Malladi, and James A. Sethian. Subjective surfaces: A method for completing missing boundaries. *Proc. Natl. Acad. Sci. USA*, 97(12):6258–6263, 2000.

[32] Alfred Schmidt. Computation of three dimensional dendrites with finite elements. *J. Comput. Phys.*, 125(2):293–312, 1996.

[33] J.A. Sethian. *Level set methods and fast marching methods. Evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science.* Cambridge Monographs on Applied and Computational Mathematics. 3. Cambridge: Cambridge University Press. xx, 378 p. , 1999.

[34] Noel J. Walkington. Algorithms for computing motion by mean curvature. *SIAM J. Numer. Anal.*, 33(6):2215–2238, 1996.