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A Congruence Relation for the Wiener Index of Graphs with a Tree–Like Structure

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Abstract

The Wiener index, defined as the sum of distances between all unordered pairs of vertices in a graph, is one of the most popular molecular descriptors. Congruence relations for the Wiener index for specific families of trees were studied by several authors. Namely, in \cite{GutmanRouvray90} it is shown that Wiener indices of any two trees on the same number of vertices and with 1-factor are congruent modulo 4. Recently, the author of \cite{Lin13} generalized this result to trees with path factors and \cite{GutmanXuLiu} generalized it to even much larger class of graphs. We continue this work by establishing congruence relations for various large families of graphs with a tree-like structure, whose “vertices” and “edges” represent some graphs of prescribed type and congruence.

1 Introduction

All graphs considered in this paper are finite, simple and connected. Let $G$ be a graph. Its vertex and edge sets are denoted by $V(G)$ and $E(G)$, respectively. Let $u, v \in V(G)$. 

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The length of a shortest path in $G$ between $u$ and $v$ is denoted by $d_G(u, v)$ (or by $d(u, v)$ when no confusion is likely).

The oldest topological index related to molecular branching is the Wiener index [14], which was introduced in 1947. It is defined as the sum of distances between all (unordered) pairs of vertices of $G$,

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u, v).$$

The Wiener index plays an important role in organic chemistry and has been extensively studied. At first it was used for predicting the boiling point of paraffins [13], but later strong correlation between the Wiener index and the chemical properties of a compound was found. Nowadays this index is a tool used for preliminary screening of drug molecules [1]. The Wiener index also predicts binding energy of protein-ligand complex at a preliminary stage. Besides applications in chemistry it was studied also from a purely graph-theoretical point of view. More details can be found in some of the many surveys [2, 4, 7, 11, 15].

In [14], Wiener proved that for a tree $T$

$$W(T) = \sum_{e=ij \in E(T)} n_e(i) n_e(j),$$

where $n_e(i)$ and $n_e(j)$ are the orders of the components of $T - ij$. In analogy to this result, we have the following vertex version [9].

**Theorem 1.** Let $T$ be a tree on $n$ vertices. Then

$$W(T) = \sum_{v \in V(T)} \sum_{1 \leq i < j \leq p} n(T_i) n(T_j) + \binom{n}{2},$$

where $T_1, T_2, \ldots, T_p$ are the components of $T - v$.

In addition, Gutman and Škrekovski [9] generalized this result by proving that for a connected graph $G$, $W(G) = \sum_{v \in V(G)} B(v) + \binom{n}{2}$. This formula shows that the Wiener index is related to the betweenness centrality $B(v)$ of the vertices $v \in V(G)$, a quantity used in theory of social networks, which measures the number of times a vertex lies on a shortest path between two other vertices.

If a tree contains a small number of branching vertices (i.e., vertices of degree at least three) it is suitable to apply the theorem of Doyle and Graver [3]. Some applications of this formula were elaborated in [5, 10].
Theorem 2 (Doyle and Graver). Let $T$ be a tree on $n$ vertices. Then

$$W(T) = \binom{n + 1}{3} - \sum_{v \in V(T)} \sum_{1 \leq i < j < k \leq p} n(T_i) n(T_j) n(T_k),$$

where $T_1, T_2, \ldots, T_p$ are the components of $T - v$.

It was of interest to several authors to obtain congruence relations for the Wiener index. The first result of this kind was proved by Gutman and Rouvray [8]. They established the congruence relation for the Wiener index of trees with perfect matchings.

Theorem 3 (Gutman and Rouvray). Let $T$ and $T'$ be two trees on the same number of vertices. If both $T$ and $T'$ have perfect matchings, then $W(T) \equiv W(T') \pmod{4}$.

A segment of a tree is its path-subtree whose terminal vertices are branching or pendant vertices. Dobrynin, Entringer and Gutman [2] obtained a congruence relation for the Wiener index in the class of $k$-proportional trees. Trees of this class have the same order, the same number of segments, and the lengths of all segments are proportional to the coefficient $k$. More precisely, if $l_1, l_2, \ldots, l_m$ and $l'_1, l'_2, \ldots, l'_m$ are the lengths of the segments of the trees $T$ and $T'$, respectively, then $l_i = kr_i$ and $l'_i = kr'_i$, $1 \leq i \leq m$, where $r_i$ and $r'_i$ are positive integers.

Theorem 4 (Dobrynin, Entringer and Gutman). Let $T$ and $T'$ be two $k$-proportional trees. Then

$$W(T) \equiv W(T') \pmod{k^3}.$$
Theorem 5 (Lin). If $T$ and $T'$ are two trees on the same number of vertices, both with $P_r$-factors, then

$$W(T) \equiv W(T') \pmod{r} \quad \text{for odd } r,$$

and

$$W(T) \equiv W(T') \pmod{2r} \quad \text{for even } r.$$

Recently Gutman, Xu and Liu [6] showed that the first congruence in the above result is a special case of a much more general result on the Szeged index and as a consequence for the Wiener index they obtained the following result (compare with Corollary 13).

Theorem 6 (Gutman, Xu and Liu). Let $\Gamma_0$ be the union of connected graphs $G_1, G_2, \ldots, G_p$, $p \geq 2$, each of order $r \geq 2$, all blocks of which are complete graphs. Denote by $\Gamma$ a graph obtained by adding $p - 1$ edges to $\Gamma_0$ so that the resulting graph is connected. Then

$$W(\Gamma) \equiv \sum_{i=1}^{n} W(G_i) \pmod{r}.$$

Here we give a straightforward argument for both congruences from Theorem 5, and in the next two sections we describe how the idea of that proof can be generalized to show that the Wiener index is in the same congruence class modulo $r$ (or $2r$ when $r$ is even) for larger families $\mathcal{G} = \mathcal{G}(\mathcal{H}, \mathcal{F})$ of graphs with a tree-like structure, whose “vertices” are graphs from a given set $\mathcal{H}$ all of congruent order, and “edges” are from a given set of graphs $\mathcal{F}$ also all of congruent order (see the beginning of Section 2 for precise definitions of these notions).

Proof of Theorem 5. Let $T$ be a tree on $n$ vertices having a $P_r$-factor. For an edge $uv \in E(T)$, denote by $T_u$ and $T_v$ the connected components of $T - \{uv\}$, such that $u \in V(T_u)$ and $v \in V(T_v)$.

Let $ij$ be an edge connecting vertices from different copies of $P_r$ of the $P_r$-factor. Then $|V(T_i)| = ar$ and $|V(T_j)| = br$ for some integers $a$ and $b$ such that $(a + b)r = n$. Let $T'$ be a tree obtained from $T$ by replacing the edge $ij$ by an edge $i'j$, where $i'$ is a neighbour of $i$ in $T_i$. Note that $W(T) = W(T_i) + W(T_j) + \sum d_T(x, y)$, and $W(T') = W(T_i) + W(T_j) + \sum d_T'(x, y)$, where the sum in both cases is taken over all $x \in V(T_i)$ and $y \in V(T_j)$. Observe that for $x \in T_i$ the difference between $d_T(x, i')$ and $d_T(x, i)$ is either 1 or $-1$. Let $s$ be the number of vertices in $V(T_i)$ for which $d_T(x, i') - d_T(x, i) = 1$. We
\[ W(T') - W(T) = \sum_x \sum_y \left( d_{T'}(x, y) - d_T(x, y) \right) = \sum_x \sum_y \left( d_{T'}(x, i') - d_T(x, i) \right) \]
\[ = |T_j| \sum_x \left( d_{T'}(x, i') - d_T(x, i) \right) \]
\[ = bs - (ar - s)) = 2brs - abr^2, \]

where \( x \in V(T_i) \) and \( y \in V(T_j) \).

Hence \( W(T') \equiv W(T) \pmod{r} \), and clearly \( W(T') \equiv W(T) \pmod{2r} \) in the case when \( r \) is even. It is left to the reader to observe that by replacing edges sequentially as described above, one can always construct a path on \( n \) vertices which is in the same congruence class modulo \( r \) as \( T \). Hence the result follows. \( \square \)

The generalizations of the above proof are given in Theorems 10 and 17. From them we infer some interesting consequences, for example Corollary 11, where \( H \) is composed of cycles, and \( F \) of paths. In Corollaries 12 and 18, \( H \) is a set of trees and \( F \) is composed of edges (paths of length 1). When all graphs of \( H \) are isomorphic to a given tree \( T \) and \( F \) contains only edges, (i.e., we consider graphs with \( T \)-factors), we get Corollaries 14 and 20. And, as a particular case, when this prescribed tree \( T \) is a path, we obtain Lin’s Theorem.

2 Congruence modulo \( r \)

In this section we generalize the first part of Theorem 5. However, the notation introduced here will be used also in the next section.

Let \( r \) and \( t \) be integers, \( r \geq 2 \) and \( 0 \leq t < r \). We will choose three things. First, let \( H = \{H_1, H_2, \ldots, H_\ell\} \) be a set of connected graphs, such that for all \( i \), \( 1 \leq i \leq \ell \), we have \( |V(H_i)| \equiv r - t \pmod{r} \). Second, let \( F = \{F_1, F_2, \ldots, F_{\ell-1}\} \) be a set of connected graphs, such that for all \( j \), \( 1 \leq j \leq \ell - 1 \), we have \( |V(F_j)| \equiv t + 2 \pmod{r} \). Third, for every \( F_j \), choose vertices \( v^1_j, v^2_j \in V(F_j) \) (we remark that chosen vertices \( v^1_j \) and \( v^2_j \) are not necessarily distinct). Now, when these three items are chosen, namely \( H, F \) and pairs of vertices in graphs of \( F \), identify the vertices \( v^1_j, v^2_j \in V(F_j) \) (we remark that chosen vertices \( v^1_j \) and \( v^2_j \) are not necessarily distinct). Denote by \( G = G(H, F) \) the class of those graphs obtained by this identification process,
which are connected.

In Fig. 1 we have one graph $G$ of $\mathcal{G}$ for given parameters $r$, $t$ and $\ell$, and for given sets $\mathcal{H}$, $\mathcal{F}$ and $\{v_j^1, v_j^2\}_{j=1}^{\ell-1}$. The vertices of $H_j$’s are depicted by full circles in Fig. 1 and the edges of $H_i$’s are thick. Observe that $|V(H_1)| \equiv |V(H_2)| \equiv |V(H_3)| \equiv |V(H_4)| \equiv 7 - 3 \pmod{7}$ and $|V(F_1)| \equiv |V(F_2)| \equiv |V(F_3)| \equiv 3 + 2 \pmod{7}$.

![Figure 1: A graph of $G$ for $r = 7$, $t = 3$, $\ell = 4$ and given $H_i$’s, $F_j$’s and $v^{h_i}$’s.](image)

In the definition above, there are $\ell$ graphs in $\mathcal{H}$, $\ell - 1$ graphs in $\mathcal{F}$, and each graph of $\mathcal{F}$ connects two graphs of $\mathcal{H}$. Since the resulting structures (graphs in $\mathcal{G}$) are connected, if we contract every $H_i$ to a single vertex and we consider $F_j$’s as edges joining pairs of these contracted vertices, then the resulting graph is a tree. In this way, $H_1, H_2, \ldots, H_\ell$ can be regarded as supernodes, $F_1, F_2, \ldots, F_{\ell-1}$ as supereges, and the corresponding graph is called an associated supergraph. For example, for the graph depicted in Fig. 1 the associated supergraph is the claw $K_{1,3}$ with central supernode $H_2$ and pendant supernodes $H_1, H_3$ and $H_4$.

Now we define a representative graph $\Gamma_G$ for $\mathcal{G}$ (recall that in $\Gamma_G$ we fixed $\mathcal{H}$, $\mathcal{F}$ and $\{v_j^1, v_j^2\}_{j=1}^{\ell-1}$). For every $i$, $1 \leq i \leq \ell$, choose one vertex of $V(H_i)$ and denote it by $u_i$. Then $\Gamma_G$ is obtained from $H_1 \cup \cdots \cup H_\ell \cup F_1 \cup \cdots \cup F_{\ell-1}$ by identifying $v_i^1$ with $u_i$ and $v_i^2$ with $u_{i+1}$. Observe that the associated supergraph for $\Gamma_G$ is a path of length $\ell - 1$ with the ordering of supernodes $(H_1, H_2, \ldots, H_\ell)$. Moreover, $H_i$ is connected with $H_{i+1}$ by the supedge $F_i$, $1 \leq i \leq \ell - 1$.

Roughly speaking, our main results state that it does not matter how we provide the identification, the Wiener index of all graphs in $\mathcal{G}$ is in the same congruence class modulo $r$. We obtain this by showing that every graph from $\mathcal{G}$ is in the same congruence class modulo $r$ as $\Gamma_G$.

First, we will consider the following operation: Let $G \in \mathcal{G}$. Choose $j$, $1 \leq j \leq \ell - 1$,
and $k$, $1 \leq k \leq 2$. Assume that the vertex $v_j^k$ of $F_j$ was identified with a vertex, say $u$, of $H_a$. So detach $v_j^k$ from this vertex. This will disconnect $G$ to two components, say $G^1$ and $G^2$. Assume that $u \in V(G^1)$. Choose $u' \in V(H_1) \cup \cdots \cup V(H_{\ell})$ such that $u' \in V(G^1)$, identify $v_j^k$ with $u'$ and denote the resulting graph by $G'$. If $u' \in V(H_b)$, then we denote this operation by $[H_a, F_j, H_b]$.

We show that the operation defined above preserves the modularity by $r$.

**Lemma 7.** Let $G, G' \in \mathcal{G}$, where $G'$ was obtained from $G$ by the operation $[H_a, F_j, H_b]$ as described above. Then $W(G) \equiv W(G') \pmod{r}$.

**Proof.** Due to the tree structure of $G$, the graphs $G^1$ and $G^2$ are connected and satisfy

$$|V(G^1)| \equiv r - t \pmod{r} \quad \text{and} \quad |V(G^2 - v_j^k)| \equiv 0 \pmod{r}.$$ 

However, the graph $G^2 - v_j^k$ may be disconnected. For this reason, denote $W_w(G^2) = \sum d_{G^2}(u, v)$, where the sum is taken over all two-element subsets of $G^2 - w$. Since both $V(G)$ and $V(G')$ are disjoint unions of $V(G^1)$ and $V(G^2 - v_j^k)$, we have

$$W(G) = W(G^1) + W_{-v_j^k}(G^2) + \sum d_G(x, y) \quad \text{and},$$

$$W(G') = W(G^1) + W_{-v_j^k}(G^2) + \sum d_{G'}(x, y), \quad (1)$$

where the sums are taken over all $x \in V(G^1)$ and $y \in V(G^2 - v_j^k)$. Here all the shortest paths from $x$ to $y$ must contain $v_j^k$ in both $G$ and $G'$. Therefore, for a fixed $x \in V(G^1)$ there is $d_x \in \mathbb{Z}$ (where $d_x = d_{G'}(x, u') - d_G(x, u)$), such that for an arbitrary $y \in V(G^2 - v_j^k)$ we have $d_{G'}(x, y) = d_G(x, y) + d_x$. Since $|V(G^2 - v_j^k)| \equiv 0 \pmod{r}$, we have

$$\sum_x \left( \sum_y d_{G'}(x, y) \right) = \sum_x \left( \sum_y \left( d_G(x, y) + d_x \right) \right) \equiv \sum_x \left( \sum_y d_G(x, y) \right) \pmod{r},$$

where $x \in V(G^1)$ and $y \in V(G^2 - v_j^k)$. By (1), $W(G) \equiv W(G') \pmod{r}$. \hfill \Box

Let $G \in \mathcal{G}$ and $1 \leq j \leq \ell - 1$. Assume that in the process of obtaining $G$, $v_j^1$ was identified with $u_a \in V(H_a)$ and $v_j^2$ was identified with $u_b \in V(H_b)$. Now detach $u_a$ from $v_j^1$ and detach $u_b$ from $v_j^2$, identify $u_a$ with $v_j^2$ and identify $u_b$ with $v_j^1$, and denote the resulting graph by $G'$. We say that $G'$ was obtained from $G$ by reversing the superedge $F_j$. 
Let $G \in \mathcal{G}$. In the next lemma we show that by a sequence of operations $[H_a, F_j, H_b]$, the graph $G$ can be transformed to a graph $G^\times$ which is either identical with $\Gamma_G$, or can be obtained from $\Gamma_G$ by reversing of some superedges $F_j$.

**Lemma 8.** Let $G \in \mathcal{G}$. By a sequence of operations $[H_a, F_j, H_b]$ the graph $G$ can be transformed to a graph from $\mathcal{G}$, denote it by $G^\times$, such that the associated supergraph for $G^\times$ is a path with $F_i$ connecting $H_i$ with $H_{i+1}$, $1 \leq i \leq \ell - 1$, where one of $v_i^1$ and $v_i^2$ is identified with $u_i$, while the other vertex is identified with $u_{i+1}$.

**Proof.** Let $G \in \mathcal{G}$. First we transform $G$ to $G'$, where the associated supergraph for $G'$ is the path $(H_1, H_2, \ldots, H_\ell)$.

Assume that we already have a subpath $(H_1, H_2, \ldots, H_\ell)$ in the associated supergraph $S$. Due to the tree structure, there is a unique path from $H_i$ to $H_{i+1}$ in $S$. Denote by $F_a$ the last edge of this path and denote by $H_b$ the end vertex of $F_a$ which is different from $H_{i+1}$. Consequently, provide the operation $[H_b, F_a, H_i]$. Observe that after this operation we have the subpath $(H_1, H_2, \ldots, H_i, H_{i+1})$ in the associated supergraph. Hence, repeating this procedure we obtain the required graph $G' \in \mathcal{G}$.

Now we transform $G'$ to $G''$, where the associated supergraph for $G''$ is the path $(H_1, H_2, \ldots, H_\ell)$, analogously as in the associated supergraph for $G'$, but in $G''$ the supervertices $H_i$ and $H_{i+1}$ are connected by the superedge $F_i$.

Assume that we already have a subpath with superedges $(H_1, F_1, H_2, F_2, \ldots, F_{i-1}, H_i)$ in the associated supergraph $S$. Suppose that $F_i$ does not connect $H_i$ with $H_{i+1}$ in $S$, but instead it connects $H_j$ with $H_{j+1}$ for some $j$, $j > i$. For every $k$, $i \leq k < j$, denote by $F_k'$ the superedge connecting $H_k$ with $H_{k+1}$ in $S$ (see Fig. 2). Now provide the operation $[H_j, F_k, H_i]$. This attaches $F_i$ to $H_i$ (see Fig. 2). Consequently, provide $[H_j-1, F_{(j-1)'}', H_{j+1}], [H_j-2, F_{(j-2)'}', H_j], \ldots, [H_1, F_{\ell'}, H_{i+2}]$. Finally, provide $[H_{j+1}, F_i, H_{i+1}]$ (see Fig. 2). After these steps we have a subpath with edges $(H_1, F_1, H_2, F_2, \ldots, F_{i-1}, H_i, F_i, H_{i+1})$ in the associated supergraph. Hence, repeating this procedure we obtain the required graph $G'' \in \mathcal{G}$.

Finally, providing $[H_i, F_i, H_i]$ we can attach the corresponding vertex $v_i^j$ to $u_i$, and providing $[H_{i+1}, F_i, H_{i+1}]$ we can attach $v_i^{3-j}$ to $u_{i+1}$. Hence, repeating this procedure we can transform $G''$ to a graph $G^\times$, such that the associated supergraph for $G^\times$ is a path $(H_1, F_1, H_2, F_2, \ldots, F_{\ell-1}, H_\ell)$, where one of $v_j^1$ and $v_j^2$ is identified with $u_i$, while the other vertex is identified with $u_{i+1}$. □
In the next lemma we show that for the graph $G^x$ from Lemma 8 it holds $W(G^x) \equiv W(\Gamma_G) \pmod{r}$.

**Lemma 9.** Let $G \in \mathcal{G}$ and $1 \leq j \leq \ell - 1$. Let $G'$ be obtained from $G$ by reversing the superedge $F_j$. Then $W(G) \equiv W(G') \pmod{r}$.

**Proof.** Assume that $v^1_j$ was identified with $u_a \in V(H_a)$ and $v^2_j$ was identified with $u_b \in V(H_b)$ in $G$. Let $G^1$ and $G^2$ be the connected components obtained after detaching the vertex $v^1_j$ from $u_a$ and $v^2_j$ from $u_b$, such that $u_a \in V(G^1)$ and $u_b \in V(G^2)$.

Having two vertices of $V(G)$, their distance in $G$ differs from that in $G'$ only if one of the vertices is in $F_j - \{v^1_j, v^2_j\}$ and the other is in $V(G^1) \cup V(G^2)$. Hence,

$$W(G) - W(G') = \sum (d_G(x, y) - d_{G'}(x, y)) + \sum (d_G(y, z) - d_{G'}(y, z)),$$

where the first sum is taken over all $x \in V(G^1)$ and $y \in V(F_j - \{v^1_j, v^2_j\})$, while the second sum is taken over all $y \in V(F_j - \{v^1_j, v^2_j\})$ and $z \in V(G^2)$. Since all the paths from $x$ to $y$ (from $y$ to $z$, respectively) must pass through $u_a$ (through $u_b$, respectively), we have

$$W(G) - W(G') = |V(G^1)| \sum \left( d_G(u_a, y) - d_{G'}(u_a, y) \right) + |V(G^2)| \sum \left( d_G(y, u_b) - d_{G'}(y, u_b) \right),$$

where the sums are taken over all $y \in V(F_j - \{v^1_j, v^2_j\})$. Since $G'$ was obtained from $G$
by reversing $F_j$, we have $d_G(u, y) = d_{F_j}(u, y) = d_G(y, u)$ and $d_G'(u, y) = d_{F_j}(y, v_j^2) = d_G(y, u)$.

This yields

$$W(G) - W(G') = \left( |V(G^1)| - |V(G^2)| \right) \sum \left( d_{F_j}(v_j^1, y) - d_{F_j}(y, v_j^2) \right).$$

Since $|V(G^1)| \equiv |V(G^2)| \equiv r - t \pmod{r}$, we have $W(G) \equiv W(G') \pmod{r}$, as required.

By Lemmas 7, 8 and 9, for every $G \in \mathcal{G}$ it holds $W(G) \equiv W(\Gamma_G) \pmod{r}$. Hence, we have the main result of this section:

**Theorem 10.** Let $G_1, G_2 \in \mathcal{G}$. Then $W(G_1) \equiv W(G_2) \pmod{r}$.

Probably the most interesting case appears when all the superedges are paths. This yields the following corollaries of Theorem 10:

**Corollary 11.** Let $H_1, H_2, \ldots, H_\ell$ be a collection of cycles with lengths congruent to $r - t \pmod{r}$. Further, let $F_1, F_2, \ldots, F_{\ell-1}$ be a collection of paths of lengths congruent to $t + 1 \pmod{r}$. Finally, let $\mathcal{G}$ be a class of connected graphs obtained by identifying each end vertex of $F_j$’s with exactly one vertex of $H_1 \cup H_2 \cup \cdots \cup H_\ell$ so that the resulting graph is connected. Then for every $G_1, G_2 \in \mathcal{G}$ we have $W(G_1) \equiv W(G_2) \pmod{r}$.

**Corollary 12.** Let $H_1, H_2, \ldots, H_\ell$ be a collection of trees with numbers of vertices congruent to $r - t \pmod{r}$. Further, let $F_1, F_2, \ldots, F_{\ell-1}$ be a collection of paths of lengths congruent to $t + 1 \pmod{r}$. Finally, let $\mathcal{G}$ be a class of connected graphs obtained by identifying each end vertex of $F_j$’s with exactly one vertex of $H_1 \cup H_2 \cup \cdots \cup H_\ell$ so that the resulting graph is connected. Then for every $G_1, G_2 \in \mathcal{G}$ we have $W(G_1) \equiv W(G_2) \pmod{r}$.

Another interesting case appears when all the superedges are simple edges.

**Corollary 13.** Let $H_1, H_2, \ldots, H_\ell$ be a collection of connected graphs with numbers of vertices congruent to $0 \pmod{r}$. Let $\mathcal{G}$ be a class of connected graphs obtained by adding $\ell - 1$ edges to $H_1 \cup H_2 \cup \cdots \cup H_\ell$. Then for every $G_1, G_2 \in \mathcal{G}$ we have $W(G_1) \equiv W(G_2) \pmod{r}$.
For graphs having $T$-factors we have the following corollary, which also follows from Theorem 6.

**Corollary 14.** Let $T$ be a tree with $r$ vertices. Further, let $G_1$ and $G_2$ be trees with the same number of vertices, both having a $T$-factor. Then $W(G_1) \equiv W(G_2) \pmod{r}$.

We remark that an instance of Corollary 14 when $T$ is a path on $r$ vertices, $r$ being odd, is exactly the first part of Theorem 5.

### 3 Congruence modulo $2r$ when $r$ is even

In this section we generalize the second part of Theorem 5. However, there are some limitations in this case.

First, an analogue of Theorem 10 does not necessarily hold if some graph of $\mathcal{H}$ is not a tree even if $t = 0$ and $\ell = 2$. To demonstrate this, let $r$ be even, $r \geq 4$, $\ell = 2$, $t = 0$, $H_1$ is a path on $r$ vertices, $H_2$ is a cycle of length $r - 1$ with one pendant vertex attached, $F_1$ is an edge, and $v_1^1$ and $v_1^2$ are the end vertices of $F_1$. Denote by $u$ the vertex of degree 3 in $H_2$. In $G$ the edge $F_1$ joins an end vertex of $H_1$ with $u$, while in $G'$ the edge $F_1$ joins an end vertex of $H_1$ with a neighbour of $u$ on the cycle, see Fig. 3 for the case $r = 4$. Then $W(G') = W(G) + r$, and so $W(G') \not\equiv W(G) \pmod{2r}$. (For the case $r = 2$ it suffices to consider the same graph as for $r = 6$.)

![Figure 3: The graphs $G$ and $G'$ demonstrating that an analogue of Theorem 10 is not true if some graph in $\mathcal{H}$ is not a tree.](image)

Next, an analogue of Theorem 10 does not necessarily hold if $t > 0$ even if $\ell = 2$ and $H_1$, $H_2$ and $F_1$ are all paths. To demonstrate this, let $r$ be even, $r \geq 4$, $\ell = 2$, $t = 1$, $H_1$ and $H_2$ are paths on $r - 1$ vertices, $F_1$ is a path on 3 vertices, and $v_1^1$ and $v_1^2$ are different end vertices of $F_1$. Denote by $u_1$ and $u_2$ the central vertices of $H_1$ and $H_2$, respectively. In $G$ the end vertices of $F_1$ are identified with $u_1$ and $u_2$, while in $G'$ the end vertices of $F_1$ are identified with $u_1$ and a neighbour of $u_2$, see Fig. 4 for the case $r = 4$. Then
\(W(G') = W(G) + r\), and so \(W(G') \not\equiv W(G) \pmod{2r}\). (For the case \(r = 2\) it suffices to consider the same graph as for \(r = 6\).)

![Figure 4: The graphs \(G\) and \(G'\) demonstrating that an analogue of Theorem 10 is not true if \(t > 0\).]

In the light of the above examples we restrict ourselves to trees and to \(t = 0\). Hence, let \(r\) be an even number, \(r \geq 2\). Analogously as in Section 2, we choose three things. First, let \(\mathcal{H} = \{H_1, H_2, \ldots, H_\ell\}\) be a set of trees, such that for all \(i\), \(1 \leq i \leq \ell\), we have \(|V(H_i)| \equiv r \pmod{r}\). Second, let \(\mathcal{F} = \{F_1, F_2, \ldots, F_{\ell-1}\}\) be a set of trees, such that for all \(j\), \(1 \leq j \leq \ell - 1\), we have \(|V(F_j)| \equiv 2 \pmod{r}\). Third, for every \(F_j\), choose vertices \(v_j^1, v_j^2 \in V(F_j)\) (we remark that chosen vertices \(v_j^1\) and \(v_j^2\) are not necessarily distinct).

Now, when these three items (namely \(\mathcal{H}, \mathcal{F}\) and \(\{v_j^1, v_j^2\}_{j=1}^{\ell-1}\)) are chosen, identify the vertices \(v_j^i, 1 \leq i \leq 2\) and \(1 \leq j \leq \ell\), with some vertices of \(H_1 \cup H_2 \cup \cdots \cup H_\ell\) so that each \(v_j^i\) will be identified with exactly one vertex of \(H_1 \cup H_2 \cup \cdots \cup H_\ell\) (if \(v_j^1 = v_j^2\) then this vertex will be identified with two vertices of \(H_1 \cup H_2 \cup \cdots \cup H_\ell\)). Denote by \(G^T = G^T(\mathcal{H}, \mathcal{F})\) the class of those graphs obtained by this identification process, which are connected.

We prove that the Wiener index of all graphs in \(G^T\) belongs to the same congruence class modulo \(2r\). For this, we improve Lemmas 7 and 9. We start with Lemma 7.

**Lemma 15.** Let \(r\) be even and \(G, G' \in G^T\), where \(G'\) was obtained from \(G\) by the operation \([H_a, F_j, H_b]\). Then \(W(G) \equiv W(G') \pmod{2r}\).

**Proof.** Assume that \(v_j^k\) is identified with \(u \in V(H_a)\) in \(G\) and it is identified with \(u' \in V(H_b)\) in \(G'\). Further, denote by \(G^1\) and \(G^2\) the two components which appear after detaching \(v_j^k\) from \(u\) in \(G\). Assume that \(u \in V(G^1)\). Then the notation is identical with that in the proof of Lemma 7. Moreover, \(|V(G^1)| = ar\) and \(|V(G^2 - v_j^k)| = br\) for some integers \(a\) and \(b\) such that \((a + b)r = |V(G)|\). Let \(u = z_0, z_1, \ldots, z_f = u'\) be a path in \(G^1\). Denote by \(G_{z_i}\) a graph obtained from \(G^1 \cup G^2\) by identifying \(z_i\) with \(v_j^k\). Then \(G_{z_0} = G\) and \(G_{z_f} = G'\). Now fix \(i\), \(0 \leq i \leq f - 1\). We prove that \(W(G_{z_{i+1}}) \equiv W(G_{z_i}) \pmod{2r}\).
First, analogously as in the proof of Lemma 7 we have the following analogue of (1):

\[ W(G_{z_{i+1}}) - W(G_{z_i}) = \sum_{x} \left( d_{G_{z_{i+1}}}(x,y) - d_{G_{z_i}}(x,y) \right), \]

where the sum is taken over all \( x \in V(G^1) \) and \( y \in V(G^2 - v_j^k) \). A shortest path from \( x \in V(G^1) \) to \( y \in V(G^2 - v_j^k) \) contains \( z_i(=v_j^k) \) in \( G_{z_i} \) and \( z_{i+1}(=v_j^k) \) in \( G_{z_{i+1}} \). Since both \( G_{z_i} \) and \( G_{z_{i+1}} \) are trees, the difference \( d_{G_{z_{i+1}}}(x,y) - d_{G_{z_i}}(x,y) = d_{G_{z_{i+1}}}(x,z_{i+1}) - d_{G_{z_i}}(x,z_i) \) is either 1 or -1. Let \( s \) be the number of vertices \( x \in V(G^1) \) for which \( d_{G_{z_{i+1}}}(x,z_{i+1}) - d_{G_{z_i}}(x,z_i) = 1 \). Then

\[ W(G_{z_{i+1}}) - W(G_{z_i}) = \sum_{x} \sum_{y} \left( d_{G_{z_{i+1}}}(x,y) - d_{G_{z_i}}(x,y) \right) \]
\[ = br \sum_{x} \left( d_{G_{z_{i+1}}}(x,z_{i+1}) - d_{G_{z_i}}(x,z_i) \right) \]
\[ = br(s - (ar - s)) = 2brs - abr^2, \]

where \( x \in V(G^1) \) and \( y \in V(G^2 - v_j^k) \). Since \( r \) is even, we conclude \( W(G_{z_{i+1}}) \equiv W(G_{z_i}) \) (mod 2r).

In this way we get \( W(G_{z_0}) \equiv W(G_{z_1}) \equiv W(G_{z_2}) \equiv \cdots \equiv W(G_{z_j}) \) (mod 2r), and hence \( W(G) \equiv W(G') \) (mod 2r).

Now we prove an analogue of Lemma 9.

**Lemma 16.** Let \( r \) be even, \( G \in \mathcal{G}^T \) and \( 1 \leq j \leq \ell - 1 \). Let \( G' \) be obtained from \( G \) by reversing the superedge \( F_j \). Then \( W(G) \equiv W(G') \) (mod 2r).

**Proof.** Assume that \( v_j^1 \) was identified with \( u_a \in V(H_a) \) and \( v_j^2 \) was identified with \( u_b \in V(H_b) \) in \( G \). Let \( G^1 \) and \( G^2 \) be the connected components obtained after detaching the vertex \( v_j^1 \) from \( u_a \) and \( v_j^2 \) from \( u_b \), such that \( u_a \in V(G^1) \) and \( u_b \in V(G^2) \). Then analogously as in the proof of Lemma 9 we obtain

\[ W(G) - W(G') = \left( |V(G^1)| - |V(G^2)| \right) \sum \left( d_{F_j}(v_j^1, y) - d_{F_j}(y, v_j^2) \right), \]

(2)

where the sum is taken over all \( y \in V(F_j - \{v_j^1, v_j^2\}) \).

Let \( y \in V(F_j - \{v_j^1, v_j^2\}) \). Recall that \( F_j \) is a tree. Thus, if \( d_{F_j}(v_j^1, v_j^2) \) is even, then \( d_{F_j}(v_j^1, y) - d_{F_j}(y, v_j^2) \) is also even. On the other hand if \( d_{F_j}(v_j^1, v_j^2) \) is odd, then \( d_{F_j}(v_j^1, y) - d_{F_j}(y, v_j^2) \) is also odd. Since \( |V(F_j)| \equiv 2 \) (mod \( r \)), there is even number of
vertices in $V(F_j - \{v_j^1, v_j^2\})$, and so the sum in (2) is even. Finally, from $|V(G^1)| \equiv |V(G^2)| \equiv 0 \pmod{r}$, we conclude $W(G) \equiv W(G') \pmod{2r}$.

By Lemmas 8, 15 and 16, for every $G \in \mathcal{G}^T$ it holds $W(G) \equiv W(\Gamma_{gT}) \pmod{2r}$. Hence, we obtain the main result of this section:

**Theorem 17.** Let $r$ be even and $G_1, G_2 \in \mathcal{G}^T$. Then $W(G_1) \equiv W(G_2) \pmod{2r}$.

Analogously as in the previous section, we present some corollaries of Theorem 17. If all the superedges are paths, we obtain:

**Corollary 18.** Let $r$ be even and let $H_1, H_2, \ldots, H_\ell$ be a collection of trees whose numbers of vertices are congruent to 0 (mod $r$). Further, let $F_1, F_2, \ldots, F_{\ell-1}$ be a collection of paths of lengths congruent to 1 (mod $r$). Finally, let $\mathcal{G}$ be a class of connected graphs obtained by identifying each end vertex of $F_j$’s with exactly one vertex of $H_1 \cup H_2 \cup \cdots \cup H_\ell$ so that the resulting graph is connected. Then for every $G_1, G_2 \in \mathcal{G}$ we have $W(G_1) \equiv W(G_2) \pmod{2r}$.

Another interesting case appears when all the superedges are simple edges.

**Corollary 19.** Let $r$ be even and let $H_1, H_2, \ldots, H_\ell$ be a collection of trees whose numbers of vertices are congruent to 0 (mod $r$). Let $\mathcal{G}$ be a class of connected graphs obtained by adding $\ell - 1$ edges to $H_1 \cup H_2 \cup \cdots \cup H_\ell$. Then for every $G_1, G_2 \in \mathcal{G}$ we have $W(G_1) \equiv W(G_2) \pmod{2r}$.

For graphs having $T$-factors we have the following corollary.

**Corollary 20.** Let $r$ be even and let $T$ be a tree with $r$ vertices. Further, let $G_1$ and $G_2$ be trees with the same number of vertices, both having a $T$-factor. Then $W(G_1) \equiv W(G_2) \pmod{2r}$.

We remark that an instance of Corollary 20 when $T$ is a path on $r$ vertices, $r$ being even, is exactly the second part of Theorem 5.

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