Regular article

Slow and fast diffusion effects in image processing

Jozef Kačur¹, Karol Mikula²

¹ Comenius University, Faculty of Mathematics and Physics, Dept. of Numerical Analysis and Optimization, Mlynska Dolina, 842 15 Bratislava, Slovakia (e-mail: kacur@fmph.uniba.sk)
² Slovak Technical University, Dept. of Mathematics, Radlinskeho 11, 813 68 Bratislava, Slovakia (e-mail: mikula@vox.svf.stuba.sk)

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Abstract. A mathematical model for a nonlinear image multiscale analysis is studied. Processing of an image is based on a solution of the strongly nonlinear parabolic partial differential equation, which can degenerate depending on values of the greylevel intensity function. The governing PDE is a generalization of the regularized (in the sense of Catté, Lions, Morel and Coll) Perona-Malik anisotropic diffusion equation. We present numerical techniques for solving the suggested initial-boundary value problem and also existence and convergence results. Numerical experiments are discussed.

1 Introduction

In the present paper we study the following nonlinear diffusion problems. Let \( u(t, x) \) be a function (representing the greylevel intensity function in image multiscale analysis – see [2, 3, 21, 30]) which satisfies one of the PDEs:

\[
\partial_t b(x, u) - \nabla (g(|\nabla G_\sigma \ast b(x, u)|) \nabla b(x, u)) = f(u_0 - u),
\]

or

\[
\partial_t b(x, u) - \nabla (g(|\nabla G_\sigma \ast b(x, u)|) \nabla b(x, u)) = f(u_0 - u),
\]

for \( t \in I = [0, T] \), \( x \in \Omega \subset \mathbb{R}^N \), where \( \Omega \) is a bounded domain with Lipschitz continuous boundary (\( N = 2 \) or \( 3 \) in practice of the image analysis). The equations are coupled with boundary and initial conditions of the form

\[
\partial_t \beta(x, u) = 0 \quad \text{on} \quad I \times \partial \Omega,
\]

\[
b(x, u(0, x)) = b(x, u_0(x)).
\]

For the data in (1.1)–(1.3) we assume that

(H1) \( g \) is a Lipschitz continuous function, \( g(0) = 1 \) and \( 0 < g(s) \to 0 \) for \( s \to \infty \),

(H2) \( G_\sigma \in C^\infty(\mathbb{R}^N) \) is a compactly supported smoothing kernel \( (\int_{\mathbb{R}^N} G_\sigma(x)dx = 1, \ G_\sigma(x) \to \delta_x \ \text{Dirac measure at point } x, \ \text{for } \sigma \to 0) \),

(H3) \( f \) is a Lipschitz continuous, nondecreasing function, \( f(0) = 0 \),

(H4) \( u_0 \in L_2(\Omega) \) (represents the processed image).

By the term \( \nabla G_\sigma \ast v \) we mean \( \int_{\mathbb{R}^N} \nabla G_\sigma(x - \xi) \tilde{v}(\xi)d\xi \), where \( \tilde{v} \) is an extension of \( v \), for which we assume

\[
\|\tilde{v}\|_{W^2_1(\mathbb{R}^N)} \leq C\|v\|_{W^2_1(\mathbb{R})}.
\]

We consider following four cases for the shape of the functions \( b \) and \( \beta \) which indicate the structure of the governing equations:

(I) \( b(x, s) \) is continuous, strictly increasing in \( s \), \( b(x, 0) = 0 \) and \( \beta(x, s) \equiv s \),

(II) \( b(x, s) \) is nondecreasing Lipschitz continuous in \( s \), \( b(x, 0) = 0 \) and \( \beta(x, s) \equiv s \),

(III) \( \beta(x, s) \) is continuous, strictly increasing in \( s \), \( \beta(x, 0) = 0 \), and \( b(x, s) \equiv s \),

(IV) \( \beta(x, s) \) is nondecreasing Lipschitz continuous in \( s \), \( \beta(x, 0) = 0 \) and \( b(x, s) \equiv s \).

The initial-boundary value problems (1.1)–(1.3) in form a or b, in all cases I–IV are generalizations of the regularized (in the sense of Catté, Lions, Morel and Coll) Perona-Malik nonlinear diffusion equation. The Perona-Malik equation as well as its regularization are also called anisotropic diffusion in the computer vision community and they are widely used for image selective smoothing and edge detection in the image processing applications. The previous papers ([7, 16, 29]) have been dealing with the case when both \( \beta(x, s) \equiv s \), \( b(x, s) \equiv s \). In such case, the image analysis depends strongly on \( \nabla u \) ([29]) or \( \nabla G_\sigma \ast u \) ([7]) which are considered as edge indicators. Such special PDEs selectively diffuse the image in the regions, where the signal is of a constant mean in spite of those
Due to the strong nonlinearity and possible degeneracy in (1.1), the proof of existence of a solution and its numerical approximation needs nonstandard techniques, especially, if we want to prove its convergence to the solution. We use a special approximation (see numerical schemes 2.1, 2.2) which in a constructive way look for the solution and can be implemented for the computational purposes. It is based on the special time discretization developed and applied in [6, 9–13, 15, 22–24] for solving the Stefan-like problems, flow in porous media (including saturated-unsaturated zones), mean curvature flow of convex curves in a plane, affine invariant multiscale shape analysis and further related free boundary problems. In the present paper, we use those ideas together with the techniques developed in [7] and [16] to obtain existence of a weak solution of (1.1)–(1.3) and to prove convergence of the suggested approximations.

In Sect. 2, we present approximation schemes for solving numerically the initial-boundary value problems (1.1)–(1.3). Section 3 is devoted to analysis of the existence of a weak solution and convergence of the approximations. In Sect. 4, we discuss numerical experiments with real and artificial images in order to show new features of the models.

2 Approximation schemes

In this Section we introduce the numerical technique for solving the problems (1.1)–(1.3) in all partial cases.

2.1 Approximation scheme (for the cases I and II):

Let \( n \in \mathbb{N} \) and \( \tau = \frac{T}{n} \) be the time-scale step. On each discrete time-scale level \( t_i = it \tau, i = 0, \ldots, n \) we look for the solution \( \theta_i \) \( (\theta_i \approx u_i, u_i \approx u(t_i, x)) \) of the regular elliptic problem

\[
\frac{\partial \theta_i}{\partial t} + \tau \nabla (g(\nabla G_\sigma \ast u_{i-1}) \nabla \theta_i) = \tau f(u_0 - u_{i-1}) \quad \text{in} \quad \Omega, \\
\partial_\sigma \theta_i = 0 \quad \text{on} \quad I \times \partial \Omega, 
\]

where \( \lambda_i \in L_\infty(\Omega) \) is the relaxation function connected with the \( \theta_i \) by the convergence condition

\[
\frac{1}{2} \tau^d \leq \lambda_i \\
\leq \min \left\{ \frac{b_n(x, u_{i-1} + \alpha(\theta_i - u_{i-1})) - b_n(x, u_{i-1})}{\theta_i - u_{i-1}}, K \right\}.
\]

where \( \alpha \in (0, 1) \) (\( \alpha \) close to 1), \( 0 < K \) (large), \( d \in (0, 1) \) are parameters of the method and

\[
b_n(x, s) := b(x, s) + \tau^d s.
\]

The function \( u_i \) is obtained by the algebraic correction

\[
b_n(x, u_i) := b_n(x, u_{i-1} + \lambda_i(\theta_i - u_{i-1})).
\]

In the case b we solve in every discrete time-scale step the equation

\[
\lambda_i(\theta_i - u_{i-1}) - \tau \nabla (g(\nabla G_\sigma \ast b_n(x, u_{i-1}) \nabla \theta_i)) = \tau f(u_0 - u_{i-1}).
\]
2.2 Approximation scheme (for the cases III and IV):

Let \( n \in \mathbb{N} \) and \( \tau = \frac{L}{n} \) be the time-scale step. On each discrete time-scale level \( t_i = i \tau, \quad i = 1, \ldots, n \) we look for the solution \( \theta_i (\theta_i \approx \beta(x, u_i), u_i \approx u(t_i, x)) \) of the regular elliptic problem

\[
\mu_i(\theta_i - \beta(x, u_i - 1)) - \tau \nabla(g(\nabla G_{\sigma} * u_{i-1})) = \tau f(u_0 - u_{i-1}) \quad \text{in } \Omega
\]

\[
\partial_i \beta_i = 0 \quad \text{on } I \times \partial\Omega, \quad (2.2.1b)
\]

where \( \mu_i \in L_\infty(\Omega) \) is the relaxation function, connected with \( \theta_i \) by the convergence condition

\[
1 - \frac{\tau^d}{2} \leq \mu_i \leq \min\left\{ \frac{\beta_{n}^{-1}(x, \beta_n(x, u_{i-1}) + \alpha(\theta_i - \beta(x, u_{i-1}))) - u_{i-1}}{\theta_i - \beta(x, u_{i-1})}, K \right\}, \quad (2.2.2)
\]

where \( \alpha \in (0, 1) \) (\( \alpha \) close to 1), \( 0 < K \) (large), \( d \in (0, 1) \) are parameters of the method and

\[
\beta_n(x, s) := \beta(x, s) + \tau^d s. \quad (2.2.3)
\]

The function \( u_i \) is obtained by the algebraic correction

\[
u_i := u_{i-1} + \mu_i(\theta_i - \beta(x, u_{i-1})). \quad (2.2.4)
\]

In the case a we solve in every discrete time-scale step the equation

\[
\mu_i(\theta_i - \beta(x, u_{i-1})) - \tau \nabla(g(\nabla G_{\sigma} * \theta_i)) = \tau f(u_0 - u_{i-1}). \quad (2.2.1a)
\]

Neither scheme 2.1, nor 2.2, is explicit with respect to \( \lambda_i, \theta_i, \mu_i, \theta_i \) respectively. However, they are powerful theoretical and practical techniques for solving the nonlinear degenerate parabolic equations. Due to the properties of \( \lambda_i, \mu_i \) and from the structure of the linear elliptic equations (2.1.1)-(2.2.1), the existence of the solution \( \theta_i \) is guaranteed by the theory of monotone operator’s equations (see e.g. [8]). By means of the relaxation functions we control the nonlinearities (degeneracies) in the equations. They correspond to \( b'(x, \xi), 1/b'(x, \xi), \) respectively, in some \( \xi \in (u_{i-1}, u_i) \). The range, given by the convergence conditions is rather large. The simplest possibility is to choose e.g. \( \lambda_i = \frac{1}{2} \tau^d = \mu_i \), and the couples (2.1.1)-(2.1.2), (2.2.1)-(2.2.2), respectively, are fulfilled. However, from the background of the method, a reasonable approximation has to force the relaxation functions to be close to the divergence quotients in the right-hand sides of the convergence conditions. For this purpose, we use iterations (described in Remarks 2.6, 2.7) similar to ones from [10, 12, 13, 24]. We can expect that some other semiimplicit ([31]), implicit ([27]) or explicit (widely used in image analysis) computational techniques can be successful for some special cases of (1.1)-(1.3). In the numerical implementation we can put also \( \alpha = 1 \) which simplifies the formulas. Then, in the optimal case (of the choice of \( \lambda_i, \mu_i \)) we actually have to solve the corresponding nonlinear elliptic problem; e.g., in the case (IV) it is

\[
\beta_n^{-1}(x, \theta_i) - \tau \nabla(g(\nabla G_{\sigma} * u_{i-1})) = u_{i-1} + \tau f(u_0 - u_{i-1}),
\]

which can be treated in an iterative way (see (2.7.1)-(2.7.4)).

Remark 2.3. In fact, both degeneracies (under the time derivative and in the divergence term) can be included using simultaneously \( b(x, s) \) and \( \beta(x, s) \) provided they are strictly increasing in \( s \). However, we can invert \( b(x, s) \) or \( \beta(x, s) \) and transform the problem into the form I or III. For completeness we present also the scheme for the general situation. It uses two relaxation functions balanced in the optimal way by two conditions. Convergence can be obtained by the similar arguments as in Sect. 3 (for technical details see also [13]).

On each discrete time-scale level \( t_i = i \tau, \quad i = 0, \ldots, n \) we look for the solution \( \theta_i (\theta_i \approx \beta(x, u_i), u_i \approx u(t_i, x)) \) of the regular elliptic problem

\[
\lambda_i(\theta_i - \beta(x, u_{i-1})) - \tau \nabla(g(\nabla G_{\sigma} * u_{i-1})) = \tau f(u_0 - u_{i-1})
\]

\[
\partial_i \theta_i = 0 \quad \text{on } I \times \partial\Omega, \quad (2.2.1b)
\]

where \( \lambda_i \in L_\infty(\Omega) \) satisfies

\[
1 - \frac{\tau^d}{2} \leq \lambda_i \leq \min\left\{ \frac{\beta_n(x, u_{i-1}) + \mu_i(\theta_i - \beta(x, u_{i-1})) - b_n(x, u_{i-1})}{\theta_i - \beta(x, u_{i-1})}, K \right\},
\]

with \( \mu_i \in L_\infty(\Omega) \)

\[
0 \leq \mu_i \leq \min\left\{ \frac{\beta_n^{-1}(x, \beta_n(x, u_{i-1}) + \alpha(\theta_i - \beta(x, u_{i-1}))) - u_{i-1}}{\theta_i - \beta(x, u_{i-1})}, K \right\},
\]

where \( \alpha \in (0, 1) \) (\( \alpha \) close to 1), \( 0 < K \) (large), \( d \in (0, 1) \) are parameters of the method. The function \( u_i \) is determined from

\[
b_n(x, u_i) := b_n(x, u_{i-1}) + \lambda_i(\theta_i - \beta(x, u_{i-1})).
\]

If \( b(x, s) \equiv s \), we can take \( \mu_i = \alpha \) to obtain Approximation scheme 2.1. When \( b(x, s) \equiv s \) we can take \( \lambda_i = \mu_i \) to obtain Approximation scheme 2.2.

In what follows, we understand the solutions of (2.2.1), and (2.2.2), respectively, in variational sense. It means, that we look for \( \theta_i \in V \), satisfying the following identities

\[
\lambda_i(\theta_i - u_{i-1}), v) + \tau (g(\nabla G_{\sigma} * u_{i-1})) \nabla \theta_i, \nabla v)
\]

\[
= \tau(f(u_0 - u_{i-1}), v) \quad (2.4a)
\]

\[
\lambda_i(\theta_i - u_{i-1}), v) + \tau (g(\nabla G_{\sigma} * b_n(x, u_{i-1})) \nabla \theta_i, \nabla v)
\]

\[
= \tau(f(u_0 - u_{i-1}), v) \quad (2.4b)
\]

\[
(\mu_i(\theta_i - \beta(x, u_{i-1})), v) + \tau (g(\nabla G_{\sigma} * \beta(x, u_{i-1})) \nabla \theta_i, \nabla v)
\]

\[
= \tau(f(u_0 - u_{i-1}), v), \quad (2.5a)
\]

\[
(\mu_i(\theta_i - \beta(x, u_{i-1})), v) + \tau (g(\nabla G_{\sigma} * u_{i-1}) \nabla \theta_i, \nabla v)
\]

\[
= \tau(f(u_0 - u_{i-1}), v), \quad (2.5b)
\]

for every \( v \in V \), where \( V \equiv W^1_2(\Omega) \) is Sobolev space.
Now we introduce iteration processes suitable for determination of relaxation functions, which are close to the difference quotients in the right hand sides of the convergence conditions.

**Remark 2.6.** The couple $\theta_i, \lambda_i$ simultaneously satisfying (2.1.1), (2.1.2) is determined iteratively by the following scheme

$$
\begin{align*}
(\lambda_{i,k-1} - (\theta_{i,k} - u_{i-1}), v) + \tau g([\nabla G_\sigma * u_{i-1}]) \nabla \theta_{i,k}, \nabla v) \\
= \tau (f(u_0 - u_{i-1}), v)
\end{align*}
(2.6.1a)

$$
\overline{\lambda}_{i,k} = \min \left\{ \beta_n(u_{i-1} + \alpha (\theta_{i,k} - u_{i-1})), \frac{\beta_n(u_{i-1} - \beta(x,u_{i-1})) - \beta_n(u_{i-1})}{\theta_{i,k} - u_{i-1}}, K \right\},
$$
(2.6.2)

$$
\lambda_{i,k} := \overline{\lambda}_{i,k}, \quad \text{for } 1 \leq k \leq k_0,
$$

$$
\lambda_{i,k} := \min(\overline{\lambda}_{i,k}, \lambda_{i,k-1}), \quad \text{for } k = k_0 + 1, \ldots,
$$
(2.6.3)

starting this process with

$$
\lambda_{i,0} = \min(\alpha(\beta_n)^{(0)}, (u_{i-1}), K).
$$
(2.6.4)

If $\theta_{i,k} = u_{i-1}$ then we put $\overline{\lambda}_{i,k} = \lambda_{i,0}.$

**Remark 2.7.** The couple $\theta_i, \mu_i$ simultaneously satisfying (2.2.1)–(2.2.2) is determined iteratively by the following scheme

$$
\begin{align*}
(\mu_{i,k-1} - (\theta_{i,k} - \beta(x,u_{i-1}))), v) + \tau g([\nabla G_\sigma * u_{i-1}]) \nabla \theta_{i,k}, \nabla v) \\
= \tau (f(u_0 - u_{i-1}), v)
\end{align*}
(2.7.1b)

$$
\overline{\mu}_{i,k} = \min \left\{ \beta_n^{-1}(x, \beta_n(x,u_{i-1}) + \alpha (\theta_{i,k} - \beta(x,u_{i-1})))), \frac{\beta_n^{-1}(x,u_{i-1}) - \beta_n^{-1}(x,u_{i-1})}{\theta_{i,k} - \beta(x,u_{i-1})}, K \right\},
$$
(2.7.2)

$$
\mu_{i,k} := \overline{\mu}_{i,k}, \quad \text{for } 1 \leq k \leq k_0,
$$

$$
\mu_{i,k} := \min(\overline{\mu}_{i,k}, \mu_{i,k-1}), \quad \text{for } k = k_0 + 1, \ldots,
$$
(2.7.3)

The iterations are starting with

$$
\mu_{i,0} = \min(\alpha / \beta_n^{(0)}, (x,u_{i-1}), K).
$$
(2.7.4)

Note that, if $\theta_{i,k} = \beta(x,u_{i-1})$ then we put $\overline{\mu}_{i,k} = \mu_{i,0}.$

By the previous constructions with $k_0 \geq 1$, the sequences $\{\mu_{i,k}\}, \{\lambda_{i,k}\}$ are forced to be monotone and hence convergent. The corresponding sequences of unknown functions $\theta_{i,k}$ in the elliptic equations converge in some functional spaces, too. The limit functions fulfill (2.1.1)–(2.1.2), (2.2.1)–(2.2.2), respectively. The proof of that fact can be obtained in the similar lines as in [12–14,24]. In practical implementations $k_0$ can be chosen in accordance with the shape of $\beta$, and $b$ (e.g. sufficiently large, if the numerical convergence of $\overline{\mu}_{i,k}, \overline{\lambda}_{i,k}$, respectively, is observed).

We denote $Q_T = \mathcal{T} \times \Omega$, the scalar product in $L_2(\Omega)$ by $(\cdot, \cdot)$ and duality between $V$ and $V^*$ by $\langle \cdot, \cdot \rangle$. We use symbols $| \cdot |, \| \cdot \|, \| \cdot \|_p, | \cdot |_{L^\infty}, | \cdot |_p$ for norms in $L_2(\Omega), V, V^*, L_\infty(\Omega)$ and $L_p(\Omega)$ (see e.g. [20]). By $\rightarrow, \rightharpoonup$ we mean the strong and weak convergence. By $C, C_1, \ldots$ we denote general (large) constants.

By means of $u_i, \theta_i$, determined by Approximation schemes 2.1, 2.2, in each discrete time-scale step, we construct Rothe’s functions

$$
u^{(n)}(t) = u_{i-1} + \frac{t - t_{i-1}}{\tau} (u_i - u_{i-1})
$$

for $t_{i-1} \leq t \leq t_i$, $i = 1, \ldots, n$

$$
\pi^{(n)}(t) = u_i, \quad \text{for } t_{i-1} < t \leq t_i, i = 1, \ldots, n \ , \ \pi^{(n)}(0) = u_0.
$$

$$
\theta^{(n)}(t) = \theta_{i-1} + \frac{t - t_{i-1}}{\tau} (\theta_i - \theta_{i-1})
$$

for $t_{i-1} \leq t \leq t_i$, $i = 1, \ldots, n$

$$
\gamma^{(n)}(t) = \theta_i, \quad \text{for } t_{i-1} < t \leq t_i, i = 1, \ldots, n, \ \gamma^{(n)}(0) = \theta_0.
$$
(2.8)

They are considered as the approximations of a weak solution of (1.1)–(1.3) defined by

**Definition 1.** A measurable function $u : Q_T \rightarrow \mathbb{R}$ is a weak solution of (1.1a)–(1.3) iff

(i) $\partial_t b(x,u) \in L_2(I,V^*)$, $\beta(x,u) \in L_2(I,V)$

(ii) $\int_I \beta(x,u) \partial_t v \geq - \int_I \beta(x,u) \partial_t v, \forall v \in V \cap L_\infty(Q_T)$ with $\beta(x,u) \in L_\infty(Q_T), v(T,x) = 0$

(iii) $\int_I \beta(x,u) \partial_t v \geq \int_I \beta(x,u) \partial_t v, \forall v \in V \cap L_\infty(Q_T), v(T,x) = 0$

In the similar way we define a weak solution of (1.1b)–(1.3).

It is clear how to understand this Definition in the partial cases I–IV.

The next Section concerns the convergence of Rothe’s functions (2.8) to corresponding weak solutions.

### 3 Convergence results

#### Case I.

In this Section, we assume some further technical assumptions:

(H5) $|f(s)| \leq C(1 + |s|)$

(H6) $C_1 s^2 - C_2 \leq b(x,s)s \leq C_3 + C_4 s^2$

(H7) $|b'_s(x + y, s) - b'_s(x, s)| \leq \omega(|y|)(1 + b'_s(x, s))$, where $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, $\omega(0) = 0$.

Let us denote

$$
\Phi_n(x,s) := \int_0^s b_n(x,z)dz
$$

and

$$
B_n(x,s) := b_n(x,s) - \int_0^s b_n(x,z)dz = b_n(x,s) - \Phi_n(x,s).
$$

Similarly we define $\Phi(x,s), B(x,s)$ replacing $b_n(x,s)$ by $b(x,s)$.

The Lax-Milgram theorem guarantees the existence of a solution $\theta_i \in V$ in (2.4a), (2.4b), respectively, for every $i = 1, \ldots, n$. 
Lemma 1. The a priori estimates

$$\max_{1 \leq i \leq n} \int_{\Omega} B_n(x, u_i) \leq C, \sum_{i=1}^n \|\theta_i\|^2 \tau \leq C,$$

$$\sum_{i=1}^n \int_{\Omega} \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1}))^2 \leq C$$

hold uniformly with respect to $n$.

Proof. Let us test (2.4a), (2.4b), respectively by $\theta_i$ and sum it up for $i = 1, \ldots, j$. Applying (2.1.4), Young’s inequality and the assumption (H5) we obtain

$$\sum_{i=1}^j (b_n(x, u_i) - b_n(x, u_{i-1}), 1_{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1})))$$

$$+ \sum_{i=1}^j (b_n(x, u_i) - b_n(x, u_{i-1}), u_i)$$

$$- \sum_{i=1}^j (b_n(x, u_i) - b_n(x, u_{i-1}), u_i - u_{i-1})$$

$$+ \sum_{i=1}^j \gamma_i(u_{i-1})|\nabla \theta_i|^2 \tau \leq C \sum_{i=1}^j (|u_{i-1}|^2 + |\theta_i|^2) + C_2,$$

(3.1)

where $\gamma_i(u_{i-1}) = g(|\nabla G_\sigma \ast u_{i-1}|) \geq 0$ ($\gamma_i(u_{i-1}) = g(|\nabla G_\sigma \ast b_n(x, u_{i-1})|) \geq 0$ in case b). From (2.1.2) and (2.1.4) we have

$$|b_n(x, u_i) - b_n(x, u_{i-1})| = |\lambda_i(\theta_i - u_{i-1})|$$

$$\leq |b_n(x, u_{i-1} + \alpha(\theta_i - u_{i-1})) - b_n(x, u_{i-1})|$$

and thus from the strict monotonicity of $b_n(x, s)$ in $s$ we obtain

$$|u_i - u_{i-1}| \leq \alpha(\theta_i - u_{i-1})$$

$$= \alpha \left| \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1})) \right|,$$

which implies the relation between the first and third term in (3.1)

$$\sum_{i=1}^j \left| \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1})) \right|$$

$$\leq \alpha \sum_{i=1}^j \int_{\Omega} \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1}))^2.$$

(3.2)

From the inequality

$$(u_i - u_{i-1})b_n(x, u_{i-1}) \leq \Phi_n(x, u_i) - \Phi_n(x, u_{i-1}),$$

we have

$$\sum_{i=1}^j (b_n(x, u_i) - b_n(x, u_{i-1}), u_i) \geq (b_n(x, u_j), u_j)$$

$$- (b_n(x, u_0), u_0) - \sum_{i=1}^j (u_i - u_{i-1}, b_n(x, u_{i-1}))$$

$$\geq (b_n(x, u_j), u_j) - (b_n(x, u_0), u_0)$$

$$- \sum_{i=1}^j \int_{\Omega} (\Phi_n(x, u_j) - \Phi_n(x, u_{i-1}))$$

$$= \int_{\Omega} B_n(x, u_j) - \int_{\Omega} B_n(x, u_0).$$

(3.3)

Due to (2.1.4) we obtain

$$|\theta_i|^2 \leq 2 \int_{\Omega} \frac{1}{\lambda_i^2} (b_n(x, u_i) - b_n(x, u_{i-1}))^2 + 2|u_{i-1}|^2$$

(3.4)

and by the asymptotical properties of $b(x, s)$ (see (H6)) we have the relation $B_n(x, s) \geq C_1 s^2 - C_2$ which implies

$$|u_i|^2 \leq C_1 \int_{\Omega} B_n(x, u_i) + C_2.$$

(3.5)

Applying (3.2)–(3.5) in (3.1) we obtain

$$\int_{\Omega} B_n(x, u_j) + (1 - \alpha) \sum_{i=1}^j \int_{\Omega} \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1}))^2$$

$$+ \sum_{i=1}^j \gamma_i(u_{i-1})|\nabla \theta_i|^2 \tau$$

$$\leq C_1 + C_2 \tau \sum_{i=1}^j \int_{\Omega} B_n(x, u_i)$$

$$+ C_3 \tau \sum_{i=1}^j \int_{\Omega} \frac{1}{\lambda_i^2} (b_n(x, u_i) - b_n(x, u_{i-1}))^2.$$

(3.6)

Then for $\tau \leq \tau_0$ due to the properties of $\lambda_i$ and by Gronwall’s argument we obtain

$$\max_{1 \leq i \leq n} \int_{\Omega} B_n(x, u_i) \leq C,$$

$$\sum_{i=1}^n \int_{\Omega} \frac{1}{\lambda_i} (b_n(x, u_i) - b_n(x, u_{i-1}))^2 \leq C.$$  

(3.7)

Using (H2), (3.7), Young’s inequality and asymptotical properties of $b$ we estimate

$$\left| \frac{\partial}{\partial x_i} G_{\sigma} \ast u_{i-1} \right| \leq \int_{\mathbb{R}^N} B_n^{\ast} (x, \frac{\partial}{\partial x_i} G_{\sigma}) + \int_{\mathbb{R}^N} B_n(x, u_{i-1})$$

$$\leq C_1 + C_2 \int_{\Omega} B_n(x, u_{i-1}) \leq C,$$

(3.8)
where $B^*_n(x,s)$ is a conjugate to the convex function $B_n(x,s) \geq 0$ (see [1, 19]). Thus
\[
\| \nabla B_n * u_{i+1} \| \leq C < \infty
\]
and so
\[
g((\nabla B_n * u_{i+1})) \geq v > 0, \quad i = 1, \ldots, n, \tag{3.9}
\]
which implies the second estimate stated in the lemma for the case a. In the case b, we use
\[
\left| \frac{\partial}{\partial x_i} G_\sigma * b_n(x, u_{i+1}) \right| \\
\leq \int \Phi_n \left( x, \frac{\partial}{\partial x_i} G_\sigma \right) + \int \Psi_n(b_n(x, u_{i+1})) \\
\leq C_3 + C_4 \int_\Omega B_n(x, u_{i+1}) \leq C,
\]
where $\Psi_n$ is the conjugate function to the potential $\Phi_n$ and we have used the equality $\Psi_n(b_n(x, u_{i+1})) = B_n(x, u_{i+1})$ which can be verified (see also [1]). So the proof is complete. □

**Consequence 1.**
\[
\sum_{i=1}^n |\theta_i - u_{i-1}|^2 \leq \mathcal{C} \tau^{-d}, \quad \sum_{i=1}^n |u_i - u_{i-1}|^2 \leq \mathcal{C} \tau^{-d}.
\]

Now, let us define the functions
\[
b_n(x, \overline{\nu}(x)) := b_n(x, u_{i+1}) + \frac{t - t_{i+1}}{\tau} (b_n(x, u_i) - b_n(x, u_{i-1})),
\]
for $t_{i+1} \leq t \leq t_i$, $i = 1, \ldots, n$.

**Lemma 2.** There exists $u \in L^2(I, V)$, with $\partial b(x, u) \in L^2(I, V^*)$ such that (in the sense of subsequences)
\[
\overline{\nu}(x) \rightarrow u \quad \text{a.e. in } Q_T, \\
\overline{\sigma}(x) \rightarrow u \quad \text{in } L^1(Q_T), \forall s < 2, \\
\overline{\nu}(x) \rightarrow u \quad \text{in } L^s(Q_T), \forall s < 2, \\
\overline{\sigma}(x) \rightarrow u \quad \text{in } L^2(I, V), \\
\partial b_n(x, \overline{\nu}(x)) \rightarrow \partial b(x, u) \quad \text{in } L^2(I, V^*).
\]

**Proof.** We use (2.1.4) in (2.4a) (in 2.4b, respectively) and sum it for $i = j + 1, \ldots, j + k$. Let us consider $v = (\theta_{j+k} - \theta_j) \tau$ as a test function and sum it again for $j = 0, \ldots, n - k$. Using the a priori estimates of Lemma 1 and Consequence 1 we successively obtain the estimate
\[
\sum_{j=0}^{n-k} (b_n(x, u_{j+k}) - b_n(x, u_j), \theta_{j+k} - \theta_j) \tau \leq C \mathcal{K} \tau, \tag{3.10}
\]
which can be rewritten into the form
\[
\int_0^{t-z} (b_n(x, \overline{\nu}(x) + z) - b_n(x, \overline{\nu}(x)), \overline{\nu}(t + z) - \overline{\nu}(t)) \leq C(z + \tau), \tag{3.11}
\]
where $k \tau \leq z \leq (k+1) \tau$. Using (2.1.4), (H6), (3.5) and the estimates of Lemma 1 and Consequence 1 we obtain
\[
\int_0^{t-z} (b_n(x, \overline{\nu}(x) + z) - b_n(x, \overline{\nu}(x)), \overline{\nu}(t + z) - \overline{\nu}(t)) \leq C_1 (z + \tau^{1-d/2}), \tag{3.12}
\]
Let us define
\[
\rho(x, z) := \min\{b'_j(x, s), 1\}
\]
and
\[
W(x, z) := \int_0^z \rho(x, z) dz.
\]
The function $W(x, z)$ is strictly monotone in $s$, and (3.12) implies
\[
\int_0^{t-z} (W(x, \overline{\nu}(x) + z) - W(x, \overline{\nu}(x)), \overline{\nu}(t + z) - \overline{\nu}(t)) \leq C_2 (z + \tau^{1-d/2}), \tag{3.13}
\]
for all $n \geq n_0$, $0 < \tau \leq z_0$. The second estimate of Lemma 1 gives us
\[
\int_{Q_T} (\overline{\nu}(x, t + y) - \overline{\nu}(x, t, x))^2 \leq C_3 (\omega(|y|) + \tau^{1-d/2}), \tag{3.14}
\]
where $|y| < y_0$ (see e.g. [26]). Then (2.1.4) and Consequence 1 imply
\[
\int_{Q_T} (\overline{\nu}(x, t + y) - \overline{\nu}(x, t, x))^2 \leq C_4 (\omega(|y|) + \tau^{1-d/2}), \tag{3.15}
\]
Then from the construction of $W(x, s)$ and from (H6), (H7) we obtain
\[
\int_{Q_T} (W(x + y, \overline{\nu}(x, t + y)) - W(x, \overline{\nu}(x), t, x))^2 \leq C_4 (\omega(|y|) + \tau^{1-d/2}), \tag{3.16}
\]
The compactness of $\{W(x, \overline{\nu}(x), t, x)\}_{n=1}^{\infty}$ in $L^2(Q_T)$ follows from (3.13) and (3.16). Since $W(x, s)$ is strictly increasing in $s$ we have (in the sense of subsequences) that $\overline{\nu}(n) \rightarrow u$ a.e. in $Q_T$ and moreover $\overline{\nu}(n) \rightarrow u$ in $L^1(Q_T), \forall s < 2$. Because of Consequence 1 we have also $\overline{\sigma}(n) \rightarrow u$ in $L^s(Q_T), \forall s < 2$. From that and from the second estimate of Lemma 1 we obtain $\overline{\sigma}(n) \rightarrow u$ in $L^2(I, V)$. By duality argument in (2.4a) (respectively (2.4b)), using (2.1.4) and a priori estimates of Lemma 1 and Consequence 1, we obtain
\[
\| \partial b_n(x, \overline{\nu}(n)) \|_{L^2(I, V^*)}^2 \leq C \int_0^t (1 + \| \overline{\sigma}(n) \|^2 + \| \overline{\nu}(n) \|^2) \leq C
\]
and hence \( \partial_t \bar{b}_n(x, \bar{\pi}^{(n)}) \to \chi \) in \( L^2(I, V^*) \) (in the sense of subsequences). Lemma 1 then implies

\[
\int_{Q_T} (\bar{b}_n(x, \bar{\pi}^{(n)}) - b_n(x, \bar{\pi}^{(n)}))^2 \leq C r^{1-d} \to 0, \quad \text{for } n \to \infty.
\]

Since \( b_n(x, s) \to b(x, s) \) locally uniformly in \( s \), \( \bar{\pi}^{(n)} \to u \) a.e. in \( Q_T \) and \( \int_Q b_n^2(x, \bar{\pi}^{(n)}) \leq C \) we have \( b_n(x, \bar{\pi}^{(n)}) \to b(x, u) \) in \( L_s(Q_T), \forall s < 2 \). Thus \( \partial_t \bar{b}_n(x, \bar{\pi}^{(n)}) \to \partial_t b(x, u) \).

Let us integrate (2.4a) respectively (2.4b) on \( (0, t), t \in I \).

We obtain

\[
\begin{align*}
&\int_{0}^{t} (\partial_t \bar{b}_n(x, \bar{\pi}^{(n)}), v) + \int_{0}^{t} (\bar{g}(|\nabla G_n \ast \bar{\pi}^{(n)}|) \nabla \bar{\theta}^{(n)}, \nabla v) \\
&= \int_{0}^{t} (f(u_0 - \bar{\pi}^{(n)}(x)), v), \forall v \in V. \quad (3.17) \\
&\int_{0}^{t} (\partial_t \bar{b}_n(x, \bar{\pi}^{(n)}), v) + \int_{0}^{t} (\bar{g}(|\nabla G_n \ast b_n(x, \bar{\pi}^{(n)})|) \nabla \bar{\theta}^{(n)}, \nabla v) \\
&= \int_{0}^{t} (f(u_0 - \bar{\pi}^{(n)}(x)), v), \forall v \in V. \quad (3.18)
\end{align*}
\]

We can take the limit for \( n \to \infty \) in previous identities and using the convergence results of Lemma 2 and the facts that

\[ g(|\nabla G_n \ast \bar{\pi}^{(n)}|) \to g(|\nabla G_n \ast u|), \quad g(|\nabla G_n \ast b_n(x, \bar{\pi}^{(n)})|) \to g(|\nabla G_n \ast b(x, u)|) \quad \text{a.e. in } Q_T \]

(whichever follow from Consequence 1 and Lemma 2) we have the following

**Theorem 1.** There exists variational solution \( u \) of the problems (1.1)–(1.3) in case I. Moreover \( \bar{\pi}^{(n)} \to u, \bar{\theta}^{(n)} \to u \) in \( L_s(Q_T), \forall s < 2 \), where \( \bar{\pi}^{(n)}, \bar{\theta}^{(n)} \) are the sequences obtained by Approximation scheme 2.1.

Using the results of [1] and [12], in both cases a and b, the stronger convergence result can be proved.

**Theorem 2.** Let \( \bar{\pi}^{(n)}, \bar{\theta}^{(n)} \) be the sequences obtained by Approximation scheme 2.1. Then

\[ \bar{\pi}^{(n)} \to u \quad \text{in } L^2(Q_T), \quad \bar{\theta}^{(n)} \to u \quad \text{in } L^2(I, V^*), \]

where \( u \) is a variational solution of the problems (1.1)–(1.3) in case I.

**Proof.** Let us test (3.17) (respectively (3.18)) by \( v = \bar{\theta}^{(n)} - u \), where \( u \) is a variational solution. We have (due to the per partes formula – see [12], Lemma 3.25 or [1], Lemma 1.5)

\[
\int_{0}^{t} (\partial_t \bar{b}_n(x, \bar{\pi}^{(n)}), u) + \int_{0}^{t} < \partial_t b(x, u), u > \\
= \int_{\Omega} B(x, u(t)) - \int_{\Omega} B(x, u(0)). \quad (3.19)
\]

Since \( b_n(x, s) \to B(x, s) \) locally uniformly for bounded \( s \), by Fatou’s argument and using Lemma 2 and (3.3) we obtain

\[
\lim_{n \to \infty} \int_{0}^{t} (\partial_t \bar{b}_n(x, \bar{\pi}^{(n)}), \bar{\theta}^{(n)}) \geq \int_{\Omega} B_n(x, \bar{\pi}^{(n)}(t)) - \int_{\Omega} B_n(x, u(0)) \\
\geq \int_{\Omega} B(x, u(t)) - \int_{\Omega} B(x, u(0)).
\]

Thus

\[
\int_{0}^{t} (\partial_t \bar{b}_n(x, \bar{\pi}^{(n)}), \bar{\theta}^{(n)} - u) \geq o(1), \quad (3.20)
\]

where the Landau symbol \( o(1) \) denotes a term \( c_n \to 0 \) for \( n \to \infty \). From the growth properties of \( f \) and from the fact that \( \bar{\theta}^{(n)} \to u \) in \( L^s(Q_T) \) we have

\[
\int_{0}^{t} (g(|\nabla G_n \ast \bar{\pi}^{(n)}|) \nabla \bar{\theta}^{(n)}, \nabla (\bar{\theta}^{(n)} - u)) = o(1). \quad (3.21)
\]

Since \( 0 < v \leq g(|\nabla G_n \ast \bar{\pi}^{(n)}|) \to g(|\nabla G_n \ast u|) \quad \text{a.e. in } Q_T \) and from \( \bar{\theta}^{(n)} \to u \) in \( L^2(I, V) \) we have

\[
\int_{0}^{t} (g(|\nabla G_n \ast \bar{\pi}^{(n)}|) \nabla u, \nabla (\bar{\theta}^{(n)} - u)) = o(1). \quad (3.22)
\]

From (3.20)–(3.22) we deduce

\[
\int_{0}^{t} |\nabla (\bar{\theta}^{(n)} - u)|^2 \to 0 \quad \text{for } n \to \infty,
\]

so \( \nabla \bar{\theta}^{(n)} \to \nabla u \) in \( L^2(Q_T) \). The same result holds also for the case b. To prove \( \bar{\theta}^{(n)} \to u \) in \( L^2(Q_T) \) we use the following argument. Let us take

\[ c_n = \frac{1}{|Q_T|} \int_{Q_T} \bar{\theta}^{(n)} \]

and construct

\[ \psi^{(n)} := \bar{\theta}^{(n)} - c_n. \]

Since \( \nabla \psi^{(n)} \to \nabla u \) in \( L^2(Q_T) \) and \( \int_{Q_T} \psi^{(n)} = 0 \) we have that \( \psi^{(n)} \) converges in \( L^2(Q_T) \). Since \( c_n \) is bounded \( (\int_{Q_T} |\bar{\theta}^{(n)}|^2 \leq C) \) we can assume \( c_n \to c \) (up to a subsequence) and hence \( \bar{\theta}^{(n)} = \psi^{(n)} + c_n \) converges in \( L^2(Q_T) \). Thus \( \bar{\theta}^{(n)} \to u \) in \( L^2(Q_T) \) since \( \bar{\theta}^{(n)} \to u \) in \( L^2(Q_T) \). Then also \( \bar{\pi}^{(n)} \to u \) in \( L^2(Q_T) \) and the proof is complete.

In the analysis of the next three cases we will be more brief, because the ideas are similar to the ones used in the previous part of this Section. We will concentrate only to the main differences.

**Case II.** In this case we consider the right hand side of the equations in the form \( f \equiv f(b(x, u_0) - b(x, u)) \) and correspondingly in Approximation scheme 2.1. Instead of (H7) we assume
Using (H8) we also obtain
\[
\int \int (b_n(x, \bar{u}^{(n)}_t (t + z) - b_n(x, \bar{u}^{(n)}_t (t), x))^2 \leq C_1(z + \tau^{(1-d)/2}) .
\]

(3.23)

Using (H8) we also obtain
\[
\int (b_n(x + y, \bar{u}^{(n)}_t (t, x + y) - b_n(x, \bar{u}^{(n)}_t (t), x))^2 \leq C_2(\omega(y)) + \tau^{(1-d)/2} ,
\]
from which and (3.23) follow the compactness of \(b_n(x, \bar{u}^{(n)}_t )\) in \(L^2(Q_T)\). The asymptotical properties of \(b\) imply that \(\int Q_T |\bar{u}^{(n)}|^2 \leq C\) and hence \(\bar{u}^{(n)} \rightarrow u\) in \(L^2(Q_T)\). From the compactness result we have \(b_n(x, \bar{u}^{(n)}_t ) \rightarrow \overline{x}\) in \(L^2(Q_T)\). Then
\[
0 \leq \int (\xi - b(x, v), u - v) \rightarrow \int (\xi - b(x, v), u - v) \geq 0
\]
for every \(v \in L^2(Q_T)\) because of the monotonicity of \(b_n\). Hence for \(v = u \pm \epsilon w\) and \(\epsilon \rightarrow 0\) we obtain \(\xi = b(x, u)\). In the similar way as in the proof of Lemma 2 we obtain \(\partial b_n(x, \bar{u}^{(n)}_t ) \rightarrow \partial b(x, u)\) in \(L^2(I, V^*)\). Due to a priori estimate \(\int J \phi(u(t))^2 \leq C\) we obtain \(\bar{u}^{(n)}_t \rightarrow u\) in \(L^2(I, V^*)\) since \(\int J \left|\bar{u}^{(n)}_t - \bar{u}^{(n)}_t\right|^2 \rightarrow 0\). The previous considerations allow us to take limit \(n \rightarrow \infty\) in (3.17), (3.18), respectively, and we conclude that \(u\) is a weak solution of (1.1)–(1.3). Then we can proceed in the similar lines as in the proof of Theorem 2 to obtain

**Theorem 3.** There exists variational solution \(u\) of the problems (1.1)–(1.3) in case II. Let \(\bar{u}^{(n)}_t\), \(\bar{u}^{(n)}_t\) be the sequences obtained by Approximation scheme 2.1. Then
\[
\bar{u}^{(n)}_t \rightarrow u \text{ in } L^2(Q_T), \quad \bar{u}^{(n)}_t \rightarrow u \text{ in } L^2(I, V).
\]

**Cases III and IV.**

We follow the ideas from [10, 13, 15] and we only sketch the results. Instead of (H6)–(H8) we assume
\[
\begin{align*}
(\text{H9}) \quad & C_1 y^2 - C_2 \leq \beta(x, s) s \leq C_3 + C_4 s^2 , \\
(\text{H10}) \quad & |\beta'_y (x, s) - \beta'_y (x, s)| \leq \omega(|y|)(1 + |\beta'_y (x, s)|), \quad \text{where } \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ is continuous}, \omega(0) = 0.
\end{align*}
\]

In case IV, we consider the right hand side of the equations in the form \(\omega \equiv f(\beta(x, u_0) - \beta(x, u))\) and correspondingly in Approximation scheme 2.2.

We denote
\[
\Phi_\beta := \int_0^t \beta(x, z) d\tau .
\]

Using the similar access as in [10] and [15] and the conjugate function to potential \(\Phi_\beta\) similarly as in the proof of Lemma 1 (to estimate the gaussian gradient term) we obtain

**Lemma 3.** The a priori estimates
\[
\max_{1 \leq i \leq n} \int \omega_i(x, u_i) \leq C, \quad \sum_{i=1}^n \|\theta_i\|^2 \leq C , \\
\sum_{i=1}^n |u_i - u_i-1|^2 \leq C
\]
hold uniformly with respect to \(n\).

Then we have also

**Consequence 2.**
\[
\sum_{i=1}^n |\theta_i - \beta(x, u_i)|^2 \leq C \tau^{-d} .
\]

By the similar ideas as in the proof of Lemma 2, [10, 15] and [13] we obtain the compactness of \(\bar{u}^{(n)}_t\) in \(L^2(Q_T)\) and then

**Lemma 4.** There exists \(u \in L^2(Q_T)\) with \(\beta(x, u) \in L^2(I, V^*)\) such that \(\bar{u}^{(n)}_t \rightarrow u\) in \(L^2(Q_T)\), \(\bar{u}^{(n)}_t \rightarrow \beta(x, u)\) in \(L^2(I, V)\), \(\partial u_n \rightarrow \partial u\) in \(L^2(I, V^*)\).

Now we can use (see [1, 15]) the relations
\[
\lim_{n \rightarrow \infty} \int_0^t <\partial u_n^{(n)}, \bar{u}^{(n)}_t > = \int_0^t \Phi_\beta(x, u(t)) - \int_0^t \Phi_\beta(x, u_0) , \\
\int_0^t <\partial u_n^{(n)}, \beta(x, u) > = \int_0^t \Phi_\beta(x, u(t)) - \int_0^t \Phi_\beta(x, u_0)
\]
to prove

**Theorem 4.** There exist weak solution of the problems (1.1)–(1.3) in cases III, IV. Let \(\bar{u}^{(n)}_t\), \(\bar{u}^{(n)}_t\) be the sequences obtained by Approximation scheme 2.2. Then
\[
\bar{u}^{(n)}_t \rightarrow u \text{ in } L^2(Q_T), \quad \bar{u}^{(n)}_t \rightarrow \beta(x, u) \text{ in } L^2(I, V).
\]

**Remark 3.11.** The same convergence results can be obtained when we use the full discretization scheme (also in space) using projection of elliptic problems (2.4)–(2.5) to finite dimensional finite elements spaces \(V_h \subset V\) with \(V_h \rightarrow V(\lambda \rightarrow 0)\) in canonical sense.

**4 Discussion on numerical experiments**

In this section we present numerical experiments demonstrating features of the models (1.1)–(1.3). We compare the results with the multiscale analysis based on the classical anisotropic diffusion equations ([7, 29]).

For computations, we use Approximation schemes 2.1, 2.2 together with the several iterations from Remarks 2.6, 2.7. For the full (scale and space) discretization of the equations one can use either the finite element method ([4, 5]) or the finite volume technique ([25, 28]). The numerical
experiments described in this section have been computed using the finite volume spatial discretization of the linear elliptic equations (2.6.1),(2.7.1). The spatial grid is given naturally by the pixel structure of the image. The spatial discretization step for the finite volume method is given as $1/r_1$ with $r_1$ number of pixels in vertical direction. The Gaussian kernel has been used in the convolution term. Because the discrete image is given by constant values on small squares (pixels), the convolution is reduced to a weighted mean value with weights given by the Gauss function. Since for $\sigma$ small the weights for pixels with bigger distance are machine zeroes, the averaging is realized only in some bounded neighbourhood. In all presented experiments we use $\sigma$ such that $7 \times 7$ pixels influence the value in the central pixel. As the Perona-Malik function we use $g(s) = 1/(1 + s^2)$.

Fig. 1. Restoration of the noisy image (left) by the anisotropic diffusion (middle) and by the anisotropic diffusion coupled with the slow diffusion effect (right)

Fig. 2. Restoration of the noisy image (left) by the anisotropic diffusion (middle) and by the anisotropic diffusion coupled with the slow diffusion effect (right)

Fig. 3. Difference in reconstruction of the detail of Boticelli's painting 'Primavera'. The restoration of the greylevel scan (left) by the anisotropic diffusion (middle) in comparison with the anisotropic diffusion coupled with the slow diffusion (right)
The first, second and fourth processed images consist of 200 × 200 pixels, the third and fifth ones have the size 570 × 350 pixels. In Figs. 1–4 we plot the noisy originals (on the left), the results of Catté, Lions, Morel and Coll anisotropic diffusion (in the middle) and the result of application of the slow and fast diffusion effects given by the models (1.1)–(1.3) (on the right). In the last Fig. 5 we present the result of processing of the color image by the anisotropic diffusion coupled with the slow diffusion effect.

Modelling the slow diffusion effect we consider the case IV. In addition to CLMC-parameters the function $\beta$ which is constant for some range of greylevels and linear in complement is used. In presented experiments we consider function $\beta(x,s) = 0$ for $s \leq a$, $\beta(x,s) = s - a$ for $s > a$ with some constant $a$ between 0 and 1 (before computations, the image intensity is transformed from integers between 0 and 255 into the real interval $[0, 1]$). In Fig. 1 we present the difference in processing of the initial noisy image (left) by the anisotropic diffusion (middle) and by the anisotropic diffusion coupled with the slow diffusion effect (right) after ten discrete scale steps with step $\tau = 0.001$ in both cases. In case of the slow diffusion, the additional parameters of the method are $a = 0.5$ (in the definition of $\beta$ function), $K = 10^6$, $d = 0.9$, $\alpha = 0.99$. The choice of $\beta$ stops the diffusion where we want to keep some fine details in the image (otherwise destroyed by the usual anisotropic diffusion). The same computational parameters have been used in the experiment documented in Fig. 2, we just start...
from different initial condition. In Fig. 3 we present reconstruction of the greylevel scan of the detail of Botticelli’s painting Primavera (left image) by the anisotropic diffusion (middle image) and by the anisotropic diffusion accompanied by the slow diffusion (image on the right). In both cases we plot the results after ten discrete scale steps with step \( \tau = 0.001 \). In the case of slow diffusion effect we use parameters \( a = 0.39, K = 10^6, d = 0.9, \alpha = 0.99 \). Using such choice of \( \beta \), the face is selectively smoothed and the details around are conserved.

In spite of this, if the smoothing of the large structural noise is desirable, the fast diffusion effect can be used. In the experiment presented in Fig. 4, the function \( b(x, s) = 0 \) for \( s \leq 0.5 \), \( b(x, s) = s - 0.5 \) for \( s > 0.5 \), i.e. we consider the case II. The scaling versions of the initial image (left) given by the anisotropic diffusion (middle) and coupled with fast diffusion smoothing (right) are plotted at scale 10\( \tau \). Further parameters were \( K = 10^6, d = 0.5, \alpha = 0.99 \).

In Fig. 5 we present application of the anisotropic diffusion coupled with the slow diffusion effect to processing of the RGB color image. We again consider Flora’s face detail as the initial condition (left part of Fig. 5). Before processing we divide the color image into red, green and blue channels. The model (1.1)–(1.3) in the case IV is applied to each channel independently using different choices of the parameters. Then the results of channel processing are put together in order to get multiscale version of the color original. The result presented in the right part of Fig. 5 has been computed using \( K = 10^6, d = 0.9, \alpha = 0.99, \tau = 0.001 \) and with different \( \beta \) functions and number of scale steps in the channels. In the red channel we use \( a = 0.5 \) and 5 scale steps, in the green channel \( a = 0.3 \) and 7 scale steps and in the blue channel \( a = 0.15 \) and 10 scale steps. Let us note that one can consider also a kind of synchronization of channels processing which will lead to a generalization of (1.1)–(1.3) to degenerate parabolic systems which can be interesting subject for further study.

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